Universität Koblenz-Landau

FB 4 Informatik

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Exercises for "Non-Classical Logics"
Exercise sheet 5

Exercise 5.1: (*3 P*)
Consider the first-order many-valued logic defined as follows:

Syntax: The signature is $\Sigma = (\Omega, \Pi)$, where $\Omega = \{0/0, m/0, M/0, +/2\}, \Pi = \{\mathsf{large}/1, \leq /2\}$; X is a set of variables.

Semantics:

- The set of truth values is $M = \{0, u, 1\}$.
- The logical operations are $\mathcal{F} = \{\top/0, \bot\ /0, \neg/1, \sim /1, \vee/2, \wedge/2\}$ with the following truth tables: $\top_M = 1, \bot_M = 0$ and:

A	¬ A	∼ A
1	0	0
u	u	1
0	1	1

$\vee_M$	0	u	1
0	0	u	1
u	u	u	1
1	1	1	1

$\wedge_M$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

- The set of quantifiers is $\mathcal{Q} = \{\forall, \exists, Q\}$ where the truth tables of the quantifiers are:

$S \subseteq M$	$\forall_M(S)$	$\exists_M(S)$	$Q_M(S)$
$\{0\}$	0	0	0
$\{u\}$	u	u	0
$\{1\}$	1	1	1
$\{0, u\}$	0	u	0
$\{0, 1\}$	0	1	0
$\{u, 1\}$	u	1	1
$\{0, u, 1\}$	0	1	0

In other words, if we assume that $0 < u < 1$ then: $\forall_M(S) = \mathsf{min}(S), \exists_M(S) = \mathsf{max}(S)$, $Q_M(S) = \begin{cases} 1 & \text{if } S = \{1\} \text{ or } S = \{u, 1\} \\ 0 & \text{otherwise} \end{cases}$.

Consider the following many-valued Σ-structure:

$\mathcal{A} = (\mathbb{N}, \{0_{\mathcal{A}}, m_{\mathcal{A}}, M_{\mathcal{A}}, +_{\mathcal{A}} : \mathbb{N}^2 \rightarrow \mathbb{N}\}, \{\mathsf{large}_{\mathcal{A}} : \mathbb{N} \rightarrow \{0, u, 1\}, \leq : \mathbb{N}^2 \rightarrow \{0, u, 1\}\})$, where:

- $0_{\mathcal{A}} = 0, m_{\mathcal{A}} = 50, M_{\mathcal{A}} = 100, +_{\mathcal{A}}$ is normal addition;
- $\mathsf{large}_{\mathcal{A}}(n) = \begin{cases} 0 & \text{if } n < 50 \\ u & \text{if } 50 \leq n < 100 \\ 1 & \text{if } n \geq 100 \end{cases}$
- $\leq (n, m)$ is 1 if n is smaller than or equal to m and 0 otherwise.

Let $\beta : X \to \mathbb{N}$ with $\beta(x) = 100, \beta(y) = 1000$. Compute:

- $\mathcal{A}(\beta)(\mathsf{large}(x + y))$.
- $\mathcal{A}(\beta)(\mathsf{large}(m) \vee (x \leq m))$.
- $\mathcal{A}(\beta)(\forall x(\neg(M \leq x) \vee \mathsf{large}(x)))$.
- $\mathcal{A}(\beta)(\forall x(\neg(m \leq x) \vee \mathsf{large}(x)))$.
- $\mathcal{A}(\beta)(\exists x((m \leq x) \wedge \mathsf{large}(x)))$.
- $\mathcal{A}(\beta)(\exists x((x \leq m) \wedge \mathsf{large}(x)))$.
- $\mathcal{A}(\beta)(Qx(\neg(M \leq x) \vee \mathsf{large}(x)))$.
- $\mathcal{A}(\beta)(Qx(\neg(m \leq x) \vee \mathsf{large}(x)))$.

Exercise 5.2: (*3 P*)
We say that an $\mathcal{L}_3$-formula F is a non-tautology iff for every 3-valued structure, $\mathcal{A}$ and every valuation $\beta : X \to \mathcal{A}$, we have $\mathcal{A}(\beta)(F) \neq 1$. We say that an $\mathcal{L}_3$-formula F is two-valued iff for every 3-valued structure, $\mathcal{A}$ and every valuation $\beta, \mathcal{A}(\beta)(F) \in \{0, 1\}$.

Which of the following statements are true? Justify your answer.

(a) If $F \equiv G$ is a tautology and F is a tautology then G is a tautology.
(b) If F id G is a tautology and F is a tautology then G is a tautology.
(c) If $F \approx G$ is a tautology and F is a tautology then G is a tautology.

(d) If $F \equiv G$ is a tautology and F is satisfiable then G is satisfiable.
(e) If F id G is a tautology and F is satisfiable then G is satisfiable.
(f) If $F \approx G$ is a tautology and F is satisfiable then G is satisfiable.

(g) If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.
(h) If F id G is a tautology and F is a non-tautology then G is a non-tautology.
(i) If $F \approx G$ is a tautology and F is a non-tautology then G is a non-tautology.

(j) If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.
(k) If F id G is a tautology and F is two-valued then G is two-valued.
(l) If $F \approx G$ is a tautology and F is two-valued then G is two-valued.

Exercise 5.3: (*2 P*)
Prove that the following formulae are $\mathcal{L}_3$ tautologies:

(a) $(\forall x q(x)) \supset q(x)[t/x]$ (for every term t).
(b) $\neg(\exists x F)$ id $\forall x(\neg F)$.

where the truth table of id is presented in the slides of lecture from 26.11.2014.

Please submit your solution until Monday, December 1, 2014, at 18:00. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name(s) on your solution.

Submission possibilities:

- By e-mail to `sofronie@uni-koblenz.de` with the keyword "Homework Non-Classical Logics" in the subject.
- Put it in the box in front of Room B 222.