Non-classical logics

Lecture 9: Modal logics (Part 2)

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Until now

History and Motivation

Syntax

Inference systems/Proofs

Syntax

- propositional variables Π
- logical symbols: $\{\vee, \wedge, \neg, \rightarrow, \leftrightarrow, \Box, \Diamond\}$

Propositional Formulas

F_Π is the set of propositional formulas over Π defined as follows:

$$
\begin{array}{llll}
F, G, H & ::= & \bot & \text{(falsum)} \\
& \mid & \top & \text{(verum)} \\
& \mid & P, \quad P \in \Pi & \text{(atomic formula)} \\
& \mid & \neg F & \text{(negation)} \\
& \mid & (F \land G) & \text{(conjunction)} \\
& \mid & (F \lor G) & \text{(disjunction)} \\
& \mid & (F \rightarrow G) & \text{(implication)} \\
& \mid & (F \leftrightarrow G) & \text{(equivalence)} \\
& \mid & \Box F & \\
& \mid & \Diamond F &
\end{array}
$$

Proof Calculi/Inference systems and proofs

Inference systems Γ (proof calculi) are sets of tuples

$$(F_1, \ldots, F_n, F_{n+1}), \;\; n \geq 0,$$

called inferences or inference rules, and written

$$\underbrace{\frac{\overbrace{F_1 \;\; \ldots \;\; F_n}^{\text{premises}}}{F_{n+1}}}_{\text{conclusion}} \;.$$

Inferences with 0 premises are also called axioms.

Proofs

A proof in Γ of a formula F from a a set of formulas N (called assumptions) is a sequence $F_1, \ldots, F_k$ of formulas where

(i) $F_k = F$,

(ii) for all $1 \leq i \leq k$: $F_i \in N$, or else there exists an inference $(F_{i_1},\ \ldots,\ F_{i_{n_i}},\ F_i)$ in Γ, such that $0 \leq i_j < i$, for $1 \leq j \leq n_i$.

Provability $\vdash_\Gamma$ of F from N in Γ:
$N \vdash_\Gamma F \;\; :\Leftrightarrow$ there exists a proof Γ of F from N.

Inference system for modal logics

Acceptable axioms:

- All axioms of propositional logic (e.g. $p \vee \neg p$)
- $(\Box A \wedge \Box(A \rightarrow B)) \rightarrow \Box B$
- $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

Acceptable inference rules

$$\frac{A \quad A \rightarrow B}{B} \qquad \text{[Modus ponens]}$$

$$\frac{A}{\Box A} \qquad \text{[Necessitation]}$$

Remark: Accepting the last inference rule is not the same with accepting $A \rightarrow \Box A$ as an axiom!

The modal system K

Axioms:

- All axioms of propositional logic (e.g. $p \vee \neg p$)
- $\Box(A \to B) \to (\Box A \to \Box B)$ (K)

Inference rules

$$\frac{A \quad A \to B}{B}$$ [Modus ponens]

$$\frac{A}{\Box A}$$ [G]

Some systems of modal logic

System	*Description*
T	$K + \Box A \to A$
D	$K + \Box A \to \Diamond A$
B	$T + \neg A \to \Box \neg \Box A$
$S4$	$T + \Box A \to \Box\Box A$
$S5$	$T + \neg \Box A \to \Box \neg \Box A$
$S4.2$	$S4 + \diamond \Box A \to \Box \Diamond A$
$S4.3$	$S4 + \Box(\Box(A \to B)) \vee \Box(\Box(B \to A))$
C	$K + \frac{A \to B}{\Box(A \to B)}$ instead of (G).

Semantics of modal logic

Two classes of models have been studied so far.

