Resolution-based Theorem Proving for SH_n-Logics

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Abstract. In this paper we illustrate by means of an example, namely SH_n-logics, a method for translation to clause form and automated theorem proving for first-order many-valued logics based on distributive lattices with operators.

1 Introduction

In this paper we present a method for translation to clause form and automated theorem proving in finitely-valued logics having as algebras of truth values distributive lattices with certain types of operators. Many non-classical logics that occur in practical applications fall in this class. One of the advantages of distributive lattices (with well-behaved operators) is the existence, in such cases, of good representation theorems, such as the Priestley representation theorem. The method for translation to clause form we present here uses the Priestley dual of the algebra of truth values. The ideas behind this method are very natural, even if the algebraic notions used may at first sight seem involved. This is why in this paper we illustrate the ideas by one example, namely SH_n-logics. SH_n-logics are a series of propositional finitely-valued logics based on Symmetric Heyting algebras of order n. The particular properties of SH_n-logics allow us to further improve the efficiency of the automated theorem proving procedure for certain types of formulae in SH_n-logics, by exploiting the structure of the Priestley dual of the algebra of truth values. This effect is difficult to explain in a general setting; the case of SH_n-logics can be considered a case-study, in which we take first steps in this direction.

The main sources of inspiration for our work are the many-valued resolution method of Baaz and Fermüller [BF95], and the results of Hähnle [Häh96]. Our method exploits the particular structure of the algebra of truth values, and leads to a reduction of the number of clauses compared to methods using the algebra of truth values, especially when the difference between the number of elements of the algebra of truth values and the number of elements of its Priestley dual is large. Resolution procedures are also discussed (in particular, a hyperresolution procedure that extends the results established for regular clauses in [Häh96]).

The paper is structured as follows. In Section 2 we briefly present the main notions needed in the paper. In Section 3 SH_n-logics are defined. In Section 4 our method for translation to clause form for SH_n-logics is presented. In Section 5

resolution-based methods for automated theorem proving are briefly presented. In Section 6 we compare our method with other approaches. Section 7 contains some conclusions.

2 Preliminaries

2.1 Partially-ordered sets and lattices

In what follows we assume known standard notions, such as partially-ordered set and lattice, order-filter and order-ideal in partially-ordered sets, meet- and join-irreducible elements, and (prime) filter and (prime) ideal in lattices. For the definitions and further information we refer to [DP90]. Given a partially-ordered set $(P, \leq)$, by $\mathcal{O}(P)$ we will denote the set of order-filters of P; for every $i \in P$, we will use the following notations: $\uparrow i = \{j \in P \mid j \geq i\}$, and $\downarrow i = \{j \in P \mid i \geq j\}$. Note also that every ideal in a finite distributive lattice is of the form $\downarrow i$ and every filter is of the form $\uparrow i$ for some element i in the lattice; such an ideal (resp. filter) is prime iff i is meet-irreducible (resp. join-irreducible).

2.2 Priestley representation

The Priestley representation theorem [DP90] states that every distributive lattice A is isomorphic to the lattice of clopen (i.e. closed and open) order filters of the ordered topological space having as points the prime filters of A, ordered by inclusion, and the topology generated by the sets of the form $X_a = \{F \mid F \text{ prime filter}, a \in F\}$ and their complements as a subbasis. The partially ordered set of all prime filters of A, ordered by inclusion, and endowed with the topology mentioned above will be denoted $D(A)$ (we will refer to it as the dual of A). $E(X)$. In particular, if A is finite, the topology on $D(A)$ is discrete, and A is isomorphic to $\mathcal{O}(D(A))$.

2.3 Many-valued logics

We now give a (semantics-based) definition of many-valued logics, which closely follows the definition in [BF95].

Let A be a (finite) set of truth values. The semantics of a many-valued logic $\mathcal{L}$ with language $(X, O, P, \Sigma, \mathcal{Q})$ (where X is a (countably) infinite set of variables; O a set of function symbols; P a set of predicate symbols; Σ a finite set of logical operators; and $\mathcal{Q}$ a finite set of (one-place) quantifiers) is given as follows:

(i) to every $\sigma \in \Sigma$ with arity n we associate a truth function $\sigma_A : A^n \to A$;
(ii) to every quantifier Q we associate a truth function $\overline{Q} : \mathcal{P}(A) \backslash \{\emptyset\} \to A$.

Definition 1. *An interpretation for a language $(X, O, P, \Sigma, \mathcal{Q})$ and a set of truth values A is a tuple (D, I, d) where D is a non-empty set, the* domain, *I is a* signature interpretation, *i.e. a map assigning a function $I(f) : D^n \to D$ to every n-ary function symbol $f \in O$, and a function $I(R) : D^n \to A$ to every n-ary predicate symbol $R \in P$, and $d : X \to D$ a* variable assignment.

Every interpretation $\mathcal{I} = (D, I, d)$ induces a valuation $v_{\mathcal{I}} : \mathsf{Fma}(\mathcal{L}) \to A$ on the set $\mathsf{Fma}(\mathcal{L})$ of formulae of $\mathcal{L}$ as follows. (For further details we refer to [BF95].)

(1) $v_{\mathcal{I}}(x) = d(x)$ for all variables $x \in X$,
(2) $v_{\mathcal{I}}(f(t_1, \ldots, t_n)) = I(f)(v_{\mathcal{I}}(t_1), \ldots, v_{\mathcal{I}}(t_n))$ for all n-ary function symbols $f \in O$, $n \geq 0$,
(3) $v_{\mathcal{I}}(R(t_1, \ldots, t_n)) = I(R)(v_{\mathcal{I}}(t_1), \ldots, v_{\mathcal{I}}(t_n))$ for all n-ary predicate symbols $R \in P$, $n \geq 0$,
(4) $v_{\mathcal{I}}(\sigma(\phi_1, \ldots, \phi_n)) = \sigma_A(v_{\mathcal{I}}(\phi_1), \ldots, v_{\mathcal{I}}(\phi_n))$ for all logical operators $\sigma \in \Sigma$,
(5) $v_{\mathcal{I}}((Qx)\phi) = \overline{Q}(\{w \mid \exists d \in D \ s.t. \ v_{\mathcal{I}_{x,d}}(\phi) = w\})$ for all quantifiers Q, where $\mathcal{I}_{x,d}$ is identical with $\mathcal{I}$ except for assigning d to the variable x.

A formula ϕ will be called *valid* iff for all interpretations $\mathcal{I}$, $v_{\mathcal{I}}(\phi) = 1$; ϕ will be called *satisfiable* iff there exists an interpretation $\mathcal{I}$ with $v_{\mathcal{I}}(\phi) = 1$.

