Non-classical logics

Lecture 6: Many-valued logics (2)

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Exam

Question: Oral or written?

When?

1. **Termin:** first two weeks after end of lectures
   (16.02.15-27.02.15)
2. **Termin:** March or April.

Doodle

Last time

Many-valued Logics

History

Motivation

Examples.

Many-valued logics

- Syntax
- Semantics
- Applications
- Proof theory / Methods for automated reasoning

1 Syntax

- propositional variables
- logical operations

Propositional Variables

Let Π be a set of propositional variables.

We use letters P, Q, R, S, to denote propositional variables.

Logical operators

Let $\mathcal{F}$ be a set of logical operators.

These logical operators could be the usual ones from classical logic

$$\{\neg/1, \vee/2, \wedge/2, \rightarrow/2, \leftrightarrow/2\}$$

but could also be other operations, with arbitrary arity.

Propositional Formulas

$F_\Pi^\mathcal{F}$ is the set of propositional formulas over Π defined as follows:

$$
\begin{array}{llll}
F, G, H & ::= & c & \text{(c constant logical operator)} \\
& | & P, \quad P \in \Pi & \text{(atomic formula)} \\
& | & f(F_1, \ldots, F_n) & (f \in \mathcal{F} \text{ with arity } n)
\end{array}
$$

$F_\Pi^\mathcal{F}$ is the smallest among all sets A with the properties:

- Every constant logical operator is in A.
- Every propositional variable is in A.
- If $f \in \mathcal{F}$ with arity n and $F_1, \ldots, F_n \in A$ then also $f(F_1, \ldots, F_n) \in A$.

Example: Classical propositional logic

If $\mathcal{F} = \{\top/0, \bot/0, \neg/1, \vee/2, \wedge/2, \rightarrow/2, \leftrightarrow/2\}$ then

$F_\Pi^\mathcal{F}$ is the set of propositional formulas over Π, defined as follows:

$$
\begin{array}{rcll}
F, G, H & ::= & \bot & \text{(falsum)} \\
& \mid & \top & \text{(verum)} \\
& \mid & P, \quad P \in \Pi & \text{(atomic formula)} \\
& \mid & \neg F & \text{(negation)} \\
& \mid & (F \wedge G) & \text{(conjunction)} \\
& \mid & (F \vee G) & \text{(disjunction)} \\
& \mid & (F \rightarrow G) & \text{(implication)} \\
& \mid & (F \leftrightarrow G) & \text{(equivalence)}
\end{array}
$$

Semantics

We assume that a set $M = \{w_1, w_2, \dots, w_m\}$ of truth values is given.

We assume that a subset $D \subseteq M$ of designated truth values is given.

1. Meaning of the logical operators

$f \in \mathcal{F}$ with arity $n \quad \mapsto \quad f_M : M^n \to M$

(truth tables for the operations in $\mathcal{F}$)

Example 1: If $\mathcal{F}$ consists of the Boolean operations and $M = B_2 = \{0, 1\}$ then specifying the meaning of the logical operations means giving the truth tables for the operations in $\mathcal{F}$

$\neg_B$	
0	1
1	0

$\vee_B$	0	1
0	0	1
1	1	1

$\wedge_B$	0	1
0	0	0
1	0	1

Semantics

We assume that a set $M = \{w_1, \dots, w_m\}$ of truth values is given.

We assume that a subset $D \subseteq M$ of designated truth values is given.

1. Meaning of the logical operators

$f \in \mathcal{F}$ with arity $n \quad \mapsto \quad f_M : M^n \to M$

(truth tables for the operations in $\mathcal{F}$)

Example 2: If $\mathcal{F}$ consists of the operations $\{\vee, \wedge, \neg\}$ and $M_3 = \{0, \text{undef}, 1\}$ then specifying the meaning of the logical operations means giving the truth tables for these operations e.g.

F	$\neg_{M_3} F$
1	0
undef	undef
0	1

$\wedge_{M_3}$	1	undef	0
1	1	undef	0
undef	undef	undef	0
0	0	0	0

$\vee_{M_3}$	1	undef	0
1	1	1	1
undef	1	undef	undef
0	1	undef	0

Semantics

We assume that a set $M = \{w_1, \ldots, w_m\}$ of truth values is given.