- Modal algebras
- Kripke models

Semantics of modal logic

Modal algebras $(B, \vee, \wedge, \neg, \rightarrow, \leftrightarrow, 0, 1, \Box, \Diamond)$ where

- $(B, \vee, \wedge, \neg, 0, 1)$ Boolean algebra, i.e. satisfies the following conditions:

$x \wedge y = y \wedge x$	$x \vee y = y \vee x$
$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	$x \vee (y \vee z) = (x \vee y) \vee z$
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
$x \wedge x = x$	$x \vee x = x$
$x \wedge (x \vee y) = x$	$x \vee (x \wedge y) = x$
$x \wedge 1 = x$	$x \vee 0 = x$
$x \wedge 0 = 0$	$x \vee 1 = 1$
$x \vee \neg x = 1$	$x \wedge \neg x = 0$

- $\rightarrow, \leftarrow$ derived operations: $x \rightarrow y := \neg x \vee y$; $x \leftrightarrow y := (x \rightarrow y) \wedge (y \rightarrow x)$

  $\Diamond x = \neg \Box \neg x$

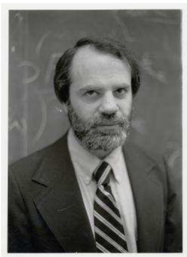
- $\Box$ has additional properties e.g. $\Box(x \wedge y) = \Box x \wedge \Box y$

Kripke Frames and Kripke Structures

Introduced by Saul Aaron Kripke in 1959.

Much less complicated and better suited to automated reasoning than modal algebras.

Saul Aaron Kripke



Born	1940 in Omaha (US)
First publication:	A Completeness Theorem in Modal Logic The Journal of Symbolic Logic, 1959
Studied at:	Harvard, Princeton, Oxford and Rockefeller University
Positions:	Harvard, Rockefeller, Columbia, Cornell, Berkeley, UCLA, Oxford since 1977 Professor at Princeton University since 1998 Emeritus at Princeton University

Kripke Frames and Kripke Structures

Definition. A Kripke frame $F = (S, R)$ consists of

- a non-empty set S (of possible worlds / states)
- an accessibility relation $R \subseteq S \times S$

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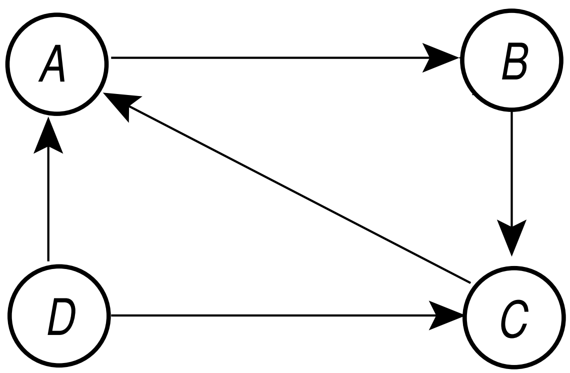
- a non-empty set S (of possible worlds / states)
- an accessibility relation $R \subseteq S \times S$

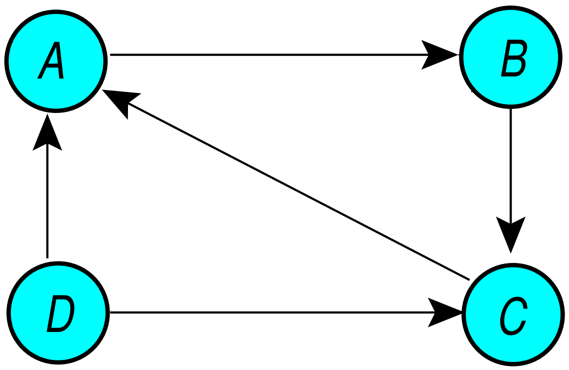
Definition. A Kripke structure $K = (S, R, \mathcal{I})$ consists of

- a Kripke frame $F = (S, R)$
- an interpretation $\mathcal{I} : \Pi \times S \to \{1, 0\}$

