Non-classical logics

Lecture 2: Classical logic, Part 2

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Last time

- Propositional logic (Syntax, Semantics)
- Problems: Checking unsatisfiability

  NP complete

  PTIME for certain fragments of propositional logic
- Normal forms (CNF/DNF)
- Translations to CNF/DNF

Decision Procedures for Satisfiability

- Simple Decision Procedures

  truth table method

- The Resolution Procedure

- Tableaux

...

Today

- Methods for checking satisfiability

    The Resolution Procedure

    Semantic Tableaux

The Propositional Resolution Calculus

Resolution inference rule:

$$\frac{C \vee A \qquad \neg A \vee D}{C \vee D}$$

Terminology: $C \vee D$: resolvent; A: resolved atom

(Positive) factorisation inference rule:

$$\frac{C \vee A \vee A}{C \vee A}$$

These are schematic inference rules; for each substitution of the schematic variables C, D, and A, respectively, by propositional clauses and atoms we obtain an inference rule.

As "$\vee$" is considered associative and commutative, we assume that A and $\neg A$ can occur anywhere in their respective clauses.

Sample Refutation

1.	$\neg P \vee \neg P \vee Q$	(given)
2.	$P \vee Q$	(given)
3.	$\neg R \vee \neg Q$	(given)
4.	R	(given)
5.	$\neg P \vee Q \vee Q$	(Res. 2. into 1.)
6.	$\neg P \vee Q$	(Fact. 5.)
7.	$Q \vee Q$	(Res. 2. into 6.)
8.	Q	(Fact. 7.)
9.	$\neg R$	(Res. 8. into 3.)
10.	$\bot$	(Res. 4. into 9.)

Soundness and Completeness

Theorem 1.6. Propositional resolution is sound.

for both the resolution rule and the positive factorization rule

the conclusion of the inference is entailed by the premises.

If N is satisfiable, we cannot deduce $\bot$ from N using the inference rules of the propositional resolution calculus.

If we can deduce $\bot$ from N using the inference rules of the propositional resolution calculus then N is unsatisfiable

Theorem 1.7. Propositional resolution is refutationally complete.

If $N \models \bot$ we can deduce $\bot$ starting from N and using the inference rules of the propositional resolution calculus.

Notation

$N \vdash_{Res} \bot$: we can deduce $\bot$ starting from N and using the inference rules of the propositional resolution calculus.

Completeness of Resolution

How to show refutational completeness of propositional resolution:

- We have to show: $N \models \bot \;\Rightarrow\; N \vdash_{Res} \bot$, or equivalently: If $N \not\vdash_{Res} \bot$, then N has a model.

- Idea: Suppose that we have computed sufficiently many inferences (and not derived $\bot$).

  Now order the clauses in N according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of valuations.

- The limit valuation can be shown to be a model of N.

Clause Orderings

1. We assume that $\succ$ is any fixed ordering on propositional variables that is *total* and well-founded.

2. Extend $\succ$ to an ordering $\succ_L$ on literals:

$$
\begin{array}{lcll}
[\neg]P & \succ_L & [\neg]Q & \text{, if } P \succ Q \\
\neg P & \succ_L & P &
\end{array}
$$

3. Extend $\succ_L$ to an ordering $\succ_C$ on clauses:
   $\succ_C = (\succ_L)_{\text{mul}}$, the multi-set extension of $\succ_L$.

   Notation: $\succ$ also for $\succ_L$ and $\succ_C$.