3 SH_n-logics

The propositional SH_n-logics were introduced by Iturrioz in [Itu82]. The language of SH_n-logics is a propositional language, whose formulae are built from propositional variables taken from a set Var, with operations $\vee$ (disjunction), $\wedge$ (conjunction), $\Rightarrow$ (intuitionistic implication), $\sim, \neg$ (a De Morgan resp. an intuitionistic negation), and a family $\{S_i \mid i = 1, \ldots, n-1\}$ of unary operations (expressing the degree of truth of a formula). The set of formulae in the variables Var will be denoted $\mathsf{Fma}(\mathsf{Var})$. Recently, Iturrioz noticed that SH_n-logics provide a source of examples for L_{t_f}-logics, which have been introduced by Rasiowa to formalize the reasoning of a poset of intelligent agents. A Hilbert style axiomatization of SH_n-logics is given in [Itu82] and [IO96]. We present it below:

Axioms: (A1) $a \Rightarrow (b \Rightarrow a)$
(A2) $(a \Rightarrow (b \Rightarrow c)) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow c))$
(A3) $(a \wedge b) \Rightarrow a$
(A4) $(a \wedge b) \Rightarrow b$
(A5) $(a \Rightarrow b) \Rightarrow ((a \Rightarrow c) \Rightarrow (a \Rightarrow (b \wedge c)))$
(A6) $a \Rightarrow (a \vee b)$
(A7) $b \Rightarrow (a \vee b)$
(A8) $(a \Rightarrow c) \Rightarrow ((b \Rightarrow c) \Rightarrow ((a \vee b) \Rightarrow c))$
(A9) $\sim\sim a \leftrightarrow a$
(A10) $S_i(a \wedge b) \leftrightarrow S_i(a) \wedge S_i(b)$
(A11) $S_i(a \Rightarrow b) \leftrightarrow (\bigwedge_{k=i}^{n} S_k(a) \Rightarrow S_k(b))$
(A12) $S_i(S_j(a)) \leftrightarrow S_j(a)$, for every $i, j = 1, \ldots, n-1$
(A13) $S_1(a) \Rightarrow a$
(A14) $S_i(\sim a) \leftrightarrow \sim S_{n-i}a$, for $i = 1, \ldots, n-1$
(A15) $S_1(a) \vee \neg S_1(a)$

where $\neg a = (a \Rightarrow \sim (a \Rightarrow a))$ and $a \leftrightarrow b$ is an abbreviation for $(a \Rightarrow b) \wedge (b \Rightarrow a)$.

Inference rules:

$$\text{(R1)} \ \frac{a, \ a \Rightarrow b}{b} \qquad \text{(R2)} \ \frac{a \Rightarrow b}{\sim b \Rightarrow \sim a} \qquad \text{(R3)} \ \frac{a \Rightarrow b}{S_1(a) \Rightarrow S_1(b)}$$

3.1 Algebraic semantics for propositional SH_n-logics

In [Itu82,Itu83], Iturrioz gave a lattice-based semantics for SH_n-logics, by means of symmetrical Heyting algebras of order n, or for short SH_n-algebras.

Definition 2. *An abstract algebra* $A = (A, 0, 1, \wedge, \vee, \Rightarrow, \neg, \sim, S_1, \ldots, S_{n-1})$ *is said to be a* symmetric Heyting algebra of order n *(SH_n-algebra for short) if:*

(1) $(A, 0, 1, \wedge, \vee, \Rightarrow, \neg)$ *is a Heyting algebra, i.e.* $(A, 0, 1, \wedge, \vee)$ *is a bounded distributive lattice; for every* $a, b \in A$, $a \Rightarrow b$ *is the pseudocomplement of* a *relative to* b, *i.e. the largest element* c *of* A *with* $a \wedge c \leq b$; *and* $\neg a = a \Rightarrow 0$;
(2) $\sim$ *is a De Morgan negation on* A, *i.e.* $\sim 0 = 1$, $\sim 1 = 0$, *and, for every* $a, b \in A$, $\sim (a \vee b) = \sim a \wedge \sim b$, $\sim (a \wedge b) = \sim a \vee \sim b$, *and* $\sim\sim a = a$;
(3) For every $a, b \in A$ *and for all* $i, j \in \{1, \ldots, n-1\}$, *the following hold:*
(S1) $S_i(a \wedge b) = S_i(a) \wedge S_i(b)$,
(S2) $S_i(a \Rightarrow b) = \bigwedge_{k=i}^{n} (S_k(a) \Rightarrow S_k(b))$,
(S3) $S_i(S_j(a)) = S_j(a)$, *for every* $i, j = 1, \ldots, n-1$,
(S4) $S_1(a) \vee a = a$,
(S5) $S_i(\sim a) = \sim S_{n-i}(a)$, *for* $i = 1, \ldots, n-1$,
(S6) $S_1(a) \vee \neg S_1(a) = 1$, *with* $\neg a = a \Rightarrow 0$.

The class of SH_n-algebras is a variety, which will be denoted $\mathsf{SH_n}$ in what follows. Iturrioz [Itu83] proved that this variety is generated by one finite SH_n-algebra, S_{n^2} (represented in the left-hand side of Figure 1), defined as follows.

Definition 3 (S_{n^2}). *Let* $n \geq 2$ *and let* S_{n^2} *be the cartesian product* $L_n \times L_n$ *where* $L_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$. *Consider the following operations on* S_{n^2}:

(1) $(x_1, y_1) \wedge (x_2, y_2) = (\mathsf{min}(x_1, x_2), \mathsf{min}(y_1, y_2))$,
(2) $(x_1, y_1) \vee (x_2, y_2) = (\mathsf{max}(x_1, x_2), \mathsf{max}(y_1, y_2))$,
(3) $\sim (x, y) = (1 - y, 1 - x)$ *for every* $(x, y) \in S_{n^2}$,
(4) $S_i(x, y) = (S_i(x), S_i(y))$, *where* $S_i(\frac{j}{n-1}) = \begin{cases} 1 & \text{if } i + j \geq n, \\ 0 & \text{if } i + j < n, \end{cases}$
(5) $(x_1, y_1) \Rightarrow (x_2, y_2) = (x_1 \to x_2, y_1 \to y_2)$,
(6) $\neg (x, y) = (x \to 0, y \to 0)$,

where $\to$ *is the Heyting relative pseudocomplementation on* L_n, *i.e.* $a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$ *and for every* $a \in L_n$, $\neg a := a \to 0 = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } a > 0 \end{cases}$.

In [Itu82] it is proved that SH_n-logics are sound and complete with respect to the variety of SH_n-algebras. Since the variety $\mathsf{SH_n}$ is generated by S_{n^2} it follows that the SH_n-logics can be regarded as many-valued logics having S_{n^2} as an algebra of truth values.

A Priestley representation theorem for SH_n-algebras is established in [Itu83], see also [SS97,SS99b]. In particular we give a description of the dual of the generator S_{n^2} of the variety of SH_n-algebras. The Priestley dual of S_{n^2}, $D(S_{n^2})$, is the set of prime filters of S_{n^2}, with the discrete topology, and ordered by inclusion. Every prime filter of S_{n^2} is of the form $\uparrow a$ where a is a join-irreducible

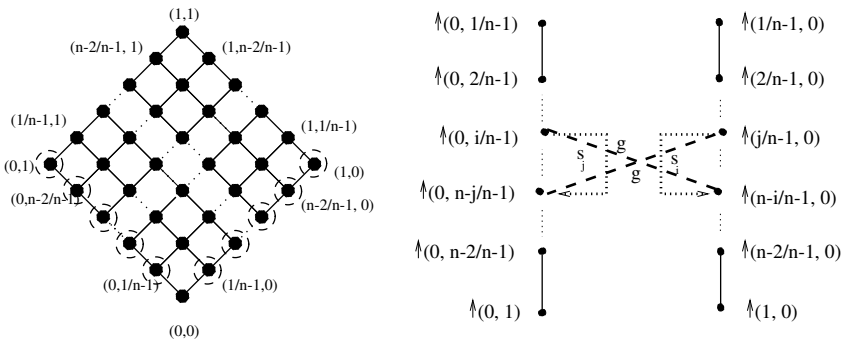

Fig. 1. S_{n^2} and its Priestley dual $D(S_{n^2})$

element of S_{n^2}. The set of join-irreducible elements of S_{n^2} (encircled in Figure 1) is $\{(0, \frac{i}{n-1}) \mid i = 1, \ldots, n-1\} \cup \{(\frac{i}{n-1}, 0) \mid i = 1, \ldots, n-1\}$.