We assume that a subset $D \subseteq M$ of designated truth values is given.

1. Meaning of the logical operators

$f \in \mathcal{F}$ with arity $n \quad \mapsto \quad f_M : M^n \to M$

(truth tables for the operations in $\mathcal{F}$)

Example 2: $\mathcal{F} = \{\vee, \wedge, \sim\}$ and $M_4 = \{\{\}, \{0\}, \{1\}, \{0,1\}\}$. The truth tables for these operations:

F	$\sim_{M_4} F$
{ }	{ }
{ 0 }	{ 1 }
{ 1 }	{ 0 }
{0, 1 }	{0, 1 }

$\wedge_{M_4}$	{ }	{0 }	{ 1 }	{0, 1}
{ }	{ }	{ 0 }	{ }	{ 0 }
{0 }	{ 0 }	{0 }	{ 0 }	{ 0 }
{1 }	{ }	{ 0 }	{ 1 }	{ 0, 1 }
{0, 1 }	{ 0 }	{0 }	{ 0, 1 }	{0, 1}

$\vee_{M_4}$	{ }	{0 }	{ 1 }	{0, 1}
{ }	{ }	{ }	{ 1 }	{ 1 }
{0 }	{ }	{0 }	{ 1 }	{ 0, 1 }
{1 }	{ 1}	{ 0, 1 }	{ 1 }	{ 0, 1 }
{0, 1 }	{ 1 }	{0, 1 }	{0, 1 }	{ 1}

Semantics

We assume that a set $M = \{w_1, \ldots, w_m\}$ of truth values is given.

We assume that a subset $D \subseteq M$ of designated truth values is given.

2. The meaning of the propositional variables

A Π-valuation is a map

$$\mathcal{A} : \Pi \to M.$$

Semantics

We assume that a set $M = \{w_1, \ldots, w_m\}$ of truth values is given.

We assume that a subset $D \subseteq M$ of designated truth values is given.

3. Truth value of a formula in a valuation

Given an interpretation of the operation symbols $(M, \{f_M\}_{f \in \mathcal{F}})$ and Π-valuation $\mathcal{A} : \Pi \to M$, the function $\mathcal{A}^*$: Σ-formulas $\to M$ is defined inductively over the structure of F as follows:

$$\begin{aligned} \mathcal{A}^*(c) &= c_M (\text{for every constant operator } c \in \mathcal{F}) \\ \mathcal{A}^*(P) &= \mathcal{A}(P) \\ \mathcal{A}^*(f(F_1, \ldots, F_n)) &= f_M(\mathcal{A}^*(F_1), \ldots, \mathcal{A}^*(F_n)) \end{aligned}$$

For simplicity, we write $\mathcal{A}$ instead of $\mathcal{A}^*$.

Example 1: Classical logic

Given a Π-valuation $\mathcal{A} : \Pi \to B_2 = \{0, 1\}$, the function $\mathcal{A}^*$: Σ-formulas $\to \{0, 1\}$ is defined inductively over the structure of F as follows:

$$\begin{aligned}
\mathcal{A}^*(\bot) &= 0 \\
\mathcal{A}^*(\top) &= 1 \\
\mathcal{A}^*(P) &= \mathcal{A}(P) \\
\mathcal{A}^*(\neg F) &= \neg_b \mathcal{A}^*(F) \\
\mathcal{A}^*(F \circ G) &= \circ_B(\mathcal{A}^*(F), \mathcal{A}^*(G))
\end{aligned}$$

with $\circ_B$ the Boolean function associated with $\circ \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$

(as described by the truth tables)