(well-founded)

Multi-Set Orderings

Let $(M, \succ)$ be a partial ordering. The multi-set extension of $\succ$ to multi-sets over M is defined by

$$
\begin{aligned}
S_1 \succ_{\mathsf{mul}} S_2 :\Leftrightarrow\ & S_1 \neq S_2 \\
& \text{and } \forall m \in M : [S_2(m) > S_1(m) \\
& \quad \Rightarrow \quad \exists m' \in M : (m' \succ m \text{ and } S_1(m') > S_2(m'))]
\end{aligned}
$$

Theorem 1.11:

a) $\succ_{\mathsf{mul}}$ is a partial ordering.

b) $\succ$ well-founded $\Rightarrow$ $\succ_{\mathsf{mul}}$ well-founded

c) $\succ$ total $\Rightarrow$ $\succ_{\mathsf{mul}}$ total

Proof:

see Baader and Nipkow, page 22–24.

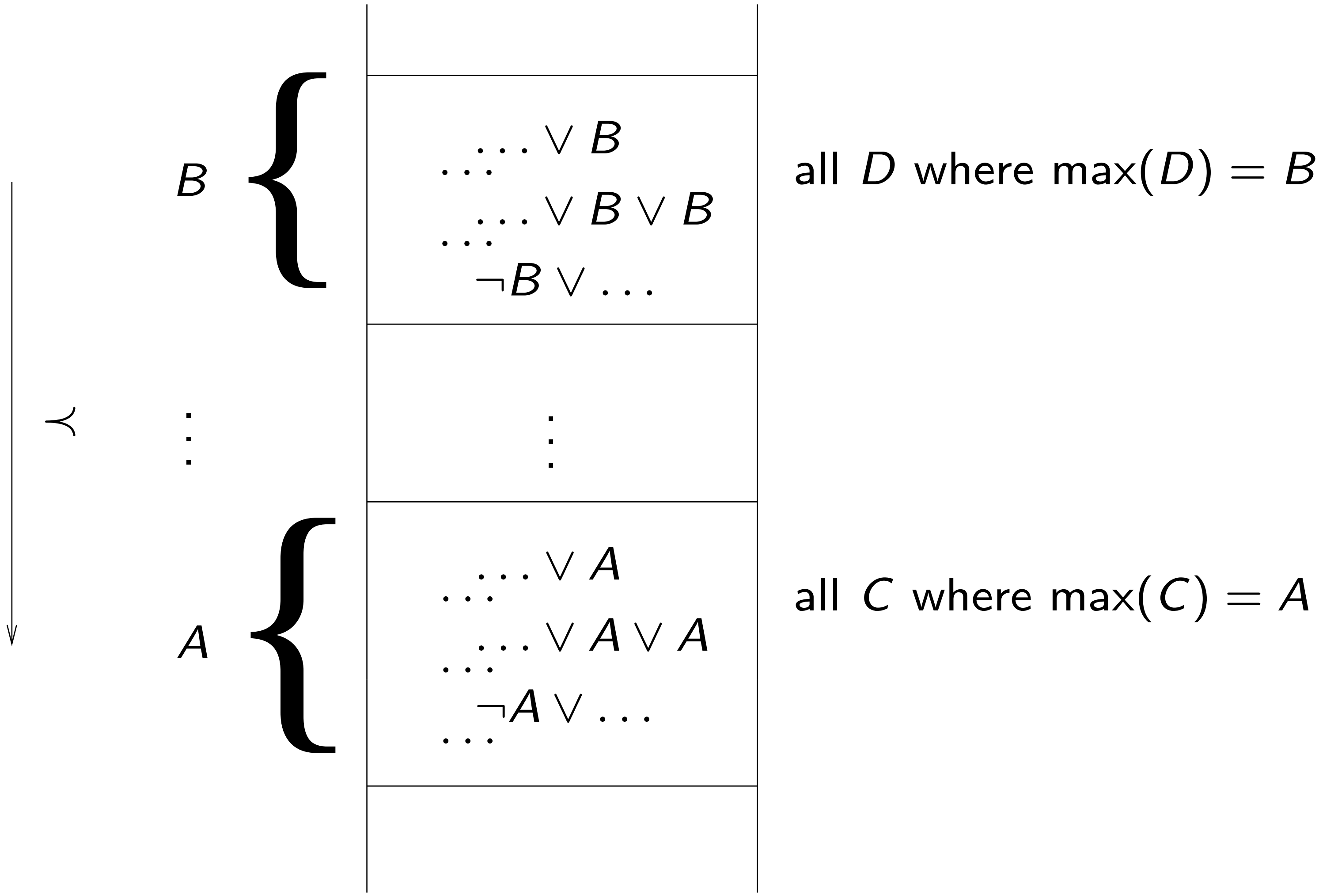
Example

Suppose $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$. Then:

