Non-classical logics

Lecture 10: Modal logics (Part 3)

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Until now

History and Motivation

Syntax

- Inference systems/Proofs

Semantics

- Frames, Kripke structures; Validity

Entailment (local, global)

- The deduction theorem (for local entailment)

Correspondence Theory

First-order definability

Properties not corresp. to schemata validity

There are some naturally occurring properties of a binary relation R that do not correspond to the validity of any modal schema.

One such properties is irreflexivity, i.e. $\forall s \neg(sRs)$.

Proof (Idea)

Assume there exists a formula F which characterizes irreflexivity.

To show:

For every frame $\mathcal{F} = (S, R)$, a frame $\mathcal{F}^* = (S^*, R^*)$ can be constructed which satisfies the same modal formulae as $\mathcal{F}$, but is irreflexive.

It would then follow that $\mathcal{F}^* \models F$, but – since in $\mathcal{F}^*$ the same formulae are true as in $\mathcal{F}$ – $(S, R) \models F$ although R is not reflexive. Contradiction.

Properties not corresp. to schemata validity

In the proof we used the following result:

Lemma. For every Kripke structure $\mathcal{K} = (S, R, I)$, a structure $\mathcal{K}^* = (S^*, R^*, I^*)$ can be constructed which satisfies the same modal formulae as $\mathcal{K}$, but R is irreflexive.

Proof: For every $s \in S$ let $s^1, s^2 \not\in S$ (different). We define:

$S^* = \{s^i \mid s \in S, i = 1, 2\}$; $\quad I^*(s^i, P) = I(s, P)$ for $i = 1, 2$.
$R^*(s^i, u^j)$ iff $R(s, u)$ for all i, j if $s \neq u$.
$R^*(s^i, s^j)$ iff $R(s, s)$ and $i \neq j$.
For every formula F and every $s \in S$ the following are equivalent:

(1) $(\mathcal{K}, s) \models F$
(2) $(\mathcal{K}^*, s^1) \models F$
(3) $(\mathcal{K}^*, s^2) \models F$

[Proof by simultaneous structural induction]

Thus, $\mathcal{K} \models F$ iff $\mathcal{K}^* \models F$.

Theorem proving in modal logics

- Inference system
- Tableau calculi
- Resolution

Proof Calculi/Inference systems and proofs

Inference systems Γ (proof calculi) are sets of tuples

$$(F_1, \ldots, F_n, F_{n+1}), \;\; n \geq 0,$$

called inferences or inference rules, and written

$$\frac{\overbrace{F_1 \;\; \ldots \;\; F_n}^{\text{premises}}}{\underbrace{F_{n+1}}_{\text{conclusion}}} \;.$$

Inferences with 0 premises are also called axioms.

Proofs

A proof in Γ of a formula F from a a set of formulas N (called assumptions) is a sequence $F_1, \ldots, F_k$ of formulas where

(i) $F_k = F$,

(ii) for all $1 \leq i \leq k$: $F_i \in N$, or else there exists an inference $(F_{i_1}, \ \ldots, \ F_{i_{n_i}}, \ F_i)$ in Γ, such that $0 \leq i_j < i$, for $1 \leq j \leq n_i$.

Provability $\vdash_\Gamma$ of F from N in Γ:

$N \vdash_\Gamma F \;\; :\Leftrightarrow$ there exists a proof Γ of F from N.

The modal system K

Axioms:

- All axioms of propositional logic (e.g. $p \vee \neg p$)
- $\Box(A \to B) \to (\Box A \to \Box B)$ (K)

Inference rules

$$\frac{A \quad A \to B}{B}$$ [Modus ponens]

$$\frac{A}{\Box A}$$ [G]

Some systems of modal logic

System	*Description*
T	$K + \Box A \to A$
D	$K + \Box A \to \Diamond A$
B	$T + \neg A \to \Box\neg\Box A$
$S4$	$T + \Box A \to \Box\Box A$
$S5$	$T + \neg\Box A \to \Box\neg\Box A$
$S4.2$	$S4 + \diamond\Box A \to \Box\Diamond A$
$S4.3$	$S4 + \Box(\Box(A \to B)) \vee \Box(\Box(B \to A))$
C	$K + \frac{A \to B}{\Box(A \to B)}$ instead of (G).