Example of Kripke frame

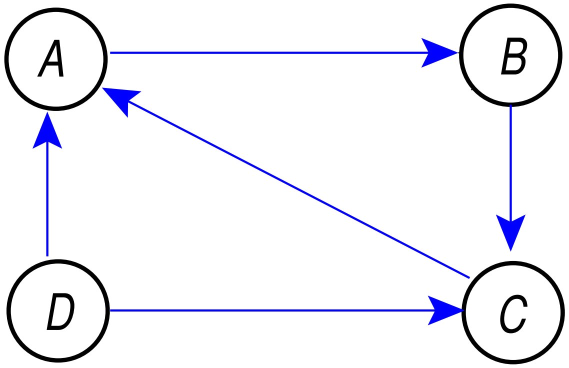


Example of Kripke frame



Set of possible worlds (states): $S = \{A, B, C, D\}$

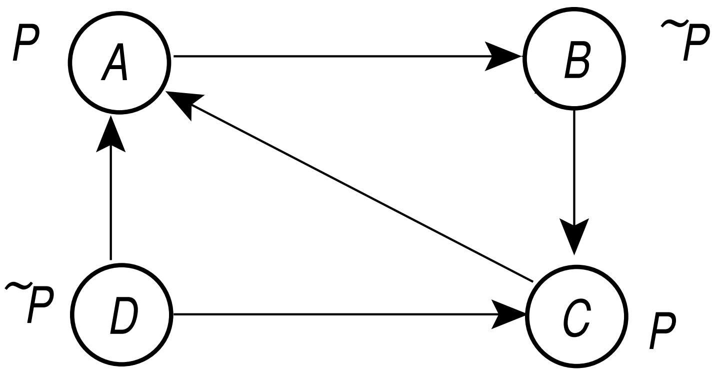
Example of Kripke frame



Set of possible worlds (states): $S = \{A, B, C, D\}$

Accessibility relation: $R = \{(A, B), (B, C), (C, A), (D, A), (D, C)\}$

Example of Kripke structure



Set of possible worlds (states): $S = \{A, B, C, D\}$

Accessibility relation: $R = \{(A, B), (B, C), (C, A), (D, A), (D, C)\}$

Interpretation: $\mathcal{I} : \Pi \times S \to \{0, 1\}$

$\mathcal{I}(P, A) = 1, \mathcal{I}(P, B) = 0, \mathcal{I}(P, C) = 1, \mathcal{I}(P, D) = 0$

Notation Instead of $(A, B) \in R$ we will sometimes write ARB.

Notation

$K = (S, R, I)$

Instead of writing $(s, t) \in R$ we will sometimes write sRt.

Modal logic: Semantics

Given: Kripke structure $K = (S, R, I)$

Valuation:

$val_K(p)(s) = I(p, s)$ for $p \in \Pi$

val_K defined for propositional operators in the same way as in classical logic

$$val_K(\Box A)(s) = \begin{cases} 1 & \text{if } val_K(A)(s') = 1 \text{ for all } s' \in S \text{ with } sRs' \\ 0 & \text{otherwise} \end{cases}$$

$$val_K(\Diamond A)(s) = \begin{cases} 1 & \text{if } val_K(A)(s') = 1 \text{ for at least one } s' \in S \text{ with } sRs' \\ 0 & \text{otherwise} \end{cases}$$

Models, Validity, and Satisfiability

$\mathcal{F} = (S, R)$, $\mathcal{K} = (S, R, I)$

F is true in $\mathcal{K}$ at a world $s \in S$:

$$(\mathcal{K}, s) \models F :\Leftrightarrow \text{val}_{\mathcal{K}}(F)(s) = 1$$

F is true in $\mathcal{K}$

$$\mathcal{K} \models F :\Leftrightarrow (\mathcal{K}, s) \models F \text{ for all } s \in S$$

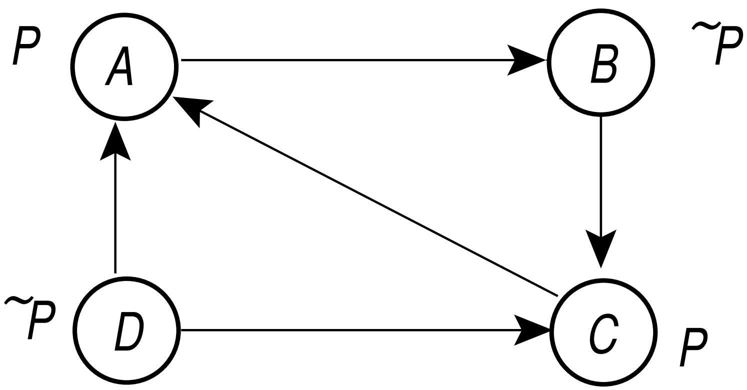
F is true in the frame $\mathcal{F} = (S, R)$

$$\mathcal{F} \models F :\Leftrightarrow (\mathcal{K}_{\mathcal{F}}) \models F \text{ for all Kripke structures } \mathcal{K}_{\mathcal{F}} = (S, R, I') \text{ defined on frame } \mathcal{F}$$