$$\begin{array}{cc} & P_0 \vee P_1 \\ \prec & P_1 \vee P_2 \\ \prec & \neg P_1 \vee P_2 \\ \prec & \neg P_1 \vee P_4 \vee P_3 \\ \prec & \neg P_1 \vee \neg P_4 \vee P_3 \\ \prec & \neg P_5 \vee P_5 \end{array}$$

Stratified Structure of Clause Sets

Let $A \succ B$. Clause sets are then stratified in this form:



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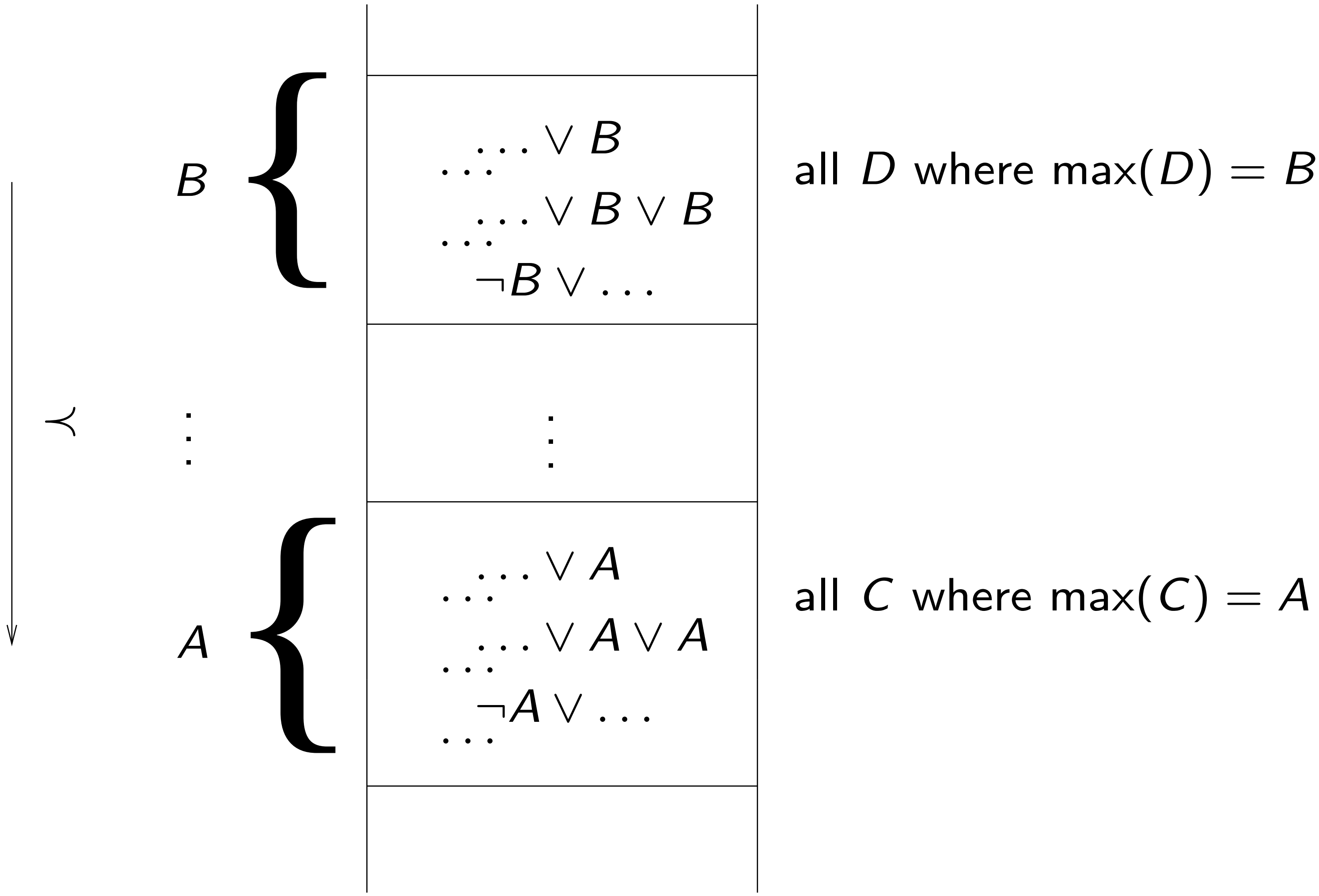

