The Modal Logic K

Contents

1 Soundness and Completeness; Decidability

We will show that the inference systems of the propositional modal logic K is sound and complete and that the modal logic K has the finite model property.

1.1 Soundness

Theorem. If the formula F is provable in the inference system for the modal logic K then F is valid in all Kripke frames.

Proof: Induction of the length of the proof, unsing the following facts:

1. The axioms are valid in every Kripke structure. Easy computation.
2. If the premises of an inference rule are valid in a Kripke structure $\mathcal{K}$, the conclusion is also valid in $\mathcal{K}$.

(MP) If $\mathcal{K} \models F, \mathcal{K} \models F \rightarrow G$ then $\mathcal{K} \models G$ (follows from the fact that for every state s of $\mathcal{K}$, if $(\mathcal{K}, s) \models F, (\mathcal{K}, s) \models F \rightarrow G$ then $(\mathcal{K}, s) \models G$).

(Gen) Assume that $\mathcal{K} \models F$. Then $(\mathcal{K}, s) \models F$ for every state s of $\mathcal{K}$.

Let t be a state of $\mathcal{K}$. $(\mathcal{K}, t) \models \Box F$ if for all t' with $(t, t') \in R$ we have $(\mathcal{K}, t') \models F$. But under the assumption that $\mathcal{K} \models F$ the latter is always the case. This shows that $(\mathcal{K}, t) \models \Box F$ for all t.

1.2 Completeness: Proof idea

Theorem. If the formula F is is valid in all Kripke frames then F is provable in the inference system for the modal logic K.

Idea of the proof: Assume that F is valid but not provable in the inference system for the modal logic K. We show that:

(1) $\neg F$ is "consistent" with the set L of all theorems of K

(2) We can construct a "canonical" Kripke structure $\mathcal{K}$ and a state w of $\mathcal{K}$ such that $(\mathcal{K}, w) \models \neg F$.

Contradiction!

We construct the Kripke structure $\mathcal{K}$ as follows:

1. We know that if F is not provable then $\neg F$ must be consistent with the set L of all theorems of K.
2. This means that $L \cup \{\neg F\}$ is consistent.
3. We show that every consistent set of formulae is contained in a maximal consistent set of formulae.
4. We choose a set S of states, in which every state is a maximal consistent set W of modal formulae (a "possible world").
5. We define a suitable relation R on S as explained on the slides.
6. Let $\mathcal{K}$ be the Kripke model defined this way.
   We prove that $(\mathcal{K}, W) \models \phi$ iff $\phi \in W$. Thus if $W_{\neg F}$ is the maximal consistent set containing $\neg F$ then $(\mathcal{K}, W_{\neg F}) \models \neg F$.

2 Decidability

Theorem. If a formula F has n subformulae, then F is valid in all frames iff F is valid in all frames having at most 2^n elements.

Idea of proof The direct implication is obvious. To prove the converse, we assume that there exists a Kripke structure $\mathcal{K} = (S, R, I)$ and a state $s_0 \in S$ such that $(\mathcal{K}, s_0) \models \neg F$. We construct a Kriple structure with at most 2^n elements where this is the case.

- We consider the family Γ of all subformulae of F.
  Γ is finite (has n elements) and is closed under subformulae.
- We now say that two states $s, s' \in S$ are equivalent (and can be merged) if for every $G \in \Gamma$, $(\mathcal{K}, s) \models G$ iff $(\mathcal{K}, s') \models G$ (i.e. if s and s' satisfy the same subformulae of F, in other words if we cannot distinguish these states if we only look where the subformulae of F in Γ are true or false).

- We merge equivalent states in S (i.e. we partition S into equivalence classes and define a new set of states $S' = S/\sim$, in which a state is the representative of an equivalence class of states in S).
- We define the relation R' on S' such that if sRs' then $[s]R'[s']$. The labelling is defined similarly.
- We now show that this new structure $\mathcal{K}' = (S/\sim, R', I)$ is a Kripke structure with $(\mathcal{K}', [s_0]) \models \neg F$.

If we analyse the structure $\mathcal{K}' = (S/\sim, R', I)$, we note that every state in $S/\sim$ is the representative of a set of states in S at which certain subformulae of F are true. If we have two different states s_1, s_2 in $S/\sim$:

- s_1 is the representative of a set of states in S at which a set $\Gamma_1 \subseteq \Gamma$ are true
- s_2 is the representative of a set of states in S at which a set $\Gamma_2 \subseteq \Gamma$ are true.

Clearly, $\Gamma_1 \neq \Gamma_2$ (otherwise s_1 and s_2 would be representatives for the same set of formulae, hence equal). We can now think of the states in $S/\sim$ as being labelled with the sets of formulae in Γ which are true in them. The number of states in $S/\sim$ is therefore smaller than or equal to the number of subsets of Γ.

Since Γ is finite, the number of states in $S/\sim$ is therefore finite (at most $2^{|\Gamma|}$).