Non-classical logics

Lecture 7: Many-valued logics (4)

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Until now

- Many-valued logic (finitely-valued; infinitely-valued)

  History and Motivation

  Syntax

  Semantics

  Functional completeness

  Automated reasoning:

    Tableaux

Today:

    Resolution

Resolution

Needed:

Method for computing a conjunctive normal form

Example: Classical propositional logic

$F: \quad (P \vee Q) \wedge ((\neg P \wedge Q) \vee R)$

P	Q	R	$(P \vee Q)$	$\neg P$	$(\neg P \wedge Q)$	$((\neg P \wedge Q) \vee R)$	F
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	0	0
1	0	1	1	0	0	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	0	1	1

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P	Q	R	$(P \vee Q)$	$\neg P$	$(\neg P \wedge Q)$	$((\neg P \wedge Q) \vee R)$	F
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	0	0
1	0	1	1	0	0	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	0	1	1

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P	Q	R	$(P \vee Q)$	$\neg P$	$(\neg P \wedge Q)$	$((\neg P \wedge Q) \vee R)$	F
0	0	0	0	1	0	0	0
0	0	1	0	1	0	1	0
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	0	0
1	0	1	1	0	0	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	0	1	1

DNF: $(\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge R)$

Example: Classical propositional logic

$F: \quad (P \vee Q) \wedge ((\neg P \wedge Q) \vee R)$

P	Q	R	$(P \vee Q)$	$\neg P$	$(\neg P \wedge Q)$	$((\neg P \wedge Q) \vee R)$	F	$\neg F$
0	0	0	0	1	0	0	0	1
0	0	1	0	1	0	1	0	1
0	1	0	1	1	1	1	1	0
0	1	1	1	1	1	1	1	0
1	0	0	1	0	0	0	0	1
1	0	1	1	0	0	1	1	0
1	1	0	1	0	0	0	0	1
1	1	1	1	0	0	1	1	0

CNF: (1) DNF of $\neg F$:

$$(\neg P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge \neg R)$$

(2) negate:

$$(P \vee Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R)$$

Signed resolution: Propositional logic

Translation to signed clause form.

$$\Psi = S{:}f(F_1, \ldots, F_n)$$

$$DNF(\Psi) := \bigvee_{\substack{v_1,\ldots,v_n \in M \\ f_M(v_1,\ldots,v_n) \in S}} F_1^{v_1} \wedge \cdots \wedge F_n^{v_n}$$

$$CNF(\Psi) := \bigwedge_{\substack{v_1,\ldots,v_n \in M \\ f_M(v_1,\ldots,v_n) \notin S}} (M \setminus \{v_1\}){:}F_1 \vee \cdots \vee (M \setminus \{v_n\}){:}F_n$$

(negate $DNF(M \setminus S{:}f(F_1, \ldots, F_n))$)

Example

$\Rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

Compute CNF for $\{0\}{:}(F_1 \rightarrow F_2)$:

DNF for $\{\frac{1}{2}, 1\}{:}(F_1 \rightarrow F_2)$: $\displaystyle\bigvee_{\substack{v_1, v_2 \in \{0, \frac{1}{2}, 1\} \\ v_1 \Rightarrow v_2 \neq 0}} \{v_1\}{:}F_1 \wedge \{v_2\}{:}F_2$

$$
\begin{array}{lllll}
(F_1^0 \wedge F_2^0) & \vee & (F_1^0 \wedge F_2^{\frac{1}{2}}) & \vee & (F_1^0 \wedge F_2^1) \\
(F_1^{\frac{1}{2}} \wedge F_2^0) & \vee & (F_1^{\frac{1}{2}} \wedge F_2^{\frac{1}{2}}) & \vee & (F_1^{\frac{1}{2}} \wedge F_2^1) \\
(F_1^1 \wedge F_2^{\frac{1}{2}}) & \vee & (F_1^1 \wedge F_2^1) & &
\end{array}
$$

