Automated theorem proving by resolution in non-classical logics

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Abstract

We present several classes of non-classical logics (many of which are practically relevant in knowledge representation) which can be translated into tractable and relatively simple fragments of classical logic. In this context, refinements of resolution can be often used successfully for automated theorem proving, and in many cases yield optimal decision procedures.

1 Introduction

The main goal of this paper is to present several situations in which non-classical logics can be translated into tractable and simple fragments of classical logic, and resolution can be used successfully for automated theorem proving.

The paper starts with a presentation of various non-classical logics, ranging from many-valued logics to description logics. It is known that checking validity of formulae in non-classical logics having an algebraic semantics often can be reduced to checking whether corresponding word problems hold in the class of algebraic models. We show that similar phenomena occur also in several description logics: checking subsumption with respect to TBoxes can be often reduced to checking whether suitably defined uniform word problems hold in classes of Boolean algebras, distributive lattices or semilattices with operators.

We then focus on methods based on translations to classical logic, which allow the use of (refinements of) resolution for automated theorem proving in various non-classical logics. We first show that various versions of many-valued resolution for finitely-valued logics can be reconstructed by using general saturation-based techniques for first-order theories of transitive relations [GSS00]. We then consider other non-classical logics which are not finitely valued, but for which nevertheless such embeddings into classical logics are possible. We present, for instance, a translation to clause form for prenex first-order Gödel logics [BFC01] which allow to use saturation-based techniques for dense total orderings, and then focus on propositional logics based on distributive lattices with operators (possibly many-sorted). We show that resolution-based decision procedures with optimal complexity can be obtained in many cases by using refinements of resolution such as ordered resolution with selection, or ordered chaining with selection.

2 Preliminaries

For basic notions of universal algebra we refer e.g. to [BS81]. We also assume known standard notions, such as partially-ordered set, (bounded) lattice, and distributive lattice, as well as (prime) filters in lattices. For definitions and more details we refer to [DP90].

Let Σ be a signature and $a : \Sigma \to \mathbb{N}$ an arity function. A Σ-algebra is a structure $\mathbf{A} = (A, \{\sigma_A\}_{\sigma \in \Sigma})$, where A is a non-empty set and for every $\sigma \in \Sigma$, $\sigma_A : A^{a(\sigma)} \to A$. Given a set X, the term algebra over Σ in the variables X will be denoted $T_\Sigma(X)$. An *equation* is an expression of the form $t_1 = t_2$ where $t_1, t_2 \in T_\Sigma(X)$; an *implication* is an expression of the form $\beta_1 \wedge \cdots \wedge \beta_m \to \alpha$, where $\beta_1, \ldots, \beta_m, \alpha$ are equations. A *conditional equation* (or *quasi-equation*) is an expression which is either an equation or an implication. A Σ-algebra $\mathbf{A} = (A, \{\sigma_A\}_{\sigma \in \Sigma})$ *satisfies* an equation $t_1 = t_2$ if t_1 and t_2 become equal for every substitution of elements in A for the variables. $\mathbf{A}$ satisfies an implication $\gamma = (t_1 = t'_1 \wedge \cdots \wedge t_m = t'_m) \to t = t'$ (notation: $\mathbf{A} \models \gamma$) if for every substitution v of elements in A for the variables in γ such that $v(t_i) = v(t'_i)$ for all $i = 1, \ldots, m$, $v(t) = v(t')$.

An equational class is the class of all algebras that satisfy a set of equations. A quasi-variety is the class of all algebras that satisfy a class of quasi-equations.