If Φ is a class of frames, F is true (valid) in Φ

$$\Phi \models F :\Leftrightarrow \mathcal{F} \models F \text{ for all } \mathcal{F} \in \Phi.$$

Example for evaluation



$(\mathcal{K}, A) \models P$ $\quad (\mathcal{K}, B) \models \neg P$ $\quad (\mathcal{K}, C) \models P$ $\quad (\mathcal{K}, D) \models \neg P$

$(\mathcal{K}, A) \models \Box\neg P$ $\quad (\mathcal{K}, B) \models \Box P$ $\quad (\mathcal{K}, C) \models \Box P$ $\quad (\mathcal{K}, D) \models \Box P$

$(\mathcal{K}, A) \models \Box\Box P$ $\quad (\mathcal{K}, B) \models \Box\Box P$ $\quad (\mathcal{K}, C) \models \Box\Box\neg P$ $\quad \ldots$

Entailment and Equivalence

In classical logic we proved:

Proposition:
F entails G iff $(F \to G)$ is valid

Does such a result hold in modal logic?

Entailment

In classical logic we proved:

Proposition:
$F \models G$ iff $(F \rightarrow G)$ is valid

Does such a result hold in modal logic?

Need to define what $F \models G$ means

Entailment

Goal: definition for $N \models F$, where N is a family of modal formulae

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Tentative 1:

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

Entailment

Goal: definition for $N \models F$, where N is a family of modal formulae

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"global entailment"

Example

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

Task: Show that $P \models_G \Box P$

Proof: Let $\mathcal{K} = (S, R, I)$ be a Kripke structure.

Assume that $\mathcal{K} \models P$, i.e. for every $s \in S$ we have $(\mathcal{K}, s) \models P$.

Then it follows that for every $s \in S$ we have $(\mathcal{K}, s) \models \Box P$.

By the definition of $\models_G$ it follows that $P \models_G \Box P$.

Example

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

Proved: $P \models_G \Box P$

Question: Is it true that $P \to \Box P$ is true in all Kripke structures?

Answer: Let $\mathcal{K} = (S, R, I)$, where

$S = \{s_1, s_2\}$, $R = \{(s_1, s_2)\}$, $I(P, s_1) = 1, I(P, s_2) = 0$.

Then $(\mathcal{K}, s_1) \models P$, $(\mathcal{K}, s_1) \not\models \Box p$.

Hence $(\mathcal{K}, s_1) \not\models P \to \Box P$.

Entailment

Goal: definition for $N \models F$, where N is a family of modal formulae

Tentative 2:

$N \models_L F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

$$\text{If } (\mathcal{K}, s) \models G \text{ for every } G \in N \text{ then } (\mathcal{K}, s) \models F$$

Entailment

Goal: definition for $N \models F$, where N is a family of modal formulae

Tentative 2:

$N \models_L F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

$$\text{If } (\mathcal{K}, s) \models G \text{ for every } G \in N \text{ then } (\mathcal{K}, s) \models F$$

"local entailment"

Entailment

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

$N \models_L F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

$$\text{If } (\mathcal{K}, s) \models G \text{ for every } G \in N \text{ then } (\mathcal{K}, s) \models F$$

Remark: The two entailment relations are different

$P \models_G \Box P$ (was shown before)

$P \not\models_L \Box P$

Entailment

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

$N \models_L F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

$$\text{If } (\mathcal{K}, s) \models G \text{ for every } G \in N \text{ then } (\mathcal{K}, s) \models F$$

Remark: The two entailment relations are different

$P \models_G \Box P$ (was shown before)

$P \not\models_L \Box P$

Proof: Let $\mathcal{K} = (S, R, I)$, where

$S = \{s_1, s_2\}$, $R = \{(s_1, s_2)\}$, $I(P, s_1) = 1, I(P, s_2) = 0$.