Thus, $D(S_{n^2})$ (the right-hand side of Figure 1) is the set $\{\uparrow (0, \frac{i}{n-1}) \mid i = 1, \ldots, n-1\} \cup \{\uparrow (\frac{i}{n-1}, 0) \mid i = 1, \ldots, n-1\}$. Additional operations $g, s_1, \ldots, s_{n-1}$ are defined for every prime filter F of S_{n^2} by $g(F) = S_{n^2} \setminus {\sim}^{-1}(F)$, and $s_i(F) = S_i^{-1}(F)$. Therefore, $g(\uparrow (0, \frac{i}{n-1})) = \uparrow (\frac{n-i}{n-1}, 0)$; $g(\uparrow (\frac{i}{n-1}, 0)) = \uparrow (0, \frac{n-i}{n-1})$; and, for every $j = 1, \ldots, n-1$, $s_j(\uparrow (0, \frac{i}{n-1})) = \uparrow (0, \frac{n-j}{n-1})$; $s_j(\uparrow (\frac{i}{n-1}, 0)) = \uparrow (\frac{n-j}{n-1}, 0)$. By the Priestley representation theorem for SH_n-algebras, since the topology of $D(S_{n^2})$ is discrete, there exists an isomorphism of SH_n-algebras $\eta_{S_{n^2}} : S_{n^2} \simeq \mathcal{O}(D(S_{n^2}))$, defined for every $x \in S_{n^2}$ by $\eta_{S_{n^2}}(x) = \{\uparrow \alpha \in D(S_{n^2}) \mid x \in \uparrow \alpha\}$.

3.2 A finite Kripke-style frame for SH_n-logics

In [IO96], Iturrioz and Orłowska give a completeness theorem for SH_n-logics with respect to a Kripke-style semantics. We will show that one can regard $D(S_{n^2})$ as a particular Kripke-style frame for SH_n-logics, and that a formula is an SH_n-theorem iff it is valid in this frame. As in [IO96] where Kripke-style models for SH_n-logics are investigated, we will assume that all meaning functions for $D(S_{n^2})$ have as values order-filters.

Definition 4. *Let $m : \mathsf{Var} \to \mathcal{O}(D(S_{n^2}))$ be a meaning function, and let $\overline{m} : \mathsf{Fma(Var)} \to \mathcal{O}(D(S_{n^2}))$ be the unique extension of m to formulae. We define:*

(i) $D(S_{n^2}) \overset{r}{\models}_{m,x} \phi$ *iff* $x \in \overline{m}(\phi)$;

(ii) $D(S_{n^2}) \overset{r}{\models}_{m} \phi$ *iff* $\overline{m}(\phi) = D(S_{n^2})$;

(iii) $D(S_{n^2}) \overset{r}{\models} \phi$ *iff* $D(S_{n^2}) \overset{r}{\models}_{m} \phi$ *for every* $m\colon \mathsf{Var} \to \mathcal{O}(D(S_{n^2}))$.

Theorem 1 ([SS97,SS99b]). *Every formula ϕ of SH_n-logic is an SH_n-theorem iff* $D(S_{n^2}) \overset{r}{\models} \phi$.

$(D(S_{n^2}), \leq, g, s_1, \ldots, s_{n-1})$ is in particular a Kripke-style frame as defined in [IO96], and the relation $\overset{r}{\models}$ defined above agrees with the satisfiability relation for SH_n-models defined in [IO96], i.e. for every meaning function $m : \mathsf{Var} \to \mathcal{O}(D(S_{n^2}))$ and every $x \in D(S_{n^2})$ the following hold:

$$
\begin{array}{ll}
D(S_{n^2}) \overset{r}{\models}_{m,x} p & \text{iff } x \in m(p), \qquad \text{for } p \in \mathsf{Var}, \\
D(S_{n^2}) \overset{r}{\models}_{m,x} \phi \vee \psi & \text{iff } D(S_{n^2}) \overset{r}{\models}_{m,x} \phi \text{ or } D(S_{n^2}) \overset{r}{\models}_{m,x} \psi, \\
D(S_{n^2}) \overset{r}{\models}_{m,x} \phi \wedge \psi & \text{iff } D(S_{n^2}) \overset{r}{\models}_{m,x} \phi \text{ and } D(S_{n^2}) \overset{r}{\models}_{m,x} \psi, \\
D(S_{n^2}) \overset{r}{\models}_{m,x} \phi \Rightarrow \psi & \text{iff for all } y, \text{ if } x \leq y \text{ and } D(S_{n^2}) \overset{r}{\models}_{m,y} \phi \text{ then } D(S_{n^2}) \overset{r}{\models}_{m,y} \psi, \\
D(S_{n^2}) \overset{r}{\models}_{m,x} \neg\phi & \text{iff for all } y, \text{ if } x \leq y \text{ then } D(S_{n^2}) \not\overset{r}{\models}_{m,y} \phi, \\
D(S_{n^2}) \overset{r}{\models}_{m,x} S_i(\phi) & \text{iff } D(S_{n^2}) \overset{r}{\models}_{m,s_i(x)} \phi, \\
D(S_{n^2}) \overset{r}{\models}_{m,x} \sim \phi & \text{iff } D(S_{n^2}) \not\overset{r}{\models}_{m,g(x)} \phi.
\end{array}
$$

3.3 First-order SH_n-logics

Following [BF95] we define first-order SH_n-logics as many-valued logics having S_{n^2} as set of truth values.

A first-order SH_n-logic is defined by specifying a set X of variables, a set O of operation symbols and a set P of predicate symbols. The set of logical connectives is $\{\vee, \wedge, \Rightarrow, \neg, \sim\}$, and they are interpreted in S_{n^2} as explained in Definition 3. We allow only two quantifiers, namely $\exists$ and $\forall$, such that the truth functions $\overline{\forall}, \overline{\exists} : \mathcal{P}(S_{n^2}) \to (S_{n^2})$ are defined for every $\emptyset \neq W \subseteq S_{n^2}$ by:

$$\overline{\forall}(W) = \bigwedge_{w \in W} w; \ \overline{\exists}(W) = \bigvee_{w \in W} w.$$

Frames and interpretations are defined in the usual way (cf. Definition 1). Due to the fact that S_{n^2} is isomorphic to $\mathcal{O}(D(S_{n^2}))$, an alternative (equivalent) notion of frame (D, M) (and resp. interpretation $\mathcal{M} = (D, M, d)$) can be defined, where the signature interpretation M assigns with every n-ary predicate symbol $R \in P$ a function $M(R) : D^n \to \mathcal{O}(D(S_{n^2}))$. For every interpretation $\mathcal{M} = (D, M, d)$ in $\mathcal{O}(D(S_{n^2}))$, the induced valuation function $v_{\mathcal{M}} : \mathsf{Fma}(\mathcal{L}) \to \mathcal{O}(D(S_{n^2}))$ is defined in the usual way; so are the notions of validity and satisfiability.

Taking into account the fact that the largest element of the lattice $\mathcal{O}(D(S_{n^2}))$ is $D(S_{n^2})$, it follows that a formula ϕ is valid iff for all interpretations $\mathcal{M}$ in $\mathcal{O}(D(S_{n^2}))$, $v_{\mathcal{M}}(\phi) = D(S_{n^2})$; and satisfiable iff $v_{\mathcal{M}}(\phi) = D(S_{n^2})$ for some interpretation $\mathcal{M}$ in $\mathcal{O}(D(S_{n^2}))$.