Soundness and Completeness

Question:

Is it true that a formula F is valid in all frames iff F is provable in the inference system for the modal logic K?

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- F provable $\Rightarrow$ F valid in all frames: soundness
- F valid in all frames $\Rightarrow$ F provable: completeness

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- F valid in all frames $\Rightarrow$ F provable: completeness

Do similar results hold for other logics (taking into account correspondence theory results we proved in the last lecture)?

Soundness

Theorem. If the formula F is provable in the inference system for the modal logic K then F is valid in all frames.

Proof:

(1) All axioms of the modal logic K are valid in all frames

(2) If $(\mathcal{K}, x) \models A$ and $(\mathcal{K}, x) \models A \rightarrow B$ then $(\mathcal{K}, x) \models B$

If $\mathcal{K} \models A$ and $\mathcal{K} \models A \rightarrow B$ then $\mathcal{K} \models B$

If $\mathcal{F} \models A$ and $\mathcal{F} \models A \rightarrow B$ then $\mathcal{F} \models B$

(3) If $\mathcal{K} \models A$ then $\mathcal{K} \models \Box A$

If $\mathcal{F} \models A$ then $\mathcal{F} \models \Box A$

Completeness

Theorem. If the formula F is is valid in all frames then F is provable in the inference system for the modal logic K.

Proof

Idea:

Assume that F is not provable in the inference system for the modal logic K.

We show that:

(1) $\neg F$ is consistent with the set L of all theorems of K

(2) We can construct a "canonical" Kripke structure $\mathcal{K}_L$ and a world w in this Kripke structure such that $(\mathcal{K}, w) \models \neg F$.

Contradiction!

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Contradiction!

Consistent sets of formulae

Let L be a set of modal formulae which:
(1) contains all propositional tautologies
(2) contains axiom K
(3) is closed under modus ponens and generalization
(4) is closed under instantiation

Definition. A subset $F \subseteq L$ is called L-inconsistent iff there exist formulae $A_1, \dots, A_n \in F$ such that

$$(\neg A_1 \vee \cdots \vee \neg A_n) \in L$$

F is called L-consistent iff it is not L-inconsistent.

Definition. A consistent set F of modal formulae is called maximal L-consistent if for every modal formula A wither $A \in F$ or $\neg A \in F$.

Consistent sets of formulae

Let L be as before. In what follows we assume that L is consistent.

Theorem. Let F be a maximal L-consistent set of formulae. Then:

(1) For every formula A, either $A \in F$ or $\neg A \in F$, but not both.

(2) $A \vee B \in F$ iff $A \in F$ or $B \in F$

(3) $A \wedge B \in F$ iff $A \in F$ and $B \in F$

(4) $L \subseteq F$

(5) F is closed under Modus Ponens

Proof. (1) $A \in F$ or $\neg A \in F$ by definition.

Assume $A \in F$ and $\neg A \in F$.
We know that $\neg A \vee \neg\neg A \in L$ (propositional tautology), so F is inconsistent.
Contradiction.

Consistent sets of formulae

Let L be as before.

Theorem. Let F be a maximal L-consistent set of formulae. Then:

(1) For every formula A, either $A \in F$ or $\neg A \in F$, but not both.

(2) $A \vee B \in F$ iff $A \in F$ or $B \in F$

(3) $A \wedge B \in F$ iff $A \in F$ and $B \in F$

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Proof. (2) "$\Rightarrow$" Assume $A \vee B \in F$, but $A, B \notin F$. Then $\neg A, \neg B \in F$. As $\neg\neg A \vee \neg\neg B \vee \neg(A \vee B) \in L$ (classical tautology) it follows that F is inconsistent.

(2) "$\Leftarrow$" Assume $A \in F$ and $A \vee B \notin F$. Then $\neg(A \vee B) \in F$. Then $\neg A \vee (A \vee B) \in L$, so F is inconsistent.

Consistent sets of formulae

Let L be as before.