CNF for $\{0\}{:}(F_1 \rightarrow F_2)$:

$$
\begin{array}{lllll}
(\{\frac{1}{2}, 1\}{:}F_1 \vee \{\frac{1}{2}, 1\}{:}F_2) & \wedge & (\{\frac{1}{2}, 1\}{:}F_1 \vee \{0, 1\}{:}F_2) & \wedge & (\{\frac{1}{2}, 1\}{:}F_1 \vee \{0, \frac{1}{2}\}{:}F_2) \\
(\{0, 1\}{:}F_1 \vee \{\frac{1}{2}, 1\}{:}F_2) & \wedge & (\{0, 1\}{:}F_1 \vee \{0, 1\}{:}F_2) & \wedge & (\{0, 1\}{:}F_1 \vee \{0, \frac{1}{2}\}{:}F_2) \\
(\{0, \frac{1}{2}\}{:}F_1 \vee \{0, 1\}{:}F_2) & \wedge & (\{0, \frac{1}{2}\}{:}F_1^1 \vee \{0, \frac{1}{2}\}{:}F_2^1) & &
\end{array}
$$

Example

$\Rightarrow$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
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Compute CNF for $\{0\}{:}(F_1 \to F_2)$:

DNF for $\{\frac{1}{2}, 1\}{:}(F_1 \to F_2)$: $\displaystyle\bigvee_{\substack{v_1, v_2 \in \{0, \frac{1}{2}, 1\} \\ v_1 \Rightarrow v_2 \neq 0}} \{v_1\}{:}F_1 \wedge \{v_2\}{:}F_2$

$$= \quad (F_1^0 \wedge F_2^{\{0,\frac{1}{2},1\}}) \vee (F_1^{\frac{1}{2}} \wedge F_2^{\{0,\frac{1}{2},1\}}) \vee (F_1^1 \wedge F_2^{\{\frac{1}{2},}$$

$$= \quad F_1^0 \vee F_1^{\frac{1}{2}} \vee (F_1^1 \wedge F_2^{\{\frac{1}{2},1\}})$$

CNF for $\{0\}{:}(F_1 \to F_2)$:

$$\{\tfrac{1}{2}, 1\}{:}F_1 \;\wedge\; \{0, 1\}{:}F_1 \;\wedge\; (\{0, \tfrac{1}{2}\}{:}F_1 \vee \{0\}{:}F_2)$$

Optimization

$$\Psi = S{:}f(F_1, \ldots, F_n)$$

$$DNF(\Psi) := \bigvee_{v_1,\ldots,v_{n-1} \in M} \{v_1\}{:}F_1 \wedge \cdots \wedge \{v_{n-1}\}{:}F_{n-1} \wedge \{v_n \mid f_M(v_1, \ldots, v_n) \in S\}{:}F_n$$

$$CNF(\Psi) := \bigwedge_{v_1,\ldots,v_{n-1} \in M} (M\backslash\{v_1\}){:}F_1 \vee \cdots \vee (M\backslash\{v_{n-1}\}){:}F_{n-1} \vee \{v_n \mid f_M(v_1, \ldots, v_n){\in}S\}{:}F$$

(negate $DNF(M\backslash S{:}f(F_1, \ldots, F_n))$)

Soundness

Signed resolution (propositional form)

$$\frac{P^{v_1} \vee C \qquad P^{v_2} \vee D}{C \vee D}$$

if $v_1 \neq v_2$

Signed factoring (propositional form)

$$\frac{C \vee P^{v} \vee P^{v}}{C \vee P^{v}}$$

Soundness

Theorem. The signed resolution inference rule is sound.

Proof (propositional case)
Let $\mathcal{A}$ be a valuation such that $\mathcal{A} \models P^{v_1} \vee C$ and $\mathcal{A} \models P^{v_2} \vee D$.

Case 1: $\mathcal{A} \models P^{v_1}$. Then $\mathcal{A}(P) = v_1$, hence $\mathcal{A}(P) \neq v_2$. Therefore, $\mathcal{A} \models D$.
Hence, $\mathcal{A} \models C \vee D$.

Case 2: $\mathcal{A} \not\models P^{v_1}$. Then $\mathcal{A} \models C$.
Hence also in this case $\mathcal{A} \models C \vee D$.

Soundness of signed factoring is obvious.