4 Translation into clause form

The main idea of our approach is to use signed literals, where the signs are "possible worlds", i.e. elements of $D(S_{n^2})$ (corresponding to prime filters of truth

values) instead of truth values (as done in [BF95]) or arbitrary sets of truth values (as done in [Häh94,Häh96]). The idea of using "valuations in $\{0,1\}$" instead of values appears already for instance in [Sco73] for the case of Łukasiewicz logics. In what follows, given an interpretation $\mathcal{M} = (D, M, d)$ in $\mathcal{O}(D(S_{n^2}))$ we use the following notations (where ϕ is a formula and α is an element of $D(S_{n^2})$):

$\boxed{\alpha}\ \phi^t$ in $\mathcal{M}$ means "ϕ is true at α" in the interpretation $\mathcal{M}$ (i.e. $\alpha \in v_{\mathcal{M}}(\phi)$)
$\boxed{\alpha}\ \phi^f$ in $\mathcal{M}$ means "ϕ is false at α" in the interpretation $\mathcal{M}$ (i.e. $\alpha \notin v_{\mathcal{M}}(\phi)$)

Note that if $\boxed{\beta}\ \phi^t$ in $\mathcal{M}$ and $\beta \leq \alpha$ then $\boxed{\alpha}\ \phi^t$ in $\mathcal{M}$.

Definition 5 (Literal, Clause). *Let L be an atomic formula and $\alpha \in D(S_{n^2})$. Then $\boxed{\alpha}\ L^t$ is a* positive literal (with sign $\boxed{\alpha}$) *and $\boxed{\alpha}\ L^f$ is a* negative literal (with sign $\boxed{\alpha}$). *A set of (positive or negative) signed literals is called a* (signed) clause. *A clause is called* negative (positive) *if it only contains negative (resp. positive) literals.*

Definition 6 (Signed CNF). *A* formula in signed conjunctive normal form (CNF) *is a finite set of signed clauses. We require that the clauses in a formula have disjoint variables.*

Definition 7 (Satisfiability). *A positive literal $\boxed{\alpha}\ L^t$ (resp. a negative literal $\boxed{\alpha}\ L^f$) is satisfied by an interpretation $\mathcal{M}$ in $\mathcal{O}(D(S_{n^2}))$ if L is true (resp. false) in $\mathcal{M}$ at α. A (positive or negative) literal is satisfiable if it is satisfied by some interpretation $\mathcal{M}$. A signed clause is satisfiable if at least one of its literals is satisfiable. A formula Φ in signed CNF is satisfiable if all clauses in Φ are simultaneously satisfiable by the same interpretation.*

We give a *structure-preserving* transformation method to clause form in first-order logic. The main idea of such methods is to introduce, for every non-atomic subformula ψ of ϕ, a new atomic formula of the form $P_\psi(\overline{x})$, where P_ψ is a new predicate symbol and $\overline{x}$ are the free variables in ψ.

Theorem 2. *The SH_n-formula ϕ is valid iff there exists no interpretation $\mathcal{M} = (D, M, d)$ in $\mathcal{O}(D(S_{n^2}))$ such that*

$$\begin{cases}
\boxed{\uparrow(0,1)}\ P_\phi^f \vee \boxed{\uparrow(1,0)}\ P_\phi^f \text{ in } \mathcal{M}, \\
\\
\begin{array}{c}(\boxed{\alpha}\ P_\psi(x_1,\ldots,x_n)^t \vee \boxed{\alpha}\ \sigma(P_{\psi_1},\ldots,P_{\psi_m})^f) \\ (\boxed{\alpha}\ P_\psi(x_1,\ldots,x_n)^f \vee \boxed{\alpha}\ \sigma(P_{\psi_1},\ldots,P_{\psi_m})^t)\end{array} \\
\quad \text{in } \mathcal{M};\ \text{for every } \alpha \in D(S_{n^2}),\ \text{for all subformulae } \psi = \sigma(\psi_1,\ldots,\psi_m) \\
\quad \text{of } \phi,\ \text{and all instantiations in } D \text{ of the free variables } x_1,\ldots,x_n \text{ of } \psi, \\
\\
\begin{array}{c}(\boxed{\alpha}\ P_\psi(x_1,\ldots,x_n)^t \vee \boxed{\alpha}\ (Qx)P_{\psi_1}(x,x_1,\ldots,x_n)^f) \\ (\boxed{\alpha}\ P_\psi(x_1,\ldots,x_n)^f \vee \boxed{\alpha}\ (Qx)P_{\psi_1}(x,x_1,\ldots,x_n)^t)\end{array} \\
\quad \text{in } \mathcal{M};\ \text{for every } \alpha \in D(S_{n^2}),\ \text{for all subformulae } \psi = (Qx)\psi_1(x,x_1,\ldots,x_n) \\
\quad \text{of } \phi,\ \text{and all instantiations in } D \text{ of the free variables } x_1,\ldots,x_n \text{ of } \psi.
\end{cases}$$

Proof: *(Sketch)* This is a consequence of the fact that a formula ϕ is valid iff there exists no interpretation $\mathcal{M}$ in $\mathcal{O}(D(S_{n^2}))$ such that ϕ is false in $\mathcal{M}$ at some world $\alpha \in D(S_{n^2})$ (and, thus, at some minimal world), taking into account the renamings mentioned before, and the fact that the minimal elements of $D(S_{n^2})$ are $\uparrow(0,1)$ and $\uparrow(1,0)$. □

The elimination rules for the operators and quantifiers use the correspondence between the operations on S_{n^2} and the associated operations on $D(S_{n^2})$ and $\mathcal{O}(D(S_{n^2}))$, as showed below.

Proposition 1. *The following hold for any atomic formulae $L, L_1, L_2, \ldots, L_n$, in any given interpretation $\mathcal{M}$:*

$(\vee,\ t)$	$\boxed{\alpha}\ (L_1 \vee L_2)^t$	*iff*	$\boxed{\alpha}\ L_1^t \vee \boxed{\alpha}\ L_2^t$;
$(\vee,\ f)$	$\boxed{\alpha}\ (L_1 \vee L_2)^f$	*iff*	$\boxed{\alpha}\ L_1^f \wedge \boxed{\alpha}\ L_2^f$.
$(\wedge,\ t)$	$\boxed{\alpha}\ (L_1 \wedge L_2)^t$	*iff*	$\boxed{\alpha}\ L_1^t \wedge \boxed{\alpha}\ L_2^t$;
$(\wedge,\ f)$	$\boxed{\alpha}\ (L_1 \wedge L_2)^f$	*iff*	$\boxed{\alpha}\ L_1^f \vee \boxed{\alpha}\ L_2^f$.
$(S_j,\ t)$	$\boxed{\alpha}\ (S_j(L_1))^t$	*iff*	$\boxed{s_j(\alpha)}\ L_1^t$;
$(S_j,\ f)$	$\boxed{\alpha}\ (S_j(L_1))^f$	*iff*	$\boxed{s_j(\alpha)}\ L_1^f$.
$(\sim,\ t)$	$\boxed{\alpha}\ (\sim(L_1))^t$	*iff*	$\boxed{g(\alpha)}\ L_1^f$;
$(\sim,\ f)$	$\boxed{\alpha}\ (\sim(L_1))^f$	*iff*	$\boxed{g(\alpha)}\ L_1^t$.
$(\neg,\ t)$	$\boxed{\alpha}\ (\neg L)^t$	*iff*	*for every* $\beta \geq \alpha$, $\boxed{\beta}\ L^f$;
$(\neg,\ f)$	$\boxed{\alpha}\ (\neg L)^f$	*iff*	*there exists a* $\beta \geq \alpha$ *with* $\boxed{\beta}\ L^t$.
$(\Rightarrow,\ t)$	$\boxed{\alpha}\ (L_1 \Rightarrow L_2)^t$	*iff*	*for every* $\beta \geq \alpha$, $\boxed{\beta}\ L_1^f \vee \boxed{\beta}\ L_2^t$.
$(\Rightarrow,\ f)$	$\boxed{\alpha}\ (L_1 \Rightarrow L_2)^f$	*iff*	$\boxed{\mathsf{max}\{\beta \mid \beta \geq \alpha\}}\ L_1^t$, $\boxed{\alpha}\ L_2^f$, *and, for every* α_1 *with* $\alpha \leq \alpha_1 < \mathsf{max}(\{\beta \mid \beta \geq \alpha\})$, $\boxed{\alpha_1}\ L_1^t \vee \boxed{s(\alpha_1)}\ L_2^f$.