Theorem. Let F be a maximal L-consistent set of formulae. Then:

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(2) $A \vee B \in F$ iff $A \in F$ or $B \in F$

(3) $A \wedge B \in F$ iff $A \in F$ and $B \in F$

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Proof. (3) Analogous to (2)

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(2) $A \vee B \in F$ iff $A \in F$ or $B \in F$

(3) $A \wedge B \in F$ iff $A \in F$ and $B \in F$

(4) $L \subseteq F$

(5) F is closed under Modus Ponens

Proof. (4) If $A \in L$ then $\neg A$ is inconsistent. Hence, $\neg A \notin F$, so $A \in F$.

(5) Assume $A \in F$, $A \to B \in F$ and $B \notin F$. Then $\neg A \vee \neg(A \to B) \vee B$ is a tautology, hence in L. Thus, F inconsistent.

Consistent sets of formulae

Theorem. Every consistent set F of formulae is contained in a maximally consistent set of formulae.

Proof. We enumerate all modal formulae: $A_0, A_1, \ldots$ and inductively define an ascending chain of sets of formulae:

$F_0 := F$

$$F_{n+1} := \begin{cases} F_n \cup \{A_n\} & \text{if this set is consistent} \\ F_n \cup \{\neg A_n\} & \text{otherwise} \end{cases}$$

It can be proved by induction that F_n is consistent for all n.

Let $F_{\max} = \bigcup_{n \in \mathbb{N}} F_n$.
Then $F_{\max}$ is maximal consistent and contains F.

Canonical models

Goal: Assume F is not a theorem. Construct a Kripke structure K and a possible world w of K such that $(K, w) \models \neg F$.

States:

State of $\mathcal{K}$: maximal consistent set of formulae.

Intuition: $(\mathcal{K}, W) \models F$ iff $F \in W$.

Interpretation: $I(P, W) = 1$ iff $P \in W$.

Accessibility relation:

Intuition:
$(\mathcal{K}, W) \models \Box F$ iff for all $W', ((W, W') \in R \rightarrow (\mathcal{K}, W') \models F$

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$(\mathcal{K}, W) \models \Box F$ iff for all $W', ((W, W') \in R \rightarrow (\mathcal{K}, W') \models F)$
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$\Box F \in W$ iff for all $W', ((W, W') \in R \rightarrow F \in W')$

$(W, W') \in R$ iff $W' \supseteq \Box^{-1}(W) = \{F \mid \Box F \in W\}$

Canonical Kripke structure

Theorem. For every maximal consistent set W and every formula F:

$\Box F \in W$ iff for all max. consistent sets $W'[(W, W') \in R$ implies $F \in W']$

Proof. "$\Rightarrow$" follows from the definition of R.

"$\Leftarrow$" Assume that for all max. consistent sets $W', (W, W') \in R$ implies $F \in W'$, i.e.

$$\{G \mid \Box G \in W\} \subseteq W' \text{ implies } F \in W'$$

Since W' is maximal consistent it then follows that

$$\{G \mid \Box G \in W\} \vdash_{\mathcal{L}} F$$

hence $\{\Box G \mid \Box G \in W\} \vdash_{\mathcal{L}} \Box F$, so $W \vdash_{\mathcal{L}} \Box F$.

Thus, as W is a maximal consistent set of formulae, $\Box F \in W$.

Canonical Kripke structure

Theorem. $(\mathcal{K}, W) \models F$ iff $F \in W$.

Proof. Induction on the structure of the formula F.

The case $F = P$ follows from the definition of $\mathcal{I}$, while the cases $F = \perp$ and $\perp$ are immediate.

The induction step for $F = \neg F_1$ is immediate; the cases $F = F_1 \text{op} F_2$, $\text{op} \in \{\vee, \wedge\}$ follow from the properties of maximal consistent sets.

For the case $F = \Box F_1$, assume inductively that the result holds for F_1.

$$
\begin{array}{lll}
(\mathcal{K}, W) \models \Box F_1 & \text{iff} & \text{for all } W' \; ((W, W') \in R \rightarrow (\mathcal{K}, W') \models F_1) \\
 & \text{iff} & \text{for all } W' \; ((W, W') \in R \rightarrow F_1 \in W') \\
 & \text{iff} & \Box F_1 \in W \qquad \text{(we used the previous theorem)}
\end{array}
$$

Completeness

Theorem. If the formula F is is valid in all frames then F is provable in the inference system for the modal logic K.