Completeness: Propositional Logic

Encoding into first-order logic with equality

Signed resolution

$$\frac{P \approx v_1 \vee C \qquad P \approx v_2 \vee D}{(C \vee D)} \qquad \text{if } v_1 \neq v_2$$

Signed factoring

$$\frac{C \vee P \approx v \vee P \approx v}{C}$$

Idea: Signed resolution can be simulated by a version of resolution which handles equality efficiently (superposition). Completeness then follows from the completeness of this refinement of resolution.
This also guarantees completeness of refinements of signed resolution with ordering and selection functions

Compact form of signed resolution

Propositional logic

Signs: sets of truth values

Resolution

$$\frac{S_1{:}P \lor C \qquad S_2{:}P \lor D}{(S_1 \cap S_2){:}P \lor C \lor D} \qquad \text{if } S_1 \cap S_2 = \emptyset$$

Simplificaton

$$\frac{C \lor \emptyset{:}P}{C}$$

Merging

$$\frac{S_1{:}P \lor S_2{:}P \lor C}{(S_1 \cup S_2){:}P \lor C}$$

First-order logic

Translation to clause form:

need to take into account also the truth tables of the quantifiers.

$S : QxF(x)$

DNF: $\bigvee_{\substack{\emptyset \neq V \subseteq M \\ Q_M(V) \in S}} (\forall x\ V : F(x) \wedge \bigwedge_{a \in V} \exists x\ \{a\} : F(x))$

CNF: computed by negating the DNF for $M \backslash S : \forall x F(x)$

CNF: $\bigwedge_{\substack{\emptyset \neq V \subseteq M \\ Q_M(V) \in (M \backslash S)}} (\exists x (M \backslash V) : F(x) \vee \bigvee_{a \in V} \forall x (M \backslash \{a\}) : F(x))$

$\mapsto$ leave out quantifiers (Skolem functions for existential quantifier)

Example

In $\mathcal{L}_3$, with truth values $M = \{0, u, 1\}$:

$\{1, u\}\forall x\ p(x)$

$$\Rightarrow \quad \bigwedge_{\substack{\emptyset \neq V \subseteq M \\ min(V) \in \{0\}}} (\exists x(M \backslash V) : F(x) \vee \bigvee_{a \in \{0\}} \forall x(M \backslash \{a\}) : F(x))$$

$$\Rightarrow \quad (\exists x\{1, u\}{:}p(x) \vee \forall x(M \backslash \{0\}){:}p(x)) \wedge \qquad V = \{0\}$$

$$(\exists x\{u\}{:}p(x) \vee \forall x\{1, u\}{:}p(x) \vee \forall x\{0, u\}{:}p(x)) \wedge \qquad V = \{0, 1\}$$

$$(\exists x\{1\}{:}p(x) \vee \forall x\{1, u\}{:}p(x) \vee \forall x\{0, 1\}{:}p(x)) \wedge \qquad V = \{0, u\}$$

$$\forall x\{1, u\}{:}p(x) \vee \forall x\{0, 1\} : p(x) \vee \forall x\{0, u\} : p(x)) \qquad V = M$$

Structure-preserving translation

In order to avoid rapid growth of the number of clauses, a structure-preserving translation to clause form is used.

Idea

$$S : F[G(x)] \quad \Rightarrow \quad S : F[P_{G(x)}(x)] \wedge \bigwedge_{a \in M} \forall x(\{a\} G(x) \leftrightarrow \{a\} : P_{G(x)}(x))$$

where $P_{G(x)}$ new predicate symbol.

$$S{:}F[\underbrace{f(F_1, \ldots, F_n)}_{G}]$$

$$\Rightarrow \quad S : F[P_G] \wedge \bigwedge_{a \in M} \forall x(DNF(\{a\} : f(F_1, \ldots, F_n) \leftrightarrow \{a\} : P_G)$$

Resolution for first-order clauses

Natural generalization of the resolution rule:

Signed resolution

$$\frac{L_1^{v_1} \vee C \qquad L_2^{v_2} \vee D}{(C \vee D)\sigma}$$

if $v_1 \neq v_2$, and $\sigma = \text{mgu}(L_1, L_2)$

Signed factoring

$$\frac{C \vee L_1^{v} \vee L_2^{v}}{(C \vee L_1^{v})\sigma}$$

if $\sigma = \text{mgu}(L_1, L_2)$

Regular logics

Many-valued logics for which an order $\leq$ exists on the sets of truth values and for which signed CNF's can be found which contain as signs only the sets

$$\uparrow i = \{j \in M \mid j \geq i\} \text{ and } \downarrow i = \{j \in M \mid j \leq i\}$$