Closure of Clause Sets under *Res*

$$Res(N) = \{C \mid C \text{ is concl. of a rule in } Res \text{ w/ premises in } N\}$$

$$Res^0(N) = N$$

$$Res^{n+1}(N) = Res(Res^n(N)) \cup Res^n(N), \text{ for } n \geq 0$$

$$Res^*(N) = \textstyle\bigcup_{n\geq 0} Res^n(N)$$

N is called saturated (wrt. resolution), if $Res(N) \subseteq N$.

Proposition 1.12

(i) $Res^*(N)$ is saturated.

(ii) *Res* is refutationally complete, iff for each set N of ground clauses:

$$N \models \bot \;\Leftrightarrow\; \bot \in Res^*(N)$$

Construction of Interpretations

Given: set N of clauses, atom ordering $\succ$.

Wanted: Valuation $\mathcal{A}$ such that

- "many" clauses from N are valid in $\mathcal{A}$;
- $\mathcal{A} \models N$, *if* N *is saturated and* $\bot \notin N$.

Construction according to $\succ$, starting with the minimal clause.

Main Ideas of the Construction

- Clauses are considered in the order given by $\prec$. We construct a model for N incrementally.

- When considering C, one already has a partial interpretation I_C (initially $I_C = \emptyset$) available.

  In what follows, instead of referring to partial valuations $\mathcal{A}_C$ we will refer to partial interpretations I_C (the set of atoms which are true in the valuation $\mathcal{A}_C$).

- If C is true in the partial interpretation I_C, nothing is done. ($\Delta_C = \emptyset$).

- If C is false, one would like to change I_C such that C becomes true.

Example

Let $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$ (max. literals in red)

	clauses C	$I_C = \mathcal{A}_C^{-1}(1)$	Δ_C	Remarks
1	$\neg P_0$			
2	$P_0 \vee P_1$			
3	$P_1 \vee P_2$			
4	$\neg P_1 \vee P_2$			
5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$			
6	$\neg P_1 \vee \neg P_4 \vee P_3$			
7	$\neg P_1 \vee P_5$			

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1	$\neg P_0$	$\emptyset$	$\emptyset$	true in $\mathcal{A}_C$
2	$P_0 \vee P_1$			
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4	$\neg P_1 \vee P_2$			
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	clauses C	$I_C = \mathcal{A}_C^{-1}(1)$	Δ_C	Remarks
1	$\neg P_0$	$\emptyset$	$\emptyset$	true in $\mathcal{A}_C$
2	$P_0 \vee P_1$	$\emptyset$	$\{P_1\}$	P_1 maximal
3	$P_1 \vee P_2$			
4	$\neg P_1 \vee P_2$			
5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$			
6	$\neg P_1 \vee \neg P_4 \vee P_3$			
7	$\neg P_1 \vee P_5$			

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Let $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$ (max. literals in red)