Then $(\mathcal{K}, s_1) \models P$, but $(\mathcal{K}, s_1) \not\models \Box P$. Hence, $P \not\models_L \Box P$.

Entailment

Theorem (The deduction theorem) The following are equivalent:

(1) $F \models_L G$

(2) $\{F, \neg G\}$ is unsatisfiable

(3) $\models (F \rightarrow G)$

(4) $\models_L (F \rightarrow G)$

Proof. $F \models_L G$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:
If $(\mathcal{K}, s) \models F$ then $(\mathcal{K}, s) \models G$
iff there is no Kripke structure $\mathcal{K} = (S, R, I)$ and no $s \in S$ wi
$(\mathcal{K}, s) \models F \wedge \neg G$
iff $\{F, \neg G\}$ is unsatisfiable

From propositional logic we know that $\{F, \neg G\}$ is unsatisfiable iff $F \rightarrow G$ is valid. This happens iff $\models_L F \rightarrow G$

Modal Logic: Valid Formulae

Valid:

- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$
- $(\Box P \wedge \Box(P \rightarrow Q)) \rightarrow \Box Q$
- $(\Box P \vee \Box Q) \rightarrow \Box(P \vee Q)$
- $(\Box P \wedge \Box Q) \leftrightarrow \Box(P \wedge Q)$
- $\Box P \leftrightarrow \neg\Diamond\neg P$
- $\Diamond(P \vee Q) \leftrightarrow (\Diamond P \vee \Diamond Q)$
- $\Diamond(P \wedge Q) \rightarrow (\Diamond P \wedge \Diamond Q)$

Modal Logic: Valid Formulae

Valid:

- $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$
- $(\Box P \wedge \Box(P \rightarrow Q)) \rightarrow \Box Q$
- $(\Box P \vee \Box Q) \rightarrow \Box(P \vee Q)$
- $(\Box P \wedge \Box Q) \leftrightarrow \Box(P \wedge Q)$
- $\Box P \leftrightarrow \neg\Diamond\neg P$
- $\Diamond(P \vee Q) \leftrightarrow (\Diamond P \vee \Diamond Q)$
- $\Diamond(P \wedge Q) \rightarrow (\Diamond P \wedge \Diamond Q)$

Not valid:

- $\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$
- $(\Diamond P \wedge \Diamond Q) \rightarrow \Diamond(P \wedge Q)$

Modal Logic: Valid Formulae

Not valid: $\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$

[explanations on the blackboard]

Exercises

1. Show that $\Diamond T$ and the schema $\Box A \to \Diamond A$ have exactly the same models.
2. Exhibit a frame in which $\Box \perp$ is valid.
3. In any model $\mathcal{K}$,
   (i) if A is a tautology then $\mathcal{K} \models A$;
   (ii) if $\mathcal{K} \models A$ and $\mathcal{K} \models A \to B$, then $\mathcal{K} \models B$;
   (iii) if $\mathcal{K} \models A$ then $\mathcal{K} \models \Box A$.

Correspondence Theory

Correspondence Theory

Main questions:

Assume that we consider a set of frames for which the accessibility relation has certain properties. Is it the case that in all frames in this class a certain modal formula holds?

Given a modal formula. Can we describe the frames in which the formula holds, e.g. by specifying certain properties of the accessibility relation?

Example

Let ReflFrames be the class of all frames $\mathcal{F} = (S, R)$ in which R is reflexive.

We will see that the following hold:

Theorem. For every formula A, the formula $\Box A \to A$ is true in all frames in ReflFrames.

Theorem. If the formula $\Box A \to A$ is true in a frame $\mathcal{F} = (S, R)$ for every formula A, then R must be reflexive.

Conditions on R

The following is a list of properties of a binary relation R that are denned by first-order sentences.