Proof. Assume F is not provable in the inference system for K. Then $L \cup \neg F$ is consistent, hence it is included in a consistenly maximal set W.

Then $\neg F \in W$, so by the previous theorem, $(\mathcal{K}, W) \models \neg F$.

This contradicts the fact that we assumed that F is valid in all Kripke structures.

Other soundness and completeness results

$T = K + \Box A \rightarrow A$.

A formula F is provable in the inference system for the modal logic T iff F is is valid in all frames (S, R) with R reflexive.

$S4 = T + \Box A \rightarrow \Box\Box A$.

A formula F is provable in the inference system for the modal logic $S4$ iff F is is valid in all frames (S, R) with R transitive.

$S5 = T + \neg\Box A \rightarrow \Box\neg\Box A$.

A formula F is provable in the inference system for the modal logic $S5$ iff F is is valid in all frames (S, R) with R is an equivalence relation.

Soundness/completeness: characteriz. classes

Theorem. Let $\mathcal{R}$ be a class of frames characterizable through the modal formulae $C_1, \dots C_n$, and let $K(\mathcal{R})$ be the class of all Kripke structures based on frames in $\mathcal{R}$.

Let S be the inference system obtained from K by adding $C_1, \dots, C_n$ as axioms.

A formula F is provable in the inference system for the modal logic S iff F is is valid in all Kripke structures $\mathcal{K} \in K(\mathcal{R})$.

Proof (Idea) It can be shown that if S is obtained from K by adding axioms $C_1, \dots, C_n$, then the canonical Kripke structure – constructed as in the case of the modal logic K – is in $K(\mathcal{R})$ (i.e. it is based on frames in $\mathcal{R}$).

Example: Let C_1 be the axiom schema $\Box A \rightarrow \Box\Box A$. Let L be the set of all theorems of $K + C_1$. Then all maximal L-consistent sets will contain all instances of this schema.

Let $(W, W') \in R$ and $(W', W'') \in R$.
Then $\Box F \in W$ implies $\Box\Box F \in W$, hence $\Box F \in W'$ (since $(W, W') \in R$)
so $F \in W''$ (as $(W', W'') \in R$). Thus, $(W, W'') \in R$, so R is transitive.

Theorem proving in modal logics

- Inference systems
- Tableau calculi
- Resolution

Tableau calculus

We use labelled formulae

TG standing for "Formula G is true"

FG standing for "Formula G is false"

Tableau calculus

Formula classes

α-Formulae	$T(A \wedge B)$, $F(A \vee B)$, $F(A \rightarrow B)$, $F(\neg A)$
β-Formulae	$T(A \vee B)$, $F(A \wedge B)$, $T(A \rightarrow B)$, $T(\neg A)$
ν-Formulae	$T\ \Box A$, $F\ \Diamond A$
π-Formulae	$T\ \Diamond A$, $F\ \Box A$

Tableau calculus

Successor formulae

α	α_1	α_2
$T(A \wedge B)$	TA	TB
$F(A \vee B)$	FA	FB
$F(A \rightarrow B)$	TA	FB
$F(\neg A)$	TA	TA

β	β_1	β_2
$T(A \vee B)$	TA	TB
$F(A \wedge B)$	FA	FB
$T(A \rightarrow B)$	TB	FA
$T(\neg A)$	FA	FA

ν	ν_0
$T\Box A$	TA
$F\Diamond A$	FA

π	π_0
$T\Diamond A$	TA
$F\Box A$	FA

Tableau calculus

Every combination of top-level operator and sign occurs in one of the above cases.

When constructing the tableau, we use signed formulae prefixed by states:

$$\sigma Z A$$

where $Z \in \{T, F\}$, A is a formula, and σ is a finite sequence of natural numbers.

For the modal logic K, σ_1 is accessible from σ iff

$$\sigma_1 = \sigma n \text{ for some natural number } n.$$

Tableau expansion rules are shown on the next slide.