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4	$\neg P_1 \vee P_2$	$\{P_1\}$	$\{P_2\}$	P_2 maximal
5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$	$\{P_1, P_2\}$	$\{P_4\}$	P_4 maximal
6	$\neg P_1 \vee \neg P_4 \vee P_3$			
7	$\neg P_1 \vee P_5$			

Example

Let $P_5 \succ P_4 \succ P_3 \succ P_2 \succ P_1 \succ P_0$ (max. literals in red)

	clauses C	$I_C = \mathcal{A}_C^{-1}(1)$	Δ_C	Remarks
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2	$P_0 \vee P_1$	$\emptyset$	$\{P_1\}$	P_1 maximal
3	$P_1 \vee P_2$	$\{P_1\}$	$\emptyset$	true in $\mathcal{A}_C$
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5	$\neg P_1 \vee P_4 \vee P_3 \vee P_0$	$\{P_1, P_2\}$	$\{P_4\}$	P_4 maximal
6	$\neg P_1 \vee \neg P_4 \vee P_3$	$\{P_1, P_2, P_4\}$	$\emptyset$	P_3 not maximal; *min. counter-ex.*
7	$\neg P_1 \vee P_5$	$\{P_1, P_2, P_4\}$	$\{P_5\}$	

$I = \{P_1, P_2, P_4, P_5\} = \mathcal{A}^{-1}(1)$: $\mathcal{A}$ is not a model of the clause set

$\Rightarrow$ there exists a counterexample.

Main Ideas of the Construction

- Clauses are considered in the order given by $\prec$.
- When considering C, one already has a partial interpretation I_C (initially $I_C = \emptyset$) available.
- If C is true in the partial interpretation I_C, nothing is done. ($\Delta_C = \emptyset$).
- If C is false, one would like to change I_C such that C becomes true.

Main Ideas of the Construction

- Changes should, however, be *monotone*. One never deletes anything from I_C and the truth value of clauses smaller than C should be maintained the way it was in I_C.

- Hence, one chooses $\Delta_C = \{A\}$ if, and only if, C is false in I_C, if A occurs positively in C (*adding A will make C become true*) and if this occurrence in C is strictly maximal in the ordering on literals (*changing the truth value of A has no effect on smaller clauses*).

Resolution Reduces Counterexamples

$$\frac{\neg P_1 \lor P_4 \lor P_3 \lor P_0 \qquad \neg P_1 \lor \neg P_4 \lor P_3}{\neg P_1 \lor \neg P_1 \lor P_3 \lor P_3 \lor P_0}$$

Construction of I for the extended clause set:

	clauses C	I_C	Δ_C	Remarks
1	$\neg P_0$	$\emptyset$	$\emptyset$	
2	$P_0 \lor P_1$	$\emptyset$	$\{P_1\}$	
3	$P_1 \lor P_2$	$\{P_1\}$	$\emptyset$	
4	$\neg P_1 \lor P_2$	$\{P_1\}$	$\{P_2\}$	
8	$\neg P_1 \lor \neg P_1 \lor P_3 \lor P_3 \lor P_0$	$\{P_1, P_2\}$	$\emptyset$	P_3 occurs twice *minimal counter-ex.*
5	$\neg P_1 \lor P_4 \lor P_3 \lor P_0$	$\{P_1, P_2\}$	$\{P_4\}$	
6	$\neg P_1 \lor \neg P_4 \lor P_3$	$\{P_1, P_2, P_4\}$	$\emptyset$	counterexample
7	$\neg P_1 \lor P_5$	$\{P_1, P_2, P_4\}$	$\{P_5\}$	

The same I, but smaller counterexample, hence some progress was made.