1. Reflexive: $\forall s\ (sRs)$
2. Symmetric: $\forall s \forall t\ (sRt \rightarrow tRs)$
3. Serial: $\forall s \exists t\ (sRt)$
4. Transitive: $\forall s \forall t \forall u\ (sRt \wedge tRu \rightarrow sRu)$
5. Euclidean: $\forall s \forall t \forall u\ (sRt \wedge sRu \rightarrow tRu)$
6. Partially functional: $\forall s \forall t \forall u\ (sRt \wedge sRu \rightarrow t = u)$
7. Functional: part. functional $+ \forall s \exists t (sRt)$
8. Weakly dense: $\forall s \forall t (sRt \rightarrow \exists u\ (sRu \wedge uRt))$
9. Weakly connected: $\forall s \forall t \forall u\ (sRt \wedge sRu \rightarrow tRu \vee t = u \vee uRt)$
10. Weakly directed: $\forall s \forall t \forall u\ (sRt \wedge sRu \rightarrow \exists v (tRv \wedge uRv))$

List of schemata of modal formulae

Corresponding to the list of properties of R is a list of schemata:

1. $\Box A \to A$
2. $A \to \Box\Diamond A$
3. $\Box A \to \Diamond A$
4. $\Box A \to \Box\Box A$
5. $\Diamond A \to \Box\Diamond A$
6. $\Diamond A \to \Box A$
7. $\Diamond A \leftrightarrow \Box A$
8. $\Box\Box A \to \Box A$
9. $\Box(A \wedge \Box A \to B) \vee \Box(B \wedge \Box B \to A)$
10. $\Diamond\Box A \to \Box\Diamond A$

Correspondence theorems

Properties of R		**Axioms**
1. Reflexive:	$\forall s\ (sRs)$	$\Box A \rightarrow A$
2. Symmetric:	$\forall s \forall t\ (sRt \rightarrow tRs)$	$A \rightarrow \Box \Diamond A$
3. Serial:	$\forall s \exists t\ (sRt)$	$\Box A \rightarrow \Diamond A$
4. Transitive:	$\forall s \forall t \forall u\ (sRt \wedge tRu \rightarrow sRu)$	$\Box A \rightarrow \Box \Box A$
5. Euclidean:	$\forall s \forall t \forall u\ (sRt \wedge sRu \rightarrow tRu)$	$\Diamond A \rightarrow \Box \Diamond A$
6. Partially functional:	$\forall s \forall t \forall u\ (sRt \wedge sRu \rightarrow t = u)$	$\Diamond A \rightarrow \Box A$
7. Functional:	part. functional $+ \forall s \exists t (sRt)$	$\Diamond A \leftrightarrow \Box A$
8. Weakly dense:	$\forall s \forall t (sRt \rightarrow \exists u\ (sRu \wedge uRt))$	$\Box \Box A \rightarrow \Box A$
9. Weakly connected:	$\forall s \forall t \forall u\ (sRt \wedge sRu \rightarrow tRu \vee t = u \vee uRt)$	$\Box (A \wedge \Box A \rightarrow B) \vee \Box (B \wedge \Box B \rightarrow A)$
10. Weakly directed:	$\forall s \forall t \forall u\ (sRt \wedge sRu \rightarrow \exists v (tRv \wedge uRv))$	$\Diamond \Box A \rightarrow \Box \Diamond A$

Theorem. Let $\mathcal{F} = (S, R)$ be a frame.

Then for each of the properties 1-10, if R satisfies the property, then the corresponding schema is valid in $\mathcal{F}$.

Theorem. If a frame $\mathcal{F} = (S, R)$ validates any one of the schemata 1-10, then R satisfies the corresponding property.