Modal propositional expansion rules

α-**Expansion** (for formulas that are essentially conjunctions: append subformulas α_1 and α_2 one on top of the other)

$$\frac{\sigma\ \alpha}{\begin{array}{c}\sigma\ \alpha_1\\ \sigma\ \alpha_2\end{array}}$$

β-**Expansion** (for formulas that are essentially disjunctions: append β_1 and β_2 horizontally, i. e., branch into β_1 and β_2)

$$\frac{\sigma\ \beta}{\sigma\ \beta_1 \mid \sigma\ \beta_2}$$

Modal propositional expansion rules

ν-**Expansion** (for formulae which are essentially of the form $\sigma\ T\Box A$: append $\sigma'\nu_0$, such that σ' accessible from σ and occurs on the branch already)

$$\frac{\sigma\ \nu}{\sigma'\ \nu_0}$$

π-**Expansion** (for formulae which are essentially of the form $\sigma\ T\Diamond A$: append $\sigma'\pi_0$, such that σ' is a simple unrestricted extension of σ, i.e. σ' is accessible from σ and no other prefix of the branch starts with σ')

$$\frac{\sigma\ \pi}{\sigma'\ \pi_0}$$

Tableau calculus

A tableau is closed if every branch contains some pair of formulas of the form $s\ TA$ and $s\ FA$.

A proof for modal logic formula consists of a closed tableau starting with the root $1\ FA$.

Example

These tableau rules can be used to analyze whether $\Box A \to \Diamond A$ is a theorem of K as follows:

1	$F \Box A \to \Diamond A$	(1)
1	$T \Box A$	(2) from 1
1	$F \Diamond A$	(3) from 1

No other proof rules can be used because the modal formulas are ν rules, which are only applicable for accessible prefixes that already occur on the branch.

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No other proof rules can be used because the modal formulas are ν rules, which are only applicable for accessible prefixes that already occur on the branch.

Intuition

The labels denote possible worlds. We can construct a Kripke model $\mathcal{K}$ with one possible world only and the empty relation.

Then $\Box A$ is true in $\mathcal{K}$, but $\Diamond A$ is false, so $\Box A \to \Diamond A$ is false in $\mathcal{K}$.

Example

Without the restriction that the prefix should already appear on the path, we could have closed the tableau as follows:

1	$F\Box A \rightarrow \Diamond A$	(1)
1	$T\Box A$	(2) from 1
1	$F\Diamond A$	(3) from 1
11	TA	(4) from 2
11	FA	(5) from 3

But this would have been wrong, since $\Box A \rightarrow \Diamond A$ is not a theorem of K.

Tableau calculus

The rules above are sound and complete for the modal logic K.

For other logics it may be necessary to change

- accessibility relation on prefixes

- the two modal rules.

Tableau calculus

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A tableau formed using the rules presented before is called a K-tableau.

Example

Prove that $\Box A \wedge \Box B \rightarrow \Box(A \wedge B)$

1	$F(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$	(1)
1	$T\Box A \wedge \Box B$	(2), α, 1_1
1	$F\Box(A \wedge B)$	(3), α, 1_2
1	$T\Box A$	(4), α, 2_1
1	$T\Box B$	(5), α, 2_1
11	$F(A \wedge B)$	(6), π, from 3

11	FA	(7), β, 6_1	11	FB	(8), β, 6_2
11	TA	(9), ν, from 4	11	TB	(10) ν, from 5
	$\bot$	7 and 9		$\bot$	10 and 8

Soundness and Completeness

Definition. A tableau is satisfiable in K if it has a path P, for which there is a Kripke structure $K = (S, R, I)$ for the modal logic K and a mapping m from prefixes of P to S such that

1. $m(s)Rm(s')$ iff prefix s' is accessible from prefix s; and
2. $(K, m(s)) \models A$ for every formula sTA on path P.
3. $(K, m(s)) \models \neg A$ for every formula sFA on path P.

In the sequel we will just abbreviate the last two cases to: $(K, m(s)) \models A$ for every (signed) formula sA on path P.

Soundness and Completeness

Soundness

If FA is satisfiable then we cannot derive $\perp$ on all branches

If we can construct a closed tableau with root FA,
then there is no Kripke structure in which A evaluates to false.