Proof: *(Sketch)* The cases $(\vee, t)$, $(\vee, f)$, $(\wedge, t)$, $(\wedge, f)$, $(S_j,\ t)$, (S_j, f), $(\sim, t)$, $(\sim, f)$, $(\neg,\ t)$, $(\neg,\ f)$, and $(\Rightarrow, t)$ follow immediately. We present the case $(\Rightarrow, f)$ in detail. We know that $\boxed{\alpha}\ (L_1 \Rightarrow L_2)^f$ iff for some $\beta \geq \alpha$, $\boxed{\beta}\ L_1^t$ and $\boxed{\beta}\ L_2^f$. By distributivity, the formula $\bigvee_{\beta \geq \alpha}(\boxed{\beta}\ L_1^t \wedge \boxed{\beta}\ L_2^f)$ can be written as

$$\bigwedge_{\substack{S_1, S_2,\ S_1 \cap S_2 = \emptyset \\ S_1 \cup S_2 = \{\beta \mid \beta \geq \alpha\}}} \left(\bigvee_{\beta_1 \in S_1} \boxed{\beta_1}\ L_1^t \vee \bigvee_{\beta_2 \in S_2} \boxed{\beta_2}\ L_2^f \right).$$

We then use the fact that, for every $\alpha \in D(S_{n^2})$, the set $\{\beta \mid \beta \geq \alpha\}$ is finite and totally ordered, hence every non-empty set $S_1 \subseteq \{\beta \mid \beta \geq \alpha\}$, contains a maximal element $\mathsf{max}(S_1)$ and every non-empty $S_2 \subseteq \{\beta \mid \beta \geq \alpha\}$ contains a minimal element $\mathsf{min}(S_2)$, and $\bigvee_{\beta_1 \in S_1} \boxed{\beta_1}\ L_1^t$ iff $\boxed{\mathsf{max}(S_1)}\ L_1^t$, resp. $\bigvee_{\beta_2 \in S_2} \boxed{\beta_2}\ L_2^f$ iff $\boxed{\mathsf{min}(S_2)}\ L_2^f$. Additionally, if $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = \{\beta \mid \beta \geq \alpha\}$ it follows that $\mathsf{max}(S_1) \neq \mathsf{min}(S_2)$, and for every $\gamma \geq \alpha$ either $\gamma \in S_1$ (hence $\gamma \leq \mathsf{max}(S_1)$) or $\gamma \in S_2$ (hence $\gamma \geq \mathsf{min}(S_2)$). Let $m = \mathsf{max}\{\beta \mid \beta \geq \alpha\}$.

If $\mathsf{max}(S_1) \neq m$ then it has an immediate successor which will be denoted by $s(\mathsf{max}(S_1))$. Since $\gamma = s(\mathsf{max}(S_1)) \notin S_1$ it follows that it belongs to S_2. Hence, if $\mathsf{max}(S_1) \neq m$ then $\mathsf{min}(S_2) \leq s(\mathsf{max}(S_1))$. Therefore, the following hold:

$$\boxed{\alpha}\,(L_1 \Rightarrow L_2)^f \text{ iff } \bigwedge_{\substack{S_1,S_2,S_1\cap S_2=\emptyset \\ S_1\cup S_2=\{\beta\mid\beta\geq\alpha\}}} \left(\bigvee_{\beta_1\in S_1}\boxed{\beta_1}\,L_1^t \vee \bigvee_{\beta_2\in S_2}\boxed{\beta_2}\,L_2^f\right)$$

$$\text{iff } \boxed{m}\,L_1^t \wedge \boxed{\alpha}\,L_2^f \wedge \bigwedge_{\substack{S_1,S_2\neq\emptyset,S_1\cap S_2=\emptyset \\ S_1\cup S_2=\{\beta\mid\beta\geq\alpha\}}} \left(\boxed{\mathsf{max}(S_1)}\,L_1^t \vee \boxed{\mathsf{min}(S_2)}\,L_2^f\right)$$

$$\text{iff } \boxed{m}\,L_1^t \wedge \boxed{\alpha}\,L_2^f \wedge \bigwedge_{\beta\geq\alpha}\bigwedge_{\substack{S_1,\mathsf{max}(S_1)=\beta, \\ S_2=\uparrow\alpha\setminus S_1}} \left(\boxed{\beta}\,L_1^t \vee \boxed{\mathsf{min}(S_2)}\,L_2^f\right)$$

$$\text{iff } \boxed{m}\,L_1^t \wedge \boxed{\alpha}\,L_2^f \wedge \bigwedge_{\substack{S_1,\mathsf{max}(S_1)=m \\ S_2=\uparrow\alpha\setminus S_1}} \left(\boxed{m}\,L_1^t \vee \boxed{\mathsf{min}(S_2)}\,L_2^f\right) \wedge$$
$$\bigwedge_{\alpha\leq\beta<m}\bigwedge_{\substack{S_1,\mathsf{max}(S_1)=\beta, \\ S_2=\uparrow\alpha\setminus S_1}} \left(\boxed{\beta}\,L_1^t \vee \boxed{\mathsf{min}(S_2)}\,L_2^f\right)$$

$$\text{iff } \boxed{m}\,L_1^t \wedge \boxed{\alpha}\,L_2^f \wedge \bigwedge_{\alpha\leq\beta<m}\bigwedge_{\beta'\neq\beta,\beta'\leq s(\beta)} \left(\boxed{\beta}\,L_1^t \vee \boxed{\beta'}\,L_2^f\right)$$

$$\text{iff } \boxed{m}\,L_1^t \wedge \boxed{\alpha}\,L_2^f \wedge \bigwedge_{\alpha\leq\beta<m} \left(\boxed{\beta}\,L_1^t \vee \boxed{s(\beta)}\,L_2^f\right).$$

□

For quantifiers we have the following results (for related results cf. [Häh98]).

Lemma 1. *Let $\mathcal{I} = (D, I, d)$ be an interpretation of $\mathcal{L}$ in S_{n^2}, $v_{\mathcal{I}}$ the induced valuation, and let $a \in S_{n^2}$ be a join-irreducible element. The following hold:*

- $(\forall_1)$ $v_{\mathcal{I}}(\forall x\phi(x, x_1, \ldots, x_n)) \geq a$ *iff* $v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) \geq a$ *for every instantiation $d \in D$ of x.*
- $(\forall_2)$ $v_{\mathcal{I}}(\forall x\phi(x, x_1, \ldots, x_n)) \not\geq a$ *iff* $v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) \not\geq a$ *for some instantiation $d \in D$ of x, depending on $d(x_1), \ldots, d(x_n)$.*
- $(\exists_1)$ $v_{\mathcal{I}}(\exists x\phi(x, x_1, \ldots, x_n)) \geq a$ *iff* $v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) \geq a$ *for some instantiation $d \in D$ of x, depending on $d(x_1), \ldots, d(x_n)$.*
- $(\exists_2)$ $v_{\mathcal{I}}(\exists x\phi(x, x_1, \ldots, x_n)) \not\geq a$ *iff* $v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) \not\geq a$ *for every instantiation $d \in D$ of x.*

Proof: $(\forall_1)$ and $(\forall_2)$ follow immediately, taking into account that

$$v_{\mathcal{I}}(\forall x\phi(x, x_1, \ldots, x_n)) = \bigwedge\{w \mid \exists d \in D \; s.t. \; v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) = w\}.$$

$(\exists_1)$ For the direct implication, assume that $v_{\mathcal{I}}(\exists x\phi(x)) = \bigvee\{w \mid \exists d \in D \; s.t. \; v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) = w\} \geq a$. From the fact that S_{n^2} is distributive and for every ϕ the set $\{w \mid \exists d \in D \; s.t. \; v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) = w\}$ is finite, it follows that $a = \bigvee\{w \mid \exists d \in D \; s.t. \; v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) = w\} \wedge a = \bigvee\{w \wedge a \mid \exists d \in D \; s.t. \; v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) = w\}$. By the fact that a is join-irreducible it then follows that $a = w \wedge a$ for some w such that there exists $d \in D$ with $v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) = w$, hence, $v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) \geq a$ for some $d \in D$ (depending on $d(x_1), \ldots, d(x_n)$). The converse follows immediately since

if $v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) \geq a$ for some $d \in D$, then also $v_{\mathcal{I}}(\exists x \phi(x)) = \bigvee\{w \mid \exists d \in D \; s.t. \; v_{\mathcal{I}_{x/d}}(\phi(x, x_1, \ldots, x_n)) = w\} \geq a$.