A general result

Property of R:

$$C(m,n,j,k): \quad \forall s_1 \forall s_2 \forall s_3 ((R^m(s_1,s_2) \wedge R^j(s_1,s_3) \rightarrow \exists s_4 (R^n(s_2,s_4) \wedge R^k(s_3,s_4)))$$

A general result

Property of R:

$$C(m,n,j,k): \quad \forall s_1 \forall s_2 \forall s_3 ((R^m(s_1,s_2) \wedge R^j(s_1,s_3) \rightarrow \exists s_4 (R^n(s_2,s_4) \wedge R^k(s_3,s_4)))$$

where $R^0(x,y) := x = y$

$$R^1(x,y) := R(x,y)$$

$$R^2(x,y) = \exists u (R(x,u) \wedge R(u,y))$$

$$R^m(x,y) = \exists u_1 \ldots u_{m-1} (R(x,u_1) \wedge \cdots \wedge R(u_{m-1},y))$$

A general result

Theorem. For every $m, n, j, k \in \mathbb{N}$, the axiom

$$\Diamond^m \Box^n P \to \Box^j \Diamond^k P$$

characterizes the class of all frames in which

$$C(m, n, j, k) : \forall s_1 \forall s_2 \forall s_3 ((R^m(s_1, s_2) \wedge R^j(s_1, s_3) \to \exists s_4 (R^n(s_2, s_4) \wedge R^k(s_3, s_4)))$$

is true.

We use the abbreviations

$$\Box^n P = \underbrace{\Box \ldots \Box}_{n \text{ times}} P$$

$$\Diamond^n P = \underbrace{\Diamond \ldots \Diamond}_{n \text{ times}} P$$

In particular, $\Box^0 P$ and $\Diamond^0 P$ stand for P

A general result

Theorem. For every $m, n, j, k \in \mathbb{N}$, the axiom $\Diamond^m \Box^n P \rightarrow \Box^j \Diamond^k P$ characterizes the class of all frames in which $C(m, n, j, k)$ is true, where:

$$C(m, n, j, k) : \forall s_1 \forall s_2 \forall s_3 ((R^m(s_1, s_2) \wedge R^j(s_1, s_3) \rightarrow \exists s_4 (R^n(s_2, s_4) \wedge R^k(s_3, s_4)))$$

Proof "$\Rightarrow$" Let (S, R) be s.t. for every I $(S, R, I) \models \Diamond^m \Box^n P \rightarrow \Box^j \Diamond^k P$. We show that R has property $C(m, n, j, k)$.

Let $s_1, s_2, s_3 \in S$ be such that $R^m(s_1, s_2) \wedge R^j(s_1, s_3)$.

Let I with $I(w, P) = 1$ if $R^n(s_2, w)$ and $I(w, P) = 0$ otherwise.

Then, for $\mathcal{K} = (S, R, I)$ we have $(\mathcal{K}, s_2) \models \Box^n P$, hence $(\mathcal{K}, s_1) \models \Diamond^m \Box^n P$.

Then, by assumption, $(\mathcal{K}, s_1) \models \Box^j \Diamond^k P$.
Since $R^j(s_1, s_3)$, it follows that there exists $s \in S$ such that $R^k(s_3, s)$ and $I(s, P) = 1$, hence by the definition of I, $R^n(s_2, s)$.

A general result

Theorem. For every $m, n, j, k \in \mathbb{N}$, the axiom $\Diamond^m \Box^n P \to \Box^j \Diamond^k P$ characterizes the class of all frames in which $C(m, n, j, k)$ is true, where:

$$C(m, n, j, k) : \forall s_1 \forall s_2 \forall s_3((R^m(s_1, s_2) \wedge R^j(s_1, s_3) \to \exists s_4(R^n(s_2, s_4) \wedge R^k(s_3, s_4)))$$

Proof "$\Leftarrow$" Assume $R \subseteq S \times S$ has the property $C(m, n, j, k)$.
Let $\mathcal{K} = (S, R, I)$ and $s_1 \in S$. We show that $(\mathcal{K}, s_1) \models \Diamond^m \Box^n P \to \Box^j \Diamond^k P$.

Assume that $(\mathcal{K}, s_1) \models \Diamond^m \Box^n P$.
Then there exists $s_2 \in S$ such that $R^m(s_1, s_2)$ and $(\mathcal{K}, s_2) \models \Box^n P$.

We want to show that $(\mathcal{K}, s_1) \models \Box^j \Diamond^k P$. Let $s_3 \in S$ be such that $R^j(s_1, s_3)$.
Since we assumed that R has property $C(m, n, j, k)$, there exists $s_4 \in S$ such that $R^n(s_2, s_4) \wedge R^k(s_3, s_4)$.