Example 2: Logic of undefinedness

Given a Π-valuation $\mathcal{A} : \Pi \to M_3 = \{0, \text{undef}, 1\}$, the function $\mathcal{A}^*$: Σ-formulas $\to \{0, \text{undef}, 1\}$ is defined inductively over the structure of F as follows:

$$\mathcal{A}^*(\bot) = 0$$

$$\mathcal{A}^*(\top) = 1$$

$$\mathcal{A}^*(P) = \mathcal{A}(P)$$

$$\mathcal{A}^*(\neg F) = \neg_{M_3}(\mathcal{A}^*(F))$$

$$\mathcal{A}^*(F \vee G) = \mathcal{A}^*(F) \vee_{M_3} \mathcal{A}^*(G)$$

$$\mathcal{A}^*(F \wedge G) = \mathcal{A}^*(F) \wedge_{M_3} \mathcal{A}^*(G)$$

Example 3: Belnap's 4-valued logic

Given a Π-valuation $\mathcal{A} : \Pi \to M_4 = \{\{\}, \{0\}, \{1\}, \{0, 1\}\}$, the function $\mathcal{A}^*$: Σ-formulas $\to \{\{\}, \{0\}, \{1\}, \{0, 1\}\}$ is defined inductively over the structure of F as follows:

$$\mathcal{A}^*(\bot) = \{0\}$$

$$\mathcal{A}^*(\top) = \{1\}$$

$$\mathcal{A}^*(P) = \mathcal{A}(P)$$

$$\mathcal{A}^*(\sim F) = \sim_{M_4} (\mathcal{A}^*(F))$$

$$\mathcal{A}^*(F \vee G) = \mathcal{A}^*(F) \vee_{M_4} \mathcal{A}^*(G)$$

$$\mathcal{A}^*(F \wedge G) = \mathcal{A}^*(F) \wedge_{M_4} \mathcal{A}^*(G)$$

Models, Validity, and Satisfiability

$M = \{w_1, \ldots, w_m\}$ set of truth values
$D \subseteq M$ set of designated truth values
$\mathcal{A} : \Pi \to M$.

F is valid in $\mathcal{A}$ ($\mathcal{A}$ is a model of F; F holds under $\mathcal{A}$):

$$\mathcal{A} \models F \;:\Leftrightarrow\; \mathcal{A}(F) \in D$$

F is valid (or is a tautology):

$$\models F \;:\Leftrightarrow\; \mathcal{A} \models F \text{ for all } \Pi\text{-valuations } \mathcal{A}$$

F is called satisfiable iff there exists an $\mathcal{A}$ such that $\mathcal{A} \models F$.

Otherwise F is called unsatisfiable (or contradictory).

The logic $\mathcal{L}_3$

Set of truth values: $M = \{1, u, 0\}$.

Designated truth values: $D = \{1\}$.

Logical operators: $\mathcal{F} = \{\vee, \wedge, \neg, \sim\}$.

Truth tables for the operators

$\vee$	0	u	1
0	0	u	1
u	u	u	1
1	1	1	1

$\wedge$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$$v(F \wedge G) = \min(v(F), v(G))$$

$$v(F \vee G) = \max(v(F), v(G))$$

Under the assumption that $0 < u < 1$.

Truth tables for negations

A	$\neg A$	$\sim A$	$\sim \neg A$	$\sim\sim A$	$\neg\neg A$	$\neg \sim A$
1	0	0	1	1	1	1
u	u	1	1	0	u	0
0	1	1	0	0	0	0

Translation in natural language:

$v(A) = 1$ gdw. A is true

$v(\neg A) = 1$ gdw. A is false

$v(\sim A) = 1$ gdw. A is not true

$v(\sim \neg A) = 1$ gdw. A is not false

First-order many-valued logic

$M = \{w_1, \ldots, w_m\}$ set of truth values

$D \subseteq M$ set of designated truth values.

1. Syntax

- non-logical symbols (domain-specific)
  $\Rightarrow$ terms, atomic formulas
- logical symbols $\mathcal{F}$, quantifiers
  $\Rightarrow$ formulae

Signature

A signature

$$\Sigma = (\Omega, \Pi),$$

fixes an alphabet of non-logical symbols, where

- Ω is a set of function symbols f with arity $n \geq 0$, written f/n,
- Π is a set of predicate symbols p with arity $m \geq 0$, written p/m.

If $n = 0$ then f is also called a constant (symbol).
If $m = 0$ then p is also called a propositional variable.
We use letters P, Q, R, S, to denote propositional variables.