$(\exists_2)$ follows immediately from $(\exists_1)$. □

Proposition 2. *Let L be a literal and let $\mathcal{M} = (D, M, d)$ be an interpretation of $\mathcal{L}$ in $\mathcal{O}(D(S_{n^2}))$. The following hold:*

- $(\forall, t)$ $\boxed{\alpha}\,(\forall x L(x, x_1, \ldots, x_n))^t$ *in* $\mathcal{M}$ *iff* $\boxed{\alpha}\,(L(x, x_1, \ldots, x_n))^t$ *in* $\mathcal{M}_{x/d}$ *for every instantiation* $d \in D$ *of* x.
- $(\forall, f)$ $\boxed{\alpha}\,(\forall x L(x, x_1, \ldots, x_n))^f$ *in* $\mathcal{M}$ *iff* $\boxed{\alpha}\,(L(f(x_1, \ldots, x_n), x_1, \ldots, x_n))^f$ *in* $\mathcal{M}$ *(where f is a new function symbol).*
- $(\exists, t)$ $\boxed{\alpha}\,(\exists x L(x, x_1, \ldots, x_n))^t$ *in* $\mathcal{M}$ *iff* $\boxed{\alpha}\,(L(f(x_1, \ldots, x_n), x_1, \ldots, x_n))^t$ *in* $\mathcal{M}$ *(where f is a new function symbol).*
- $(\exists, f)$ $\boxed{\alpha}\,(\exists x L(x, x_1, \ldots, x_n))^f$ *in* $\mathcal{M}$ *iff* $\boxed{\alpha}\,(L(x, x_1, \ldots, x_n))^f$ *in* $\mathcal{M}_{x/d}$ *for every instantiation* $d \in D$ *of* x.

Proof: We know that all the elements $\alpha \in D(S_{n^2})$ are of the form $\uparrow a$, where a is a join-irreducible element in S_{n^2}. For every $\mathcal{M} = (D, M, d)$, let $v_{\mathcal{M}} : \mathsf{Fma}(\mathcal{L}) \to \mathcal{O}(D(S_{n^2}))$ be the valuation function induced by $\mathcal{M}$. Let $\mathcal{I} = (D, I, d)$ be the associated interpretation in S_{n^2} (obtained by composition with $\eta^{-1}_{S_{n^2}}$ when necessary). For every $\psi \in \mathsf{Fma}(\mathcal{L})$ we have $v_{\mathcal{M}}(\psi) = \eta_{S_{n^2}}(v_{\mathcal{I}}(\psi)) = \{\uparrow a \mid v_{\mathcal{I}}(\psi) \in \uparrow a\} = \{\uparrow a \mid v_{\mathcal{I}}(\psi) \geq a\}$. Thus, $\boxed{\alpha}\,\psi^t$ in $\mathcal{M}$ iff $\alpha = \uparrow a \in v_{\mathcal{M}}(\psi)$ iff $v_{\mathcal{I}}(\psi) \geq a$. Therefore, using Lemma 1 we have:

- $(\forall, t)$ $\boxed{\alpha}\,(\forall x L(x, x_1, \ldots, x_n))^t$ in $\mathcal{M}$ iff $v_{\mathcal{I}}(\forall x L(x, x_1, \ldots, x_n)) \geq a$, which, by Lemma 1 happens iff $v_{\mathcal{I}_{x/d}}(L(x, x_1, \ldots, x_n)) \geq a$ for every instantiation d of x. Thus, $\boxed{\alpha}\,(\forall x L(x, x_1, \ldots, x_n))^t$ in $\mathcal{M}$ iff for every instantiation d of x, $\boxed{\alpha}\,(L(x, x_1, \ldots, x_n))^t$ in $\mathcal{M}_{x/d}$.
- $(\forall, f)$ $\boxed{\alpha}\,(\forall x L(x, x_1, \ldots, x_n))^f$ in $\mathcal{M}$ iff $\alpha \notin v_{\mathcal{M}}(\forall x L(x, x_1, \ldots, x_n))$. By Lemma 1 this happens iff $v_{\mathcal{I}_{x/d}}(L(x, x_1, \ldots, x_n)) \not\geq a$ for some instantiation d of a, depending on the values of the other free variables in L.

$(\exists, t)$, $(\exists, f)$ follow analogously. □

After this translation, from any formula ϕ we obtain a formula Φ in clause form, containing literals of the form $\boxed{\alpha}\,L^t$ and $\boxed{\alpha}\,L^f$ where L is an atom and $\alpha \in D(S_{n^2})$.

Theorem 3. *ϕ is a theorem iff Φ is unsatisfiable.*

Proof: Follows from Theorem 2, Proposition 1 and Proposition 2. □

Proposition 3. *The number of clauses generated from a given formula ϕ is in $\mathcal{O}(n^2 l)$, where l is the number of subformulae of ϕ, and n^2 the number of truth values. If the formula ϕ does not contain the connectives $\Rightarrow$ and $\neg$, then the number of clauses generated from ϕ is in $\mathcal{O}(nl)$.*

Proof: *(Sketch)* It is easy to see that for every $\alpha \in D(A)$ the split degree of the rules in Proposition 1 and Proposition 2 is 3 in case of $\vee$ and $\wedge$; 2 for $S_1, \ldots, S_{n-1}$, $\sim$, and $\forall, \exists$; and in $\mathcal{O}(n)$ for $\neg$ and $\Rightarrow$. The maximal number of clauses is generated by the subformulae of the form $\psi = \psi_1 \Rightarrow \psi_2$. In this case, for every $\alpha \in D(S_{n^2})$ the number of clauses generated by $(\boxed{\alpha}\ P^f_\psi \vee \boxed{\alpha}\ \psi^t)$ and $(\boxed{\alpha}\ P^t_\psi \vee \boxed{\alpha}\ \psi^f)$ is bounded by $2|\{\beta \mid \beta \geq \alpha\}| + 1$. Thus, the number of clauses generated from a formula ϕ has as upper bound $1 + 2 \cdot l \cdot \Sigma_{i=1}^{n-1}(2i+1) = 1 + 2 \cdot l \cdot (n^2 - 1)$, hence is (in the worst case) in $\mathcal{O}(n^2 l)$. The estimations for formulae not containing $\Rightarrow$ or $\neg$ follow from the fact that the branching factor is constant for $\sim$, $S_1, \ldots, S_{n-1}$, $\vee, \wedge$, $\exists, \forall$. □

4.1 Further improvements

The structure of $D(S_{n^2})$ can be used in order to further reduce the number of clauses[1]. For this, we use the fact that $D(S_{n^2})$ consists of two branches and that the transformation rules for the operations in $\{\vee, \wedge, \neg, \Rightarrow\}$ preserve the branch of $D(S_{n^2})$. For formulae that do not contain the De Morgan negation $\sim$, all the clauses generated by the renaming of subformulae only contain signs in one of the branches of $D(S_{n^2})$. It is sufficient to give a refutation for the clauses corresponding to one of the branches of $D(S_{n^2})$. For the other branch a similar refutation can be constructed by simply renaming the nodes, and they can be then combined to a refutation by resolution for the clause form of ϕ.