Factorization Reduces Counterexamples

$$\frac{\neg P_1 \lor \neg P_1 \lor P_3 \lor P_3 \lor P_0}{\neg P_1 \lor \neg P_1 \lor P_3 \lor P_0}$$

Construction of I for the extended clause set:

	clauses C	I_C	Δ_C	Remarks
1	$\neg P_0$	$\emptyset$	$\emptyset$	
2	$P_0 \lor P_1$	$\emptyset$	$\{P_1\}$	
3	$P_1 \lor P_2$	$\{P_1\}$	$\emptyset$	
4	$\neg P_1 \lor P_2$	$\{P_1\}$	$\{P_2\}$	
9	$\neg P_1 \lor \neg P_1 \lor P_3 \lor P_0$	$\{P_1, P_2\}$	$\{P_3\}$	
8	$\neg P_1 \lor \neg P_1 \lor P_3 \lor P_3 \lor P_0$	$\{P_1, P_2, P_3\}$	$\emptyset$	true in $\mathcal{A}_C$
5	$\neg P_1 \lor P_4 \lor P_3 \lor P_0$	$\{P_1, P_2, P_3\}$	$\emptyset$	
6	$\neg P_1 \lor \neg P_4 \lor P_3$	$\{P_1, P_2, P_3\}$	$\emptyset$	true in $\mathcal{A}_C$
7	$\neg P_3 \lor P_5$	$\{P_1, P_2, P_3\}$	$\{P_5\}$	

The resulting $I = \{P_1, P_2, P_3, P_5\}$ is a model of the clause set.

Construction of Candidate Models Formally

Let $N, \succ$ be given. We define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N,\ C = C' \vee A,\ A \succ C',\ I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

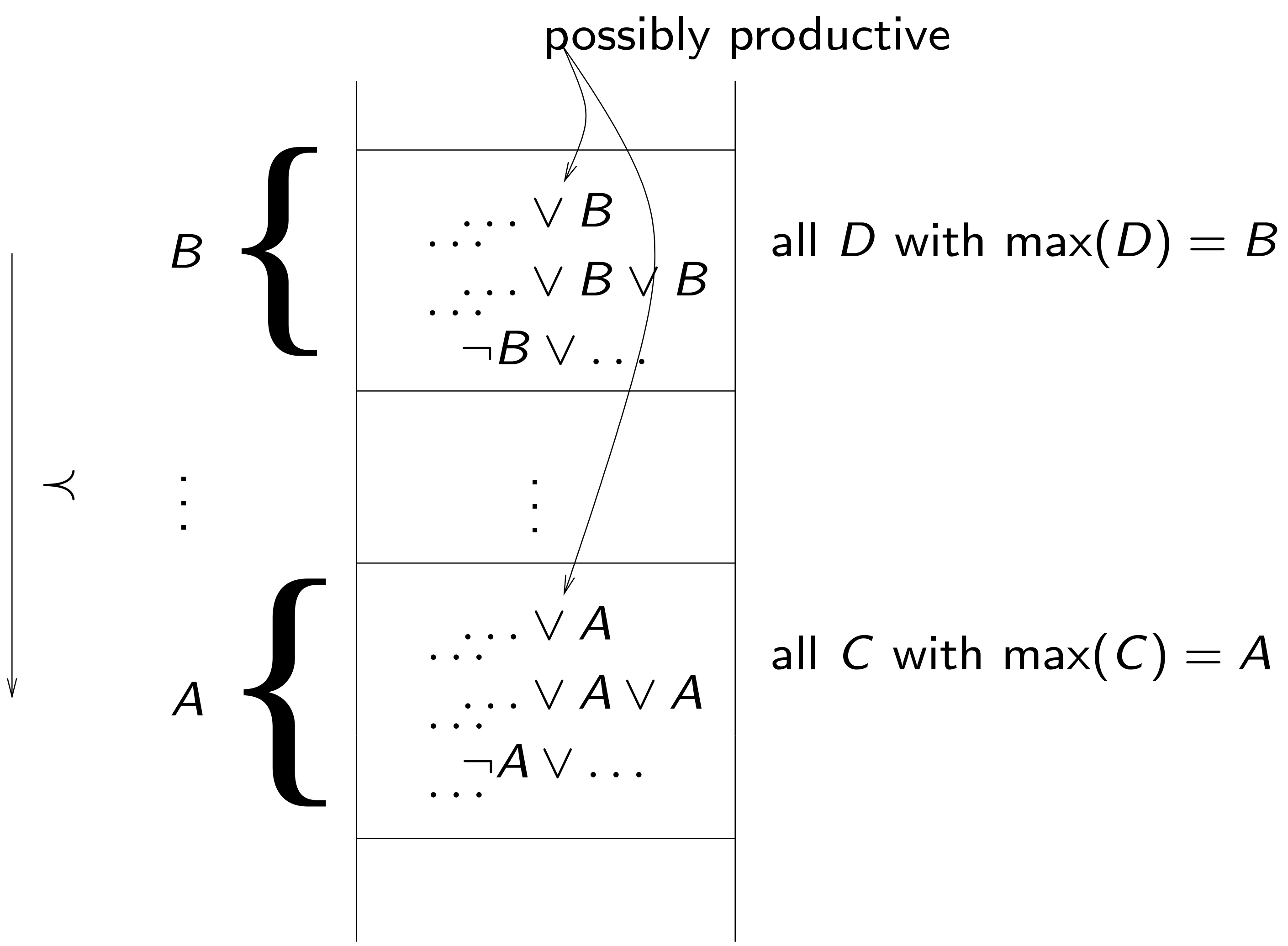
We say that C produces A, if $\Delta_C = \{A\}$.