Variables, Terms

As in classical logic

Atoms

Atoms (also called atomic formulas) over Σ are formed according to this syntax:

$$
\begin{array}{rcll}
A, B & ::= & p(s_1, \ldots, s_m) & ,\ p/m \in \Pi \\
\Big[& \mid & (s \approx t) & \text{(equation)} \quad \Big]
\end{array}
$$

In what follows we will only consider variants of first-order logic without equality.

Logical Operations

$\mathcal{F}$ set of logical operations

$\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ set of quantifiers

First-Order Formulas

$\mathsf{F}_\Sigma(X)$ is the set of first-order formulas over Σ defined as follows:

$$
\begin{array}{rcll}
F, G, H & ::= & c & (c \in \mathcal{F}, \text{ constant}) \\
& \mid & A & (\text{atomic formula}) \\
& \mid & f(F_1, \ldots, F_n) & (f \in \mathcal{F} \text{ with arity } n) \\
& \mid & QxF & (Q \in \mathcal{Q} \text{ is a quantifier})
\end{array}
$$

Bound and Free Variables

In QxF, $Q \in \mathcal{Q}$, we call F the scope of the quantifier Qx.
An *occurrence* of a variable x is called bound, if it is inside the scope of a quantifier Qx.
Any other occurrence of a variable is called free.

Formulas without free variables are also called closed formulas or sentential forms.

Formulas without variables are called ground.

Semantics

$M = \{1, \ldots, m\}$ set of truth values

$D \subseteq M$ set of designated truth values.

Truth tables for the logical operations:

$\{f_M : M^n \to M | f/n \in \mathcal{F}\}$

"Truth tables" for the quantifiers:

$\{Q_M : \mathcal{P}(M) \to M | Q \in \mathcal{Q}\}$

Examples: If $M = B_2 = \{0, 1\}$ then

$\forall_{B_2} : \mathcal{P}(\{0,1\}) \to \{0,1\} \qquad \forall_{B_2}(X) = \min(X)$

$\exists_{B_2} : \mathcal{P}(\{0,1\}) \to \{0,1\} \qquad \exists_{B_2}(X) = \max(X)$

Structures

An M-valued Σ-algebra (Σ-interpretation or Σ-structure) is a triple

$$\mathcal{A} = (U,\ (f_{\mathcal{A}} : U^n \to U)_{f/n\in\Omega},\ (p_{\mathcal{A}} : U^m \to M)_{p/m\in\Pi})$$

where $U \neq \emptyset$ is a set, called the universe of $\mathcal{A}$.

Normally, by abuse of notation, we will have $\mathcal{A}$ denote both the algebra and its universe.

By $\Sigma\text{-Alg}^M$ we denote the class of all M-valued Σ-algebras.

Assignments

Variable assignments $\beta : X \to \mathcal{A}$

and extensions to terms $\mathcal{A}(\beta) : T_\Sigma \to \mathcal{A}$ as in classical logic.

Truth Value of a Formula in $\mathcal{A}$ with Respect to β

$\mathcal{A}(\beta) : \mathsf{F}_\Sigma(X) \to M$ is defined inductively as follows:

$$\begin{aligned}
\mathcal{A}(\beta)(c) &= c_M \\
\mathcal{A}(\beta)(p(s_1, \ldots, s_n)) &= p_{\mathcal{A}}(\mathcal{A}(\beta)(s_1), \ldots, \mathcal{A}(\beta)(s_n)) \in M \\
\mathcal{A}(\beta)(f(F_1, \ldots, F_n)) &= f_M(\mathcal{A}(\beta)(F_1), \ldots, \mathcal{A}(\beta)(F_n)) \\
\mathcal{A}(\beta)(QxF) &= Q_M(\{\mathcal{A}(\beta[x \mapsto a])(F) \mid a \in U\})
\end{aligned}$$