Soundness and Completeness

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Theorem. If there is a closed K-tableau with root $1FA$, then A is valid in all Kripke structures of K.

Soundness and Completeness

Soundness

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then there is no Kripke structure in which A evaluates to false.

Theorem. If there is a closed K-tableau with root $1FA$, then A is valid in all Kripke structures of K.

In order to prove the theorem we will first prove the following lemma

Lemma. Let T_0 is a K-satisfiable tableau, and let T be the extension of T_0 with one of the extension rules. Then T is a K-satisfiable tableau as well.

Soundness and Completeness

Lemma. Let T_0 is a K-satisfiable tableau, and let T be the extension of T_0 with one of the extension rules. Then T is a K-satisfiable tableau as well.

Proof. We only consider the ν and π rules.

T_0 is satisfiable in K if it has a path P, for which there is a Kripke structure $K = (S, R, I)$ for the modal logic K and a mapping m from prefixes of P to S such that

1. $m(\sigma)Rm(\sigma')$ if prefix σ' is accessible from prefix σ; and
2. $(K, m(\sigma)) \models A$ for every formula sTA on path P.
3. $(K, m(\sigma)) \models \neg A$ for every formula sFA on path P.

Assume first that formula $\sigma\nu$ occurs on path P and the path is extended by the ν rule to $P \cup \{\sigma'\nu_0\}$, where σ' occurs already in P and is accessible from σ.
Then $m(\sigma)Rm(\sigma')$ and $(\mathcal{K}, m(\sigma)) \models \nu$.

From this it immediately follows that $(\mathcal{K}, m(\sigma')) \models \nu_0$.

Soundness and Completeness

Lemma. Let T_0 is a K-satisfiable tableau, and let T be the extension of T_0 with one of the extension rules. Then T is a K-satisfiable tableau as well.

Proof. (continued)
Assume now that formula $\sigma\pi$ occurs on path P and the path is extended by the π rule to $P \cup \{\sigma'\pi_0\}$, where no other prefix of P starts with σ' and σ' is accessible from σ. Then $m(\sigma)Rm(\sigma')$ and $(\mathcal{K}, m(\sigma)) \models \pi$.

From this it immediately follows that there exists $s \in S$ such that $(\mathcal{K}, s) \models \pi_0$.

We extend the map m by defining $m(\sigma') = s$.

(1) By the conditions on the π-rule, we know that σ' is accessible from a prefix ρ on the path P iff $\rho = \sigma$.

(2) Moreover, for every prefix ρ on the path P, ρ is not accessible from σ'.

These properties ensure that for every two prefixes on the path $P \cup \{\sigma'\pi_0\}$ we have:

$m(\rho_1)Rm(\rho_2)$ if ρ_2 is accessible from ρ_1. Thus, T is K-satisfiable.

Soundness and Completeness

Lemma. Let T_0 is a K-satisfiable tableau, and let T be the extension of T_0 with one of the extension rules. Then T is a K-satisfiable tableau as well.

Theorem. If there is a closed K-tableau with root $1FA$, then A is valid in all Kripke structures of K.

Proof. Let T be the closed K-Tableau with root $1FA$. Assume there exists a Kripke-Structure $\mathcal{K} = (S, R, I)$ and $s \in S$ such that $(\mathcal{K}, s) \models \neg A$.

Then the root of T, $1FA$, is a K-satisfiable tableau if we define $m(1) = s$. By the previous Lemma the extension of a K-satisfiable tableau with one of the extension rules is a K-satisfiable tableau as well.

It then follows that T is K-satisfiable, which contradicts the fact that T is closed.

Soundness and Completeness

Completeness

Weak form:

Show that if A is valid then there exists a closed tableau with root $1FA$.

Soundness and Completeness

Completeness

Weak form:

Show that if A is valid then there exists a closed tableau with root $1FA$.

Stronger form:

Would like to show that if $N \models A$ then, if we consider the formulae in N as "axioms" and assume that FA then we can construct a closed tableau.

Soundness and Completeness

Completeness (weak form)

Theorem. If A is valid then there exists a closed tableau with root $1FA$.