The candidate model for N (wrt. $\succ$) is given as $I_N^{\succ} := \bigcup_C \Delta_C$.

We also simply write I_N, or I, for $I_N^{\succ}$ if $\succ$ is either irrelevant or known from the context.

Structure of $N, \succ$

Let $A \succ B$; producing a new atom does not affect smaller clauses.



Model Existence Theorem

Theorem 1.14 (Bachmair & Ganzinger):
Let $\succ$ be a clause ordering, let N be saturated wrt. *Res*, and suppose that $\bot \not\in N$. Then $I_N^{\succ} \models N$.

Corollary 1.15:
Let N be saturated wrt. *Res*. Then $N \models \bot \Leftrightarrow \bot \in N$.

Model Existence Theorem

Proof:

Suppose $\bot \notin N$, but $I_N^\succ \not\models N$. Let $C \in N$ minimal (in $\succ$) such that $I_N^\succ \not\models C$. Since C is false in I_N, C is not productive. As $C \neq \bot$ there exists a maximal atom A in C.

Case 1: $C = \neg A \vee C'$ (i.e., the maximal atom occurs negatively)
$\Rightarrow I_N \models A$ and $I_N \not\models C'$
$\Rightarrow$ some $D = D' \vee A \in N$ produces A. As $\frac{D' \vee A \qquad \neg A \vee C'}{D' \vee C'}$, we infer that $D' \vee C' \in N$, and $C \succ D' \vee C'$ and $I_N \not\models D' \vee C'$
$\Rightarrow$ contradicts minimality of C.

Case 2: $C = C' \vee A \vee A$. Then $\frac{C' \vee A \vee A}{C' \vee A}$ yields a smaller counterexample $C' \vee A \in N$. $\Rightarrow$ contradicts minimality of C.

Ordered Resolution with Selection

Ideas for improvement:

1. In the completeness proof (Model Existence Theorem) one only needs to resolve and factor maximal atoms
   $\Rightarrow$ if the calculus is restricted to inferences involving maximal atoms, the proof remains correct
   $\Rightarrow$ *order restrictions*

2. In the proof, it does not really matter with which negative literal an inference is performed
   $\Rightarrow$ choose a negative literal don't-care-nondeterministically
   $\Rightarrow$ *selection*

Selection Functions

A selection function is a mapping

$$S : C \quad \mapsto \quad \text{set of occurrences of } \textit{negative} \text{ literals in } C$$

Example of selection with selected literals indicated as $\boxed{X}$:

$$\boxed{\neg A} \vee \neg A \vee B$$

$$\boxed{\neg B_0} \vee \boxed{\neg B_1} \vee A$$

Ordered resolution

In the completeness proof, we talk about (strictly) maximal literals of clauses.