Regular logics

Many-valued logics for which an order $\leq$ exists on the sets of truth values and for which signed CNF's can be found which contain as signs only the sets

$$\uparrow i = \{j \in M \mid j \geq i\} \text{ and } \downarrow i = \{j \in M \mid j \leq i\}$$

Example

Łukasiewicz logics $\mathcal{L}_n$

- Set of truth values $M = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\}$
- Logical operations: $\vee, \wedge, \neg, \Rightarrow$
  - $\vee_{Ł_n} = \max$
  - $\wedge_{Ł_n} = \min$
  - $\neg_{Ł_n} x = 1 - x$
  - $x \Rightarrow_{Ł_n} y = \min(1, 1 - x + y)$
- First-order version: $\mathcal{Q} = \{\forall, \exists\}$

Łukasiewicz logics

Łukasiewicz implication $x \Rightarrow_{Ł_n} y = \min(1, 1 - x + y)$

$\mathcal{L}_n$

$\Rightarrow$	0	$\frac{1}{n-1}$	$\frac{2}{n-1}$	$\dots$	$\frac{n-2}{n-1}$	1
0	1	1	1	$\dots$	1	1
$\frac{1}{n-1}$	$\frac{n-2}{n-1}$	1	1	$\dots$	1	1
$\frac{2}{n-1}$	$\frac{n-3}{n-1}$	$\frac{n-2}{n-1}$	1	$\dots$	1	1
$\dots$						
1	0	$\frac{1}{n-1}$	$\frac{2}{n-1}$	$\dots$	$\frac{n-2}{n-1}$	1

Example

$$\uparrow i : (F_1 \wedge F_2) \mapsto (\uparrow i : F_1) \wedge (\uparrow i : F_2)$$

$$\uparrow i : (F_1 \vee F_2) \mapsto (\uparrow i : F_1) \vee (\uparrow i : F_2)$$

$$\uparrow i : \neg F \quad \mapsto \downarrow (1 - i) : F$$

$$\uparrow i : F_1 \Rightarrow F_2 \quad \mapsto \bigvee_{j \in M} (\downarrow j : F_1 \wedge \uparrow (i + j - 1) : F_2$$

Similar for $\downarrow i : F$

signed CNFs can be obtained using the transformation rules above (and possibly negation).

Conclusions

- Finitely-valued logics: natural generalization of classical logic

- Tableau calculi
- Resolution

extend in a natural way

Similar results also for logics with infinitely many truth values?

Infinitely-Valued Logics

Łukasiewicz logics

Łukasiewicz logics

$\mathcal{L}_n,\ n \in \mathbb{N}$	$W_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, 1\}$
$\mathcal{L}_{\aleph_0}$	$W_{\aleph_0} = [0,1] \cap \mathbb{Q}$
$\mathcal{L}_{\aleph_1}$	$W_{\aleph_1} = [0,1]$

Logical operations: $\vee, \wedge, \neg, \Rightarrow$

- $\vee = \max$
- $\wedge = \min$
- $\neg x = 1 - x$
- $x \Rightarrow y = \min(1, 1 - x + y)$

Łukasiewicz logics

Łukasiewicz implication $x \Rightarrow_{Ł_n} y = \min(1, 1 - x + y)$

$\mathcal{L}_n$

$\Rightarrow$	0	$\frac{1}{n-1}$	$\frac{2}{n-1}$	$\ldots$	$\frac{n-2}{n-1}$	1
0	1	1	1	$\ldots$	1	1
$\frac{1}{n-1}$	$\frac{n-2}{n-1}$	1	1	$\ldots$	1	1
$\frac{2}{n-1}$	$\frac{n-3}{n-1}$	$\frac{n-2}{n-1}$	1	$\ldots$	1	1
$\ldots$						
1	0	$\frac{1}{n-1}$	$\frac{2}{n-1}$	$\ldots$	$\frac{n-2}{n-1}$	1

Łukasiewicz logics

Theorems.