First-order version of $\mathcal{L}_3$

$M = \{0, u, 1\}$

$D = \{1\}$

$\mathcal{F} = \{\vee, \wedge, \neg, \sim\}$

truth values as the propositional version

$\mathcal{Q} = \{\forall, \exists\}$

$$\forall_M(S) = \begin{cases} 1 & \text{if } S = \{1\} \\ 0 & \text{if } 0 \in S \\ u & \text{otherwise} \end{cases} \qquad \exists_M(S) = \begin{cases} 1 & \text{if } 1 \in S \\ 0 & \text{if } S = \{0\} \\ u & \text{otherwise} \end{cases}$$

Interpretation of quantifiers

$$
\begin{array}{llll}
\mathcal{A}(\beta)(\forall x F(x)) = 1 & \textit{iff} & \text{for all } a \in U_{\mathcal{A}}, & \mathcal{A}(\beta[x \mapsto a])(F(x)) = 1 \\
\mathcal{A}(\beta)(\forall x F(x)) = 0 & \textit{iff} & \text{for some } a \in U_{\mathcal{A}}, & \mathcal{A}(\beta[x \mapsto a])(F(x)) = 0 \\
\mathcal{A}(\beta)(\forall x F(x)) = u & & \text{otherwise} &
\end{array}
$$

$$
\begin{array}{llll}
\mathcal{A}(\beta)(\exists x F(x)) = 1 & \textit{iff} & \text{for some } a \in U_{\mathcal{A}}, & \mathcal{A}(\beta[x \mapsto a])(F(x)) = 1 \\
\mathcal{A}(\beta)(\exists x F(x)) = 0 & \textit{iff} & \text{for all } a \in U_{\mathcal{A}}, & \mathcal{A}(\beta[x \mapsto a])(F(x)) = 0 \\
\mathcal{A}(\beta)(\forall x F(x)) = u & & \text{otherwise} &
\end{array}
$$

Models, Validity, and Satisfiability

F is valid in $\mathcal{A}$ under assignment β:

$$\mathcal{A}, \beta \models F \quad :\Leftrightarrow \quad \mathcal{A}(\beta)(F) \in D$$

F is valid in $\mathcal{A}$ ($\mathcal{A}$ is a model of F):

$$\mathcal{A} \models F \quad :\Leftrightarrow \quad \mathcal{A}, \beta \models F, \text{ for all } \beta \in X \to U_{\mathcal{A}}$$

F is valid:

$$\models F \quad :\Leftrightarrow \quad \mathcal{A} \models F, \text{ for all } \mathcal{A} \in \Sigma\text{-alg}$$

F is called satisfiable iff there exist $\mathcal{A}$ and β such that $\mathcal{A}, \beta \models F$.
Otherwise F is called unsatisfiable.

Entailment

$N \models F$:$\Leftrightarrow$ for all $\mathcal{A} \in \Sigma$-alg and $\beta \in X \to U_{\mathcal{A}}$:
if $\mathcal{A}(\beta)(G) \in D$, for all $G \in N$, then $\mathcal{A}(\beta)(F) \in D$.

Models, Validity, and Satisfiability in $\mathcal{L}_3$

F is valid in $\mathcal{A}$ under assignment β:

$$\mathcal{A}, \beta \models_3 F \quad :\Leftrightarrow \quad \mathcal{A}(\beta)(F) = 1$$

F is valid in $\mathcal{A}$ ($\mathcal{A}$ is a model of F):

$$\mathcal{A} \models_3 F \quad :\Leftrightarrow \quad \mathcal{A}, \beta \models_3 F, \text{ for all } \beta \in X \to U_{\mathcal{A}}$$

F is valid (or is a tautology):

$$\models_3 F \quad :\Leftrightarrow \quad \mathcal{A} \models_3 F, \text{ for all } \mathcal{A} \in \Sigma\text{-alg}$$

F is called satisfiable iff there exist $\mathcal{A}$ and β such that $\mathcal{A}, \beta \models_3 F$.
Otherwise F is called unsatisfiable.

Entailment in $\mathcal{L}_3$

$N \models_3 F$:$\Leftrightarrow$ for all $\mathcal{A} \in \Sigma$-alg and $\beta \in X \to U_{\mathcal{A}}$:
if $\mathcal{A}(\beta)(G) = 1$, for all $G \in N$, then $\mathcal{A}(\beta)(F) = 1$.