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Bounded lattices with additional operators occur often as algebraic models of non-classical logics. The additional operations are usually interpretations of logical connectives such as the modal connectives for necessity ($\Box$) or possibility ($\Diamond$), or various types of negation ($\sim$) or implication ($\rightarrow$). The operators that correspond to these connectives often commute with part of the lattice structure, i.e. satisfy equations such as, for instance:

$$\Box(1) = 1, \qquad \Box(x \wedge y) = \Box(x) \wedge \Box(y), \tag{1}$$
$$\Diamond(0) = 0, \qquad \Diamond(x \vee y) = \Diamond(x) \vee \Diamond(y), \tag{2}$$
$$\sim 0 = 1, \qquad \sim(x \vee y) = \sim x \wedge \sim y, \tag{3}$$
$$\sim 1 = 0, \qquad \sim(x \wedge y) = \sim x \vee \sim y, \tag{4}$$
$$(0 \Rightarrow z) = 1, \qquad ((x \vee y) \Rightarrow z) = (x \Rightarrow z) \wedge (y \Rightarrow z), \tag{5}$$
$$(x \Rightarrow 1) = 1, \qquad (x \Rightarrow (y \wedge z)) = (x \Rightarrow y) \wedge (x \Rightarrow z). \tag{6}$$

We want to make the class of algebras we consider broad enough to encompass operations which satisfy equations such as (1)–(6), but also operations between different lattices, such as Galois connections, i.e. pairs (f, g) of maps $f : L_1 \rightarrow L_2$, $g : L_2 \rightarrow L_1$, with the property that

$$f(0) = 0, f(x \vee y) = f(x) \vee f(y), \qquad g(1) = 1, g(x \wedge y) = g(x) \wedge g(y), \tag{7}$$
$$f(x) \leq y \text{ iff } x \leq g(y) \qquad \text{for all } x \in L_1, y \in L_2. \tag{8}$$

Therefore, we consider classes of many-sorted algebraic structures, with many-sorted operations. We now formally define operators that have properties such as (1)–(8) above.

Definition 2.1 *Let S be a set of sorts, $\{\mathbf{L}_s\}_{s \in S}$ be an S-sorted family of bounded lattices $\mathbf{L}_s = (L_s, \vee, \wedge, 0, 1)$ and let $s_1, \dots, s_n, s \in S$. A join hemimorphism of type $s_1 \dots s_n \rightarrow s$ is a function $f : L_{s_1} \times \dots \times L_{s_n} \rightarrow L_s$ such that for every $i, 1 \leq i \leq n$,*

(1) $f(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) = 0$,
(2) $f(a_1, \dots, a_{i-1}, b_1 \vee b_2, a_{i+1}, \dots, a_n) =$
$= f(a_1, \dots, a_{i-1}, b_1, a_{i+1}, \dots, a_n) \vee f(a_1, \dots, a_{i-1}, b_2, a_{i+1}, \dots, a_n)$.

We say that a map $f : L_{s_1} \times \dots \times L_{s_n} \rightarrow L_s$ is a join hemimorphism of type $s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n} \rightarrow s^{\varepsilon}$, where $\varepsilon_1, \dots, \varepsilon_n, \varepsilon \in \{-1, +1\}$, if $f : L_{s_1}^{\varepsilon_1} \times \dots \times L_{s_n}^{\varepsilon_n} \rightarrow L_s^{\varepsilon}$ is a join hemimorphism, where $\mathbf{L}^{+1} := \mathbf{L}$ and $\mathbf{L}^{-1} := \mathbf{L}^d$, the dual of $\mathbf{L} = (L, \vee, \wedge, 0, 1)$, i.e. the lattice $(L, \vee^d, \wedge^d, 0^d, 1^d)$, where for every $x, y \in L$, $x \vee^d y = x \wedge y$, $x \wedge^d y = x \vee y$; $0^d = 1$; and $1^d = 0$.

Definition 2.2 *Let $\{\mathbf{L}_s\}_{s \in S}$ be an S-sorted family of bounded lattices and let f, g be two n-ary operators such that $f : \mathbf{L}_{s_1}^{\varepsilon_1} \times \dots \times \mathbf{L}_{s_i} \times \dots \times \mathbf{L}_{s_n}^{\varepsilon_n} \rightarrow L_s$ and $g : \mathbf{L}_{s_1}^{\varepsilon_1} \times \dots \times \mathbf{L}_s^d \times \dots \times \mathbf{L}_{s_n}^{\varepsilon_n} \rightarrow L_{s_i}^d$ are join hemimorphisms. We say that g is an i-residuation[1] associated with f if for all $a_1 \in L_{s_1}, \dots, a_n \in L_{s_n}, a \in L_s$:*

$$f(a_1, \dots, a_n) \leq a \textit{ if and only if } a_i \leq g(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_n).$$