Resolution Calculus $Res_S^{\succ}$

$$\frac{C \vee A \qquad D \vee \neg A}{C \vee D}$$ [ordered resolution with selection]

if

(i) $A \succ C$;

(ii) nothing is selected in C by S;

(iii) $\neg A$ is selected in $D \vee \neg A$,
or else nothing is selected in $D \vee \neg A$ and $\neg A \succeq \max(D)$.

Note: For positive literals, $A \succ C$ is the same as $A \succ \max(C)$.

Resolution Calculus $Res_S^{\succ}$

$$\frac{C \vee A \vee A}{(C \vee A)} \qquad \text{[ordered factoring]}$$

if A is maximal in C and nothing is selected in C.

Search Spaces Become Smaller

1	$A \vee B$	
2	$A \vee \boxed{\neg B}$	
3	$\neg A \vee B$	
4	$\neg A \vee \boxed{\neg B}$	
5	$B \vee B$	Res 1, 3
6	B	Fact 5
7	$\neg A$	Res 6, 4
8	A	Res 6, 2
9	$\bot$	Res 8, 7

we assume $A \succ B$ and S as indicated by $\boxed{X}$. The maximal literal in a clause is depicted in red.

With this ordering and selection function the refutation proceeds strictly deterministically in this example. Generally, proof search will still be non-deterministic but the search space will be much smaller than with unrestricted resolution.

$\text{Res}_S^{\succ}$: Construction of Candidate Models

Let $N, \succ$ be given. We define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C \quad := \quad \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C \quad := \quad \begin{cases} \{A\}, & \text{if } C \in N,\ C = C' \vee A,\ A \succ C',\ I_C \not\models C \\ & \text{and nothing is selected in } C \\ \emptyset, & \text{otherwise} \end{cases}$$

We say that C produces A, if $\Delta_C = \{A\}$.

The candidate model for N (wrt. $\succ$) is given as $I_N^{\succ} := \bigcup_C \Delta_C$.

We also simply write I_N, or I, for $I_N^{\succ}$ if $\succ$ is either irrelevant or known from the context.

Model Existence Theorem

Theorem 1.14^s (Bachmair & Ganzinger):
Let $\succ$ be a clause ordering, let N be saturated wrt. $Res_S^{\succ}$, and suppose that $\bot \notin N$. Then $I_N^{\succ} \models N$.

Corollary 1.15^s:
Let N be saturated wrt. $Res_S^{\succ}$. Then $N \models \bot \Leftrightarrow \bot \in N$.

Model Existence Theorem

Proof:

Suppose $\bot \notin N$, but $I_N^{\succ} \not\models N$. Let $C \in N$ minimal (in $\succ$) such that $I_N^{\succ} \not\models C$. Since C is false in I_N, C is not productive. As $C \neq \bot$ there exists a maximal atom A in C.

Case 1: $C = \neg A \vee C'$

(i.e., the maximal atom occurs negatively or $\neg A$ is selected in C)

$\Rightarrow I_N \models A$ and $I_N \not\models C'$

$\Rightarrow$ some $D = D' \vee A \in N$ produces A. As $\frac{D' \vee A \qquad \neg A \vee C'}{D' \vee C'}$, we infer that $D' \vee C' \in N$, and $C \succ D' \vee C'$ and $I_N \not\models D' \vee C'$

$\Rightarrow$ contradicts minimality of C.

Case 2: $C = C' \vee A \vee A$. Then $\frac{C' \vee A \vee A}{C' \vee A}$ yields a smaller counterexample $C' \vee A \in N$. $\Rightarrow$ contradicts minimality of C.