1. For $n, m \in \mathbb{N}$, s.t. $(m-1)|(n-1)$, we have Tautologies($\mathcal{L}_n$) $\subseteq$ Tautologies($\mathcal{L}_m$)
2. Tautologies($\mathcal{L}_{\aleph_0}$) $=$ Tautologies($\mathcal{L}_{\aleph_1}$)
3. Tautologies($\mathcal{L}_{\aleph_0}$) $= \bigcap\{$Tautologies($\mathcal{L}_n$) $\mid n \geq 2, n \in \mathbb{N}\}$

Proofs

Theorem. For $n, m \in \mathbb{N}$, s.t. $(m-1)|(n-1)$, we have Tautologies($\mathcal{L}_n$) $\subseteq$ Tautologies($\mathcal{L}_m$)

Proof

Assume $(m-1)|(n-1)$. Then $W_m \subseteq W_n$. Assume $F \in$ Tautologies($\mathcal{L}_n$). Then F evaluates to 1 under every valuation into W_n, hence also under every valuation into W_m, so $F \in$ Tautologies($\mathcal{L}_m$)

Proofs

Theorem. For $n, m \in \mathbb{N}$, s.t. $(m-1)|(n-1)$, we have Tautologies($\mathcal{L}_n$) $\subseteq$ Tautologies($\mathcal{L}_m$)

Remark: the converse also holds

If Tautologies($\mathcal{L}_n$) $\subseteq$ Tautologies($\mathcal{L}_m$) then $(m-1)|(n-1)$.

(This will be discussed in the next exercise session.)

Proofs

Theorem.
Tautologies($\mathcal{L}_{\aleph_0}$) $=$ Tautologies($\mathcal{L}_{\aleph_1}$)

Proof.
" $\supseteq$ " : Since $[0,1] \cap \mathbb{Q} \subseteq [0,1]$, it is clear that
Tautologies($\mathcal{L}_{\aleph_1}$) $\subseteq$ Tautologies($\mathcal{L}_{\aleph_0}$)

Proofs

Theorem.

Tautologies($\mathcal{L}_{\aleph_0}$) = Tautologies($\mathcal{L}_{\aleph_1}$)

Proof.

" $\subseteq$ " : Let $F \in$ Tautologies($\mathcal{L}_{\aleph_0}$). Then for every assignment of values in $[0,1] \cap \mathbb{Q}$ to the propositional variables $\{P_1, \ldots, P_n\}$ of F evaluates to 1.

We can associate a function $f_F : [0,1]^n \to [0,1]$ with F which is defined as follows: For all $(x_1, \ldots, x_n) \in [0,1]^n$ let $\mathcal{A} : \{P_1, \ldots, P_n\} \to [0,1]$ be defined by $\mathcal{A}(P_i) = x_i$. We define $f_F(x_1, \ldots, x_n) := \mathcal{A}(F)$

It can be proved by structural induction that f_F is a continuous function.

Let $(a_1, \ldots, a_n) \in [0,1]^n$. It is now sufficient to choose sequences of rational numbers converging to $a_1, \ldots, a_n$ respectively. $f_F(a_1, \ldots, a_n)$ is the limit of the sequence defined this way, hence its value is 1.

Proofs

$\text{Tautologies}(\mathcal{L}_{\aleph_0}) = \bigcap\{\text{Tautologies}(\mathcal{L}_n) \mid n \geq 2, n \in \mathbb{N}\}$

Proof.

" $\subseteq$ " : Follows from the fact that $W_n \subseteq [0, 1] \cap \mathbb{Q}$ for every $n \in \mathbb{N}$.

Proofs

Tautologies($\mathcal{L}_{\aleph_0}$) $= \bigcap\{$Tautologies($\mathcal{L}_n$) $\mid n \geq 2, n \in \mathbb{N}\}$

Proof. " $\supseteq$ "

Let F be a formula with prop. variables $\{P_1, \dots, P_k\}$ s.t. $F \not\in$ Tautologies($\mathcal{L}_{\aleph_0}$). Then there exists $\mathcal{A} : \{P_1, \dots, P_k\} \to [0, 1] \cap \mathbb{Q}$ s.t. $\mathcal{A}(F) \neq 1$.