Example 2.3

1. *The operator $\Diamond$ on a modal algebra $\mathbf{B}$ is a join hemimorphism. The operator $\Box$ on $\mathbf{B}$ is a join hemimorphism on the dual $\mathbf{B}^d$ of $\mathbf{B}$.*

2. *Let $\mathbf{L}$ be a lattice. The operation $\Rightarrow$ satisfying the conditions (5) and (6) is a join hemimorphism of type $\mathsf{lat}, \mathsf{lat}^d \rightarrow \mathsf{lat}^d$.*

3. *Let $\mathbf{L_1}, \mathbf{L_2}$ be two lattices and let $f : \mathbf{L_1} \rightarrow \mathbf{L_2}$ and $g : \mathbf{L_2} \rightarrow \mathbf{L_1}$ be a Galois connection (i.e. conditions (7) and (8) are satisfied). Let $(\mathbf{L}_1, \mathbf{L}_2)$ be the 2-sorted algebra with sorts $S = \{\mathsf{l}_1, \mathsf{l}_2\}$. Then f is a join hemimorphism of type $l_1 \rightarrow l_2$, g is a join hemimorphism of type $l_2^d \rightarrow l_1^d$, and g is the 1-residuation associated with f.*

4. *Let $(\mathbf{L}, \mathbf{C}_{n+1})$ be the 2-sorted algebra with sorts $S = \{\mathsf{lat}, \mathsf{num}\}$, where L is a bounded lattice, and $C_{n+1} = (\{0, 1, \dots, n\}, \vee, \wedge, 0, n)$ is the $n+1$-element chain. A function $f : \mathbf{L} \rightarrow \mathbf{C}_{n+1}$ that associates with every element of L an element of $\{0, 1, \dots, n\}$ such that $f(x \vee y) = f(x) \vee f(y)$ and $f(0) = 0$ is a join hemimorphism of type $\mathsf{lat} \rightarrow \mathsf{num}$.*

[1] Two i-residuations associated with the same operator coincide.

3 Non-classical logics

3.1 Many-valued logics

Let $\mathbf{L} = (X, O, P, \Sigma, \{Q_1, \dots, Q_k\})$ be a first-order language consisting of a (countably) infinite set X of variables, a set O of function symbols, a set P of predicate symbols, a set Σ of logical operators, and a finite set of (one-place) quantifiers $Q_1, \dots, Q_k$. Terms, ground terms, atomic formulae and formulae are defined in the usual way. Let A be a set of truth values. We associate truth functions with logical operators and quantifiers as follows:

- to every $\sigma \in \Sigma$ with arity n we associate a truth function $\sigma_A : A^n \to A$,
- to every quantifier Q we associate a truth function $\overline{Q} : \mathcal{P}(A) \backslash \{\emptyset\} \to A$.

A many-valued logic with language $\mathbf{L}$ and set of truth values A is a pair $\mathcal{L} = (\mathbf{L}, \mathcal{A})$ consisting of a first-order language $\mathbf{L} = (X,\ O,\ P,\ \Sigma,\ \{Q_1,\ \dots,\ Q_k\})$ and a set of truth values endowed with truth functions for all logical operators and quantifiers in $\mathbf{L}$, $\mathcal{A} = (A, \{\sigma_A\}_{\sigma \in \Sigma}, \{\overline{Q_i}\}_{i=1,\dots,k}))$. Many-valued logics with a finite set of truth values are called finitely valued logics, those with an infinite set of truth values are called infinitely valued logics.