Proof. (Idea)

We prove the contrapositive. Assume that every tableau for $1FA$ has an open saturated path P.

Let P_0 the set of all signed formulae with prefixes occurring on P.

Then for every ν-formula $\sigma\nu$, the path contains also the consequence of the ν-rule, $\sigma'\nu_0$, where σ' occurs in P and is accessible from σ.

We construct a Kripke model $\mathcal{K} = (S, R, I)$ for P as follows:

- S is the set of all prefixes occurring on P;
- R is the accessibility relation on the set of prefixes;
- If A propositional variable: $I(A, \sigma) = 1$ iff σTA occurs on P.

Soundness and Completeness

Completeness (weak form)

Theorem. If A is valid then there exists a closed tableau with root $1FA$.

Proof. (Continued)

One can prove by induction on the structure of the signed formulae that for every formula σC on P, $(\mathcal{K}, \sigma) \models C$.

Soundness and Completeness

Completeness (weak form)

Theorem. If A is valid then there exists a closed tableau with root $1FA$.

Proof. (Continued)

One can prove by induction on the structure of the signed formulae that for every formula σC on P, $(\mathcal{K}, \sigma) \models C$.

Example 1:

If $\sigma_0 T\Box B$ occurs in P, then for every prefix $\sigma \in S$ which is reachable from σ_0 also σTB occurs in P.

Induction hypothesis: $(\mathcal{K}, \sigma) \models B$ (and this holds for all $\sigma \in S$ with $\sigma_0 R \sigma$. Thus, $(\mathcal{K}, \sigma_0) \models \Box B$.

Soundness and Completeness

Completeness (weak form)

Theorem. If A is valid then there exists a closed tableau with root $1FA$.

Proof. (Continued)

One can prove by induction on the structure of the signed formulae that for every formula σC on P, $(\mathcal{K}, \sigma) \models C$.

Example 2:

If $\sigma_0 F \Box B$ occurs in P, there exists a prefix σ accessible from σ_0 such that σFB occurs in P.

By induction hypothesis, $(\mathcal{K}, \sigma) \models FB$ (i.e. $(\mathcal{K}, \sigma) \models \neg B$, hence $(\mathcal{K}, \sigma_0) \models F\Box B$.

Completeness

Completeness (strong form)

Would like to show that if $N \models A$ then, if we consider the formulae in N as "axioms" and assume that FA then we can construct a closed tableau.

We defined "local entailment" and "global entailment"

$\mapsto$ We distinguish L-completeness and G-completeness

Entailment

Global entailment:

$N \models_G F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$:

$$\text{If } \mathcal{K} \models G \text{ for every } G \in N \text{ then } \mathcal{K} \models F$$

Local entailment:

$N \models_L F$ iff for every Kripke structure $\mathcal{K} = (S, R, I)$ and every $s \in S$:

$$\text{If } (\mathcal{K}, s) \models G \text{ for every } G \in N \text{ then } (\mathcal{K}, s) \models F$$

L-Completeness

Let N be a set of modal formulae.

Definition A K-tableau is an K-L-Tableau over N if for every formula $B \in N$ the following rule can be used:

$$\frac{}{1TB}$$

Theorem. Let N be a set of modal formulae and A a modal logic formula. Then $N \models_L A$ iff there exists a closed K-L-Tableau with root $1FA$.

G-Completeness

Let N be a set of modal formulae.

Definition A K-tableau is an K-G-Tableau over N if for every formula $B \in N$ and for every prefix σ on the current path the following rule can be used:

$$\frac{}{\sigma TB}$$

Theorem. Let N be a set of modal formulae and A a modal logic formula. Then $N \models_G A$ iff there exists a closed K-G-Tableau with root $1FA$.

Tableau calculi

Sound and complete tableau calculi can be devised for a large class of systems of propositional modal logic.

Main challenge: Prove termination (can construct "saturated" or closed model in a finite number of steps)

"Blocking techniques"

Theorem proving in modal logics

- Inference system (soundness and completeness results)
- Tableau calculi (soundness and completeness results)
- Translation to first order logic (+ e.g. Resolution)

next time