Assume that $\mathcal{A}(P_1) = \frac{q_1}{p_1}, \dots, \mathcal{A}(P_k) = \frac{q_k}{p_k}$

Let $m = \text{lcm}(p_1, \dots, p_k)$. Then it is easy to see that $\mathcal{A}(P_i) \in W_{m+1}$ for all $1 \leq i \leq k$.

We thus constructed a valuation $\mathcal{A} : \{P_1, \dots, P_k\} \to W_m$ such that $\mathcal{A}(F) \neq 1$. Hence, $F \not\in$ Tautologies($\mathcal{L}_m$), so

$$F \not\in \bigcap\{\text{Tautologies}(\mathcal{L}_n) \mid n \geq 2, n \in \mathbb{N}\}$$

“Fuzzy” logics

$W = [0, 1]$

Question: How to define conjunction?

Answer: Desired conditions

$f : [0, 1]^2 \rightarrow [0, 1]$ such that:

- f associative and commutative
- for all $0 \leq A \leq B \leq 1$ and all $0 \leq C \leq 1$ we have $f(A, C) \leq f(B, C)$
- for all $0 \leq C \leq 1$ we have $f(C, 1) = C$.

Definition A function with the properties above is called a t-norm.

Examples of t-norms

Gödel t-norm	$f_G(x, y) = \min(x, y)$
Łukasiewicz t-norm	$f_Ł(x, y) = \max(0, x + y - 1)$
Product t-norm	$f_P(x, y) = x \cdot y$

Left-continuous t-norm

Definition. A t-norm f is left-continuous if for every $x, y \in [0, 1]$ and every sequence $\{x_n\}_{n \in \mathbb{N}}$ with $0 \leq x_n \leq x$ and $lim_{n \to \infty} x_n = x$ we have $lim_{n \to \infty} f(x_n, y) = f(x, y)$.

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The following t-norms are left continuous:

Gödel t-norm	$f_G(x, y) = \min(x, y)$
Łukasiewicz t-norm	$f_{Ł}(x, y) = \max(0, x + y - 1)$
Product t-norm	$f_P(x, y) = x \cdot y$

Left continuous t-norms

With every left continuous t-norm f we can associate the following operations:

- $x \circ_f y = f(x, y)$
- $x \oplus_f y = 1 - f(1 - x, 1 - y)$
- $x \Rightarrow_f y = \max\{z \mid f(x, z) \leq y\}$
- $\neg_f x = x \Rightarrow_f 0$

Remark: Left continuity ensures that $\max\{z \mid f(x, z) \leq y\}$ exists.

Validity: $D = \{1\}$

Left continuous t-norms

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- $x \Rightarrow_f y = \max\{z \mid f(x, z) \leq y\}$
- $\neg_f x = x \Rightarrow_f 0$

Łukasiewicz t-norm

$x \circ_{\text{Ł}} y = \max(0, x + y - 1)$

$x \oplus_{\text{Ł}} y = 1 - \max(0, 1 - x - y)$

$x \Rightarrow_f y = \min(1, 1 - x + y)$

$\neg x = \min(1, 1 - x) = 1 - x$

Left continuous t-norms

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Łukasiewicz t-norm

$$x \circ_Ł y = \max(0, x+y-1) \qquad x \wedge_Ł y = x \circ_Ł (x \Rightarrow y)$$

$$x \oplus_Ł y = 1 - \max(0, 1-x-y) \qquad x \vee_Ł y = \neg_Ł((\neg_Ł x) \wedge_Ł (\neg_Ł y))$$

$$x \Rightarrow_f y = \min(1, 1-x+y)$$

$$\neg x = \min(1, 1-x) = 1-x$$

Left continuous t-norms

With every left continuous t-norm f we can associate the following operations:

- $x \circ_f y = f(x, y)$
- $x \oplus_f y = 1 - f(1 - x, 1 - y)$
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- $\neg_f x = x \Rightarrow_f 0$

Gödel t-norm

$$x \circ_G y = \min(x, y)$$

$$x \oplus_G y = \max(x, y)$$

$$x \Rightarrow_G y = \max\{z \mid x \wedge z \leq y\} = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

$$\neg_G x = \max\{z \mid x \wedge z = 0) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$