The set of signed clauses that would be generated in this case corresponds to the following conjunction of formulae:

$$\begin{cases} \boxed{\uparrow(1,0)}\ P^f_\phi \text{ in } \mathcal{M}, \\ \\ (\boxed{\uparrow(i,0)}\ P_\psi(x_1,\ldots,x_n)^t \vee \boxed{\uparrow(i,0)}\ \sigma(P_{\psi_1},\ldots,P_{\psi_m})^f) \\ (\boxed{\uparrow(i,0)}\ P_\psi(x_1,\ldots,x_n)^f \vee \boxed{\uparrow(i,0)}\ \sigma(P_{\psi_1},\ldots,P_{\psi_m})^t) \\ \text{in } \mathcal{M};\ \text{for every } i = 1,\ldots,n-1,\ \text{for all subformulae } \psi = \sigma(\psi_1,\ldots,\psi_m) \\ \text{of } \phi \text{ and all instantiations in } D \text{ of the free variables } x_1,\ldots,x_n \text{ of } \psi, \\ \\ (\boxed{\uparrow(i,0)}\ P_\psi(x_1,\ldots,x_n)^t \vee \boxed{\uparrow(i,0)}\ (Qx)P_{\psi_1}(x,x_1,\ldots,x_n)^f) \\ (\boxed{\uparrow(i,0)}\ P_\psi(x_1,\ldots,x_n)^f \vee \boxed{\uparrow(i,0)}\ (Qx)P_{\psi_1}(x,x_1,\ldots,x_n)^t) \\ \text{in } \mathcal{M};\ \text{for every } i = 1,\ldots,n-1,\ \text{for all subformulae } \psi = (Qx)\psi_1(x,x_1,\ldots,x_n) \\ \text{of } \phi \text{ and all instantiations in } D \text{ of the free variables } x_1,\ldots,x_n \text{ of } \psi. \end{cases}$$

[1] We thank Luisa Iturrioz who suggested that it may be possible to further improve the efficiency of automated theorem proving for SH_n-logics by exploiting the structure of $D(S_{n^2})$; the remarks below show that this is possible; a more thorough research, also for more general logics, is planned for future work.

5 Automated Theorem Proving

In [SS98], where we focus on theorem proving for sets of signed clauses, we note that the translation to clause form described above is a translation to (many-sorted) classical logic ($\boxed{\alpha}\, L^t$ stands for $\mathsf{holds}(L, \alpha)$, and $\boxed{\alpha}\, L^f$ for $\neg\mathsf{holds}(L, \alpha)$). This also explains the fact that the polarity of formulae can be used to further reduce the number of clauses generated in the translation to clause form.

Therefore, checking the satisfiability of the sets of signed clauses obtained after the clause form transformation described in Section 4 does not present a problem. For instance, as shown in [SS98], the validity of ϕ can be checked by applying classical resolution to the set Φ^c of classical clauses obtained from the signed CNF of ϕ by replacing $\boxed{\alpha}\, L^t$ with $\mathsf{holds}(L, \alpha)$, and $\boxed{\alpha}\, L^f$ with $\neg\mathsf{holds}(L, \alpha)$, to which the set of clauses $\mathsf{Her} = \{\{\neg\mathsf{holds}(x, \alpha), \mathsf{holds}(x, \beta)\} \mid \alpha \leq \beta\}$, expressing the heredity of truth, is adjoined.

Alternatively, versions of *signed resolution* can be used, which generalize existing signed resolution rules for regular logics [Häh96]. We illustrate these ideas by presenting a version of negative hyperresolution for signed clauses, inspired by the method for regular clauses in [Häh96].

Negative Hyperresolution

$$\frac{\left\{\boxed{x_1}\, p_1^f\right\} \cup D_1, \ldots, \left\{\boxed{x_n}\, p_n^f\right\} \cup D_n, \left\{\boxed{y_1}\, p_1^t, \ldots, \boxed{y_n}\, p_n^t\right\} \cup E}{D_1 \cup \cdots \cup D_n \cup E}$$

provided that $n \geq 1$, $y_i \leq x_i$ for all $i = 1, \ldots, n$ and $D_1, \ldots, D_n, E$ are negative.

Proposition 4. *A set Φ of ground signed clauses is unsatisfiable if and only if $\Box$ can be derived from Φ by a finite number of applications of many-valued negative hyperresolution.*

Proof: *(Sketch)* It is easy to see that if $\Box$ can be derived from Φ by a finite number of applications of many-valued negative hyperresolution then Φ is unsatisfiable (this follows from the fact that if $\mathcal{M}$ is a model for the negative clauses $C_1, \ldots C_n$ and for the positive clause C then $\mathcal{M}$ is a model of any of their resolvents). The proof of completeness closely follows the proof in [Häh96] or [AB70]. Let $nl(\Phi)$ be the total number of literals in Φ and $nc(\Phi)$ the total number of clauses in Φ. It is obvious that $nl(\Phi) \geq nc(\Phi)$. We will proceed by induction on the difference $k(\Phi) = nl(\Phi) - nc(\Phi)$. If $\Box \in \Phi$ then the conclusion is obvious. Therefore in what follows we will assume that $\Box \notin \Phi$.

Case 1. $k(\Phi) = 0$: In this case $nl(\Phi) = nc(\Phi)$. Since $\Box \notin \Phi$, Φ must consist only of unit clauses. Since Φ is unsatisfiable, there must exist two clauses $\{\boxed{\alpha}\, p^t\}$ and $\{\boxed{\beta}\, p^f\}$ in Φ such that $\alpha \leq \beta$. Then $\Box$ can be derived from the clauses $\{\boxed{\alpha}\, p^t\}$ and $\{\boxed{\beta}\, p^f\}$ where $\alpha \leq \beta$ by negative hyperresolution.

Case 2. $k(\Phi) > 0$ (i.e. $nl(\Phi) > nc(\Phi)$).

Subcase 2a: Assume that all non-positive clauses consist of one literal. Then it follows that all negative literals in Φ appear in unit clauses. Since Φ is unsatisfiable, there is a positive clause in Φ which immediately produces the empty clause with suitable negative unit clauses.
Subcase 2b: There is a non-positive non-unit clause $C = \{ \boxed{\beta}\, p^f \} \cup D \in \Phi$ with $D \neq \Box$. Let $\Phi' = \Phi \backslash \{C\}$, $\Phi_1 = \Phi' \cup D$, and $\Phi_2 = \Phi' \cup \{\{ \boxed{\beta}\, p^f \}\}$. Since Φ is unsatisfiable it follows that both Φ_1, Φ_2 are unsatisfiable. Moreover, Φ_1 and Φ_2 contain the same number of clauses as Φ, but they have fewer literals. Hence, $k(\Phi_1) < k(\Phi)$ and $k(\Phi_2) < k(\Phi)$. Since Φ_1 and Φ_2 are both unsatisfiable, by the induction hypothesis, $\Box$ can be deduced from Φ_1 and $\Box$ can be deduced from Φ_2 by hyperresolution. Consider a hyperresolution deduction of $\Box$ from Φ_1. If we replace each occurrence of D by C in this proof then we obtain a valid hyperresolution deduction with last clause $\Box$ or $\{ \boxed{\beta}\, p^f \}$. If the last clause is $\Box$, we have already a hyperresolution deduction of $\Box$ from Φ. If the last clause is $\{ \boxed{\beta}\, p^f \}$, then its deduction from Φ can be extended to a deduction of $\Box$ from Φ using the fact that there is a deduction of $\Box$ from Φ_2. $\Box$

The completeness theorem (for the non-ground case) uses an extension of the lifting lemma to signed literals (for details cf. e.g. [SS97]).