Definition 3.1 (Frame, Interpretation) *A* frame *for $\mathcal{L} = (\mathbf{L}, \mathcal{A})$ is a pair (D, I) where D is a non-empty set, the* domain*, and I is a signature interpretation, i.e. a function assigning a function $I(f) : D^n \to D$ to every n-ary function symbol $f \in O$, and a function $I(R) : D^n \to A$ to every n-ary predicate symbol $R \in P$. An* interpretation *$\mathcal{I}$ for $\mathcal{L}$ (or* interpretation for $\mathbf{L}$ in A*) is a triple (D, I, d) where (D, I) is a frame and d is a* variable assignment $d : X \to D$.

Every interpretation $\mathcal{I} = (D, I, d)$ extends in a canonical way to terms, and induces a valuation function on formulae, $v_{\mathcal{I}} : \mathsf{Fma}(\mathcal{L}) \to A$, as follows:

(1) $v_{\mathcal{I}}(R(t_1, \dots, t_n)) = I(R)(v_{\mathcal{I}}(t_1), \dots, v_{\mathcal{I}}(t_n))$ for all n-ary $R \in P$, $n \geq 0$,

(2) $v_{\mathcal{I}}(\sigma(\phi_1, \dots, \phi_n)) = \sigma_A(v_{\mathcal{I}}(\phi_1), \dots, v_{\mathcal{I}}(\phi_n))$ for all n-ary $\sigma \in \Sigma$,

(3) $v_{\mathcal{I}}((Qx)\phi) = \overline{Q}(\{w \mid \exists d \in D \ s.t. \ v_{\mathcal{I}_{x,d}}(\phi) = w\})$ for all quantifiers Q, where $\mathcal{I}_{x,d}$ is identical to $\mathcal{I}$ except for assigning d to the variable x.

Assume that a subset A_d of A of *designated truth values* for the logic $\mathcal{L}$ is additionally specified. A formula ϕ is valid in a logic $\mathcal{L}$ (with set A_d of designated truth values) if and only if $v_{\mathcal{I}}(\phi) \in A_d$ for all interpretations $\mathcal{I}$ for the language of $\mathcal{L}$ in A. A formula ϕ is *satisfiable* in $\mathcal{L}$ if and only if there is an interpretation $\mathcal{I}$ with $v_{\mathcal{I}}(\phi) \in A_d$.

Examples of finitely valued logics are classical logic (with set of truth values $\{t, f\}$), the Łukasiewicz logics of order n, the Post logics of order n (set of truth values $\{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$). Validity and satisfiability in propositional finitely-valued logics is obviously decidable. (It is easy to see that satisfiability of formulae is in NP and validity is in co-NP.)

Typical examples of infinitely-valued logics are the so-called fuzzy logics. Fuzzy logics are many-valued logics having the interval $[0, 1]$ as set of truth values; premise combination $\circ$ is modeled by t-norms. Every continuous t-norm $\circ$ on $[0, 1]$ has a unique right residuum $\rightarrow$. By choosing *the Gödel t-norm*, $x \circ y = \min(x, y)$; *the Łukasiewicz t-norm*, $x \circ y = \max(0, x + y - 1)$; or *the product t-norm*, $x \circ y = x \cdot y$ (product of reals), we can define the Gödel logic G_∞, the Łukasiewicz logic Ł, or the product logic L_Π, respectively.

For every $n \in \mathbb{N}$, n-valued variants Ł$_n$ and G_n of the propositional Łukasiewicz and Gödel logics, with set of truth values $\{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$, can be defined: premise combination $\circ$ is modeled by the Łukasiewicz t-norm and resp. the Gödel t-norm, and $\rightarrow$ is again the unique right residuum of $\circ$. (Product logic is only defined for the set of truth values $[0, 1]$, since $\{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$ is not closed under product.)

First order versions of the above-mentioned fuzzy logics can be obtained by defining the truth functions for quantifiers $Q_\forall = \mathsf{inf}$ and $Q_\exists = \mathsf{sup}$. Obviously, first order many-valued logics are in general undecidable (since first order classical logic is undecidable). While the complexity of satisfiability and validity in propositional Gödel, Łukasiewicz, and product logic is the same as for two-valued logic, the situation is different in the first-order case. The following results are well-known (for proofs we refer e.g. to [Mun87], [Háj98] and