From $R^n(s_2, s_4)$ and $(\mathcal{K}, s_2) \models \Box^n P$ we infer that $I(P, s_4) = 1$.

From this and the fact that $R^k(s_3, s_4)$ it follows that $(\mathcal{K}, s_3) \models \Diamond^k P$.
It follows therefore that $(\mathcal{K}, s_1) \models \Box^j \Diamond^k P$. QED

Exercise

(1) Complete the proofs of the correspondence theorems.

(2) Give a property of R that is necessary and sufficient for $\mathcal{F}$ to validate the schema $A \to \Box A$. Do the same for $\Box \bot$.

First-order definability

The correspondence theorems go a long way toward explaining the great success that the relational semantics enjoyed upon its introduction by Kripke.

Frames are much easier to deal with than modal algebras, and many modal schemata were shown to have their frames characterised by simple first-order properties of R.

For a time it seemed that propositional modal logic corresponded in strength to first-order logic, but that proved not to be so. Here are a couple of illustrations.

Examples of schemata non-definable in FOL

Example 1. The schema

$$W : \Box(\Box A \to A) \to \Box A$$

is valid in frame (S, R) iff:

(i) R is transitive, and

(ii) there is no sequence $s_0, ..., s_n, ...$ in S with $s_0 R s_1 R s_2 \dots s_n R s_{n+1} \dots$ for all $n \geq 0$
i.e. iff R^{-1} is well-founded.

(for a proof cf. [Boolos, 1979, p.82])

It can be shown by the Compactness Theorem of first-order logic that there exists a frame satisfying (i) and (ii) that satisfies the same first-order sentences as a frame in which (ii) fails.

Hence there can be no set of first-order sentences that defines the class of frames of this schema.

Examples of schemata non-definable in FOL

Example 2. The class of frames of the so-called McKinsey schema

$$M : \Box\Diamond A \rightarrow \Diamond\Box A$$

is not defined by any set of first-order sentences

[Goldblatt, 1975; van Benthem, 1975]

Second order definability

Propositional modal logic corresponds to a fragment of second-order logic [Thomason, 1975].

Properties not corresp. to schemata validity

There are some naturally occurring properties of a binary relation R that do not correspond to the validity of any modal schema.

One such properties is irreflexivity, i.e. $\forall s \neg(sRs)$.

Proof (Idea)

Assume there exists a formula F which characterizes irreflexivity.

To show:

For every frame $\mathcal{F} = (S, R)$, a frame $\mathcal{F}^* = (S^*, R^*)$ can be constructed which satisfies the same modal formulae as $\mathcal{F}$, but is irreflexive.

It would then follow that $\mathcal{F}^* \models F$, but – since in $\mathcal{F}^*$ the same formulae are true as in $\mathcal{F}$ – $(S, R) \models F$ although R is not reflexive. Contradiction.

Properties not corresp. to schemata validity

In the proof we used the following result:

Lemma. For every Kripke structure $\mathcal{K} = (S, R, I)$, a structure $\mathcal{K}^* = (S^*, R^*, I^*)$ can be constructed which satisfies the same modal formulae as $\mathcal{K}$, but R is irreflexive.

Proof: For every $s \in S$ let $s^1, s^2 \not\in S$ (different). We define:

$S^* = \{s^i \mid s \in S, i = 1, 2\}$; $\quad I^*(s^i, P) = I(s, P)$ for $i = 1, 2$.
$R^*(s^i, u^j)$ iff $R(s, u)$ for all i, j if $s \neq u$.
$R^*(s^i, s^j)$ iff $R(s, s)$ and $i \neq j$.
For every formula F and every $s \in S$ the following are equivalent:

(1) $(\mathcal{K}, s) \models F$
(2) $(\mathcal{K}^*, s^1) \models F$
(3) $(\mathcal{K}^*, s^2) \models F$

[Proof by simultaneous structural induction]

Thus, $\mathcal{K} \models F$ iff $\mathcal{K}^* \models F$.

Next time: Theorem proving in modal logics

- Inference system
- Tableau calculi
- Resolution