It may be that translation to classical logic combined with the use of classical resolution is simpler and more satisfactory for automated theorem proving in this kind of logics than devising specialized signed resolution schemes. In [GSS99] the use of techniques from classical logic such as superposition and ordered chaining in automated theorem proving in many-valued logics is studied.

6 Comparison with other methods

We now briefly compare the method for translation to clause form described here with other existing methods. Consider first the very general structure-preserving methods for translation to clause form that use the algebra of truth values given in [BF95]. The split degree of rules of the type $(\Box, v)^+$ induced by a k-ary operator $\Box$, given a truth value v, is at most $|W|^{k-1}$, if W is the set of truth values. Since in a structure-preserving translation to clause form such rules have to be considered for every truth value v, the clause form of a formula with n occurrences of at most r-ary operators and m occurrences of quantifiers contains no more than $n|W|^r + m2^{|W|} + 1$ clauses if optimal rules are used (cf. [BF95]). In the presence of operators that are at most binary, and if the only quantifiers are $\forall$ and $\exists$, the number of clauses generated this way is in the worst case quadratic in the number of truth values, hence, for SH_n-logics it would generate $\mathcal{O}(ln^4)$ clauses for formulae containing at least one binary operator. (This upper bound is actually reached if the very general rules in [BF95] are used as presented there.) Additional improvements may be achieved by special minimization techniques, as done for instance in [Sal96]. The method of Salzer starts with an arbitrary conjunctive normal form (e.g. constructed from the truth table, as in

[BF95]), saturates it under sets-as-signs resolution, and selects a minimal subset of the saturated set of clauses by any covering-set-algorithm. The results in [Sal96] show that the conjunctive normal forms obtained this way are optimal. However, the method does not bring much information about the general form of the conjunctive normal forms, or on the kind of signs actually needed in such conjunctive normal forms for whole classes of logics.

One important advantage of the method we presented here is that only elements of the Priestley dual of the set of truth values are used as signs, thus fewer signs have to be taken into account in the process of translation to clause form. Our method is especially efficient in situations when the difference between the number of elements of the algebra of truth values and the number of elements of its Priestley dual is large. In the case of SH_n-logics the number of elements of the algebra of truth values S_{n^2} is n^2, whereas the number of elements of $D(S_{n^2})$ is $2(n-1)$. Even for linearly-ordered sets of truth values the set of clauses introduced by the Heyting negation (determined by a technique similar to that used in Proposition 1) is the same as that obtained with Salzer's algorithm [Sal96], i.e. linear in the number of truth values.

Moreover, the signed formulae we use, namely $\boxed{\alpha}\ \phi^t$ and $\boxed{\alpha}\ \phi^f$, are very similar to the "positive and negative regular formulae" of the form $\boxed{\geq i}\ \phi$ resp. $\boxed{\leq i}\ \phi$ introduced in [Häh96] for regular logics. The only difference is that in [Häh96] totally ordered sets are considered, whereas we consider duals of finite distributive lattices[2]. In the particular case of totally ordered lattices of truth values, Hähnle's notions of positive and negative literal are recovered: $\boxed{\geq i}\ \phi$ corresponds to $\boxed{\alpha}\ \phi^t$ and $\boxed{\leq i-1}\ \phi$ to $\boxed{\alpha}\ \phi^f$, where $\alpha = \uparrow i$, which justifies the terminology "literal with positive (negative) polarity" in [Häh96].

7 Conclusions

In this paper we illustrated by one example, namely SH_n-logics, an efficient transformation procedure to a signed clause form, and a refutation procedure based on negative hyperresolution for many-valued logics based on distributive lattices with operators. This method extends in a natural way the work of Hähnle [Häh96,Häh98] for regular logics, and, we think, will help in making automated theorem proving in logics based on distributive lattices with operators more efficient. A detailed presentation of these ideas for SH_n-logics, an extension to a more general framework, and the description of a Prolog implementation can be found in [SS97]. The detailed presentation of a general method based on these ideas is the subject of an extended paper, [SS99a].

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[2] For the linearly-ordered case, the negation of "$\boxed{\geq i}\ \phi$ is true" is "$\boxed{\leq i-1}\ \phi$ is true", in cases when the set of truth values is not linearly ordered this does not necessarily hold.

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References

[AB70] R. Anderson and W. Bledsoe. A linear format for resolution with merging and a new technique for establishing completeness. *Journal of the ACM*, 17:525–534, 1970.

[BF95] M. Baaz and C.G. Fermüller. Resolution-based theorem proving for many-valued logics. *Journal of Symbolic Computation*, 19:353–391, 1995.

[DP90] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 1990.

[GSS99] H. Ganzinger and V. Sofronie-Stokkermans. Chaining calculi and resolution-based theorem proving in many-valued logic. In preparation, 1999.

[Häh94] R. Hähnle. Short conjunctive normal forms in finitely valued logics. *Journal of Logic and Computation*, 4(6):905–927, 1994.

[Häh96] R. Hähnle. Exploiting data dependencies in many-valued logics. *Journal of Applied Non-Classical Logics*, 6(1):49–69, 1996.

[Häh98] R. Hähnle. Commodious axiomatization of quantifiers in multiple-valued logic. *Studia Logica*, 61(1):101–121, 1998.

[IO96] L. Iturrioz and E. Orłowska. A Kripke-style and relational semantics for logics based on Łukasiewicz algebras. Conference in honour of J. Łukasiewicz, Dublin, 1996.

[Itu82] L. Iturrioz. Modal operators on symmetrical Heyting algebras. In T. Traczyk, editor, *Universal Algebra and Applications*, Banach Center Publications, Vol. 9, pages 289–303. PWN-Polish Scientific Publishers, 1982.

[Itu83] L. Iturrioz. Symmetrical Heyting algebras with operators. *Zeitschrift f. math. Logik und Grundlagen d. Mathematik*, 29:33–70, 1983.

[Sal96] G. Salzer. Optimal axiomatizations for multiple-valued operators and quantifiers based on semilattices. In M. McRobbie and J. Slaney, editors, *Proc. 13th Conference on Automated Deduction, New Brunswick/NJ, USA*, LNCS 1104, pages 688–702. Springer Verlag, 1996.

[Sco73] D.S. Scott. Background to formalization. In H. Leblanc, editor, *Truth, Syntax and Modality. Proceedings of the Temple University Conference on Alternative Semantics*, Studies in Logic and the Foundations of Mathematics, Vol. 68, pages 244–273, 1973.

[SS97] V. Sofronie-Stokkermans. *Fibered Structures and Applications to Automated Theorem Proving in Certain Classes of Finitely-Valued Logics and to Modeling Interacting Systems.* PhD thesis, RISC-Linz, J.Kepler University Linz, Austria, 1997.

[SS98] V. Sofronie-Stokkermans. On translation of finitely-valued logics to classical first-order logic. In H. Prade, editor, *Proceedings of ECAI 98*, pages 410–411. John Wiley & Sons, 1998.

[SS99a] V. Sofronie-Stokkermans. Automated theorem proving by resolution for finitely-valued logics based on distributive lattices with operators. Submitted, 1999.

[SS99b] V. Sofronie-Stokkermans. Priestley duality for SHn-algebras and applications to the study of Kripke-style models for SHn-logics. Multiple-Valued Logic – An International Journal, To appear, 1999.