Observations

- Every $\mathcal{L}_3$-tautology is also a two-valued tautology.
- Not every two-valued tautology is an $\mathcal{L}_3$-tautology.
  Example: $F \vee \neg F$.

Entailment

$N \models F$:$\Leftrightarrow$ for all $\mathcal{A} \in \Sigma$-alg and $\beta \in X \rightarrow U_{\mathcal{A}}$:

if $\mathcal{A}(\beta)(G) \in D$, for all $G \in N$, then $\mathcal{A}(\beta)(F) \in D$.

Entailment

$N \models F :\Leftrightarrow$ for all $\mathcal{A} \in \Sigma$-alg and $\beta \in X \rightarrow U_{\mathcal{A}}$:

if $\mathcal{A}(\beta)(G) \in D$, for all $G \in N$, then $\mathcal{A}(\beta)(F) \in D$.

Goal: Define a version of implication '$\Rightarrow$' such that

$F \models G$ iff $\models F \Rightarrow G$

Weak implication

The logical operations $\supset$ and $\equiv$ are introduced as defined operations:

Weak implication

$$F \supset G := \sim F \vee G$$

Weak equivalence

$$F \equiv G := (F \supset G) \wedge (G \supset F)$$

$F \supset G$	1	u	0
1	1	u	0
u	1	1	1
0	1	1	1

$F \equiv G$	1	u	0
1	1	u	0
u	u	1	1
0	0	1	1

Strong implication

The logical operations $\rightarrow$ and $\leftrightarrow$ are introduced as defined operations:

Strong implication

$$F \rightarrow G := \neg F \vee G$$

Strong equivalence

$$F \leftrightarrow G := (F \rightarrow G) \wedge (G \rightarrow F)$$

$F \rightarrow G$	1	u	0
1	1	u	0
u	1	u	u
0	1	1	1

$F \leftrightarrow G$	1	u	0
1	1	u	0
u	u	u	u
0	0	u	1

Comparisons

Implications

$A \supset B$	1	u	0
1	1	u	0
u	1	1	1
0	1	1	1

$A \rightarrow B$	1	u	0
1	1	u	0
u	1	u	u
0	1	1	1

Equivalences

$A \equiv B$	1	u	0
1	1	u	0
u	u	1	1
0	0	1	1

$A \leftrightarrow B$	1	u	0
1	1	u	0
u	u	u	u
0	0	u	1

Equivalences

$A \supset B := \;\sim A \vee B$ $\qquad$ $A \to B := \neg A \vee B$

$A \equiv B := (A \supset B) \wedge (B \supset A)$ $\qquad$ $A \leftrightarrow B := (A \to B) \wedge (B \to A)$

$A \approx B := (A \equiv B) \wedge (\neg A \equiv \neg B)$ $\qquad$ $A \Leftrightarrow B := (A \leftrightarrow B) \wedge (\neg A \leftrightarrow \neg B)$

$A \text{ id } B := \;\sim\sim (A \approx B)$

A	B	$A \equiv B$	$A \leftrightarrow B$	$A \approx B$	$A \Leftrightarrow B$	$A \text{ id } B$
1	1	1	1	1	1	1
1	u	u	u	u	u	0
1	0	0	0	0	0	0
u	1	u	u	u	u	0
u	u	1	u	1	u	1
u	0	1	u	u	u	0
0	1	0	0	0	0	0
0	u	1	u	u	u	0
0	0	1	1	1	1	1

Some $\mathcal{L}_3$ tautologies

$\neg\neg A$ id A

$\sim\sim A \equiv A$

$\neg \sim A \equiv A$

$(A \wedge B) \vee C$ id $(A \vee C) \wedge (B \vee C)$

$(A \vee B) \wedge C$ id $(A \wedge C) \vee (B \wedge C)$

$\neg(A \vee B)$ id $\neg A \wedge \neg B$

$\neg(A \wedge B)$ id $\neg A \vee \neg B$

$\sim (A \vee B)$ id $\sim A \wedge \sim B$

$\sim (A \wedge B)$ id $\sim A \vee \sim B$

$\neg(\forall x A)$ id $\exists x \neg A$

$\neg(\exists x A)$ id $\forall x \neg A$

$\sim (\forall x A)$ id $\exists x \sim A$

$\sim (\exists x A)$ id $\forall x \sim A$

No occurrence of $\neg$

Lemma. Let F be a formula which does not contain the strong negation $\neg$. Then the following are equivalent:

(1) F is an $\mathcal{L}_3$-tautology.

(2) F is a two-valued tautology (negation is identified with $\sim$)

Proof.

"$\Rightarrow$" Every $\mathcal{L}_3$-tautology is a 2-valued tautology (the restriction of the operators $\vee, \wedge, \sim$ to $\{0, 1\}$ coincides with the Boolean operations $\vee, \wedge, \neg$).

"$\Leftarrow$" Assume that F is a two-valued tautology. Let $\mathcal{A}$ be an $\mathcal{L}_3$-structure and $\beta : X \to \mathcal{A}$ be a valuation. We construct a two-valued structure $\mathcal{A}'$ from $\mathcal{A}$, which agrees with $\mathcal{A}$ except for the fact that whenever $p_{\mathcal{A}}(\overline{x}) = u$ we define $p_{\mathcal{A}'}(\overline{x}) = 0$. Then $\mathcal{A}'(\beta)(F) = 1$. It can be proved that

$\mathcal{A}(\beta)(F) = 1 \Rightarrow \mathcal{A}'(\beta)(F) = 1$

$\mathcal{A}(\beta)(F) \in \{0, u\} \Rightarrow \mathcal{A}'(\beta)(F) = 0.$

Hence, $\mathcal{A}(\beta)(F) = 1$.

Exercises

1. **Let F be a formula which does not contain $\sim$.**
   Then F is not a tautology.

Exercises

1. **Let F be a formula which does not contain $\sim$.**
   Then F is not a tautology.

   Proof. Take the valuation which maps all propositional variables to u.

Exercises

1. **Let F be a formula which does not contain $\sim$.**
   Then F is not a tautology.

   Proof. Take the valuation which maps all propositional variables to u.

2. **Prove that for every term t, $\forall x q(x) \supset q(x)[t/x]$ is an $\mathcal{L}_3$-tautology.**

3. **Show that $\forall x q(x) \rightarrow q(x)[t/x]$ is not a tautology.**

Exercises

1. **Let F be a formula which does not contain $\sim$.**
   Then F is not a tautology.

   Proof. Take the valuation which maps all propositional variables to u.

2. **Prove that for every term t, $\forall x q(x) \supset q(x)[t/x]$ is an $\mathcal{L}_3$-tautology.**

3. **Show that $\forall x q(x) \to q(x)[t/x]$ is not a tautology.**

   Solution. $q \to q$ is not a tautology.

Exercises

4. **Which of the following statements are true?**

   If $F \equiv G$ is a tautology and F is a tautology then G is a tautology.

   If $F \equiv G$ is a tautology and F is satisfiable then G is satisfiable.

   If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.

   If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.

F is a non-tautology iff for every 3-valued structure, $\mathcal{A}$ and every valuation $\beta, \mathcal{A}(\beta)(F) \neq 1$.

F is two-valued iff for every 3-valued structure, $\mathcal{A}$ and every valuation $\beta, \mathcal{A}(\beta)(F) \in \{0, 1\}$.

Exercises

4. **Which of the following statements are true?**

   If $F \equiv G$ is a tautology and F is a tautology then G is a tautology.

   true

   If $F \equiv G$ is a tautology and F is satisfiable then G is satisfiable.

   If $F \equiv G$ is a tautology and F is a non-tautology then G is a non-tautology.

   If $F \equiv G$ is a tautology and F is two-valued then G is two-valued.

F is a non-tautology iff for every 3-valued structure, $\mathcal{A}$ and every valuation $\beta, \mathcal{A}(\beta)(F) \neq 1$.

F is two-valued iff for every 3-valued structure, $\mathcal{A}$ and every valuation $\beta, \mathcal{A}(\beta)(F) \in \{0, 1\}$.

Exercises

4. **Which of the following statements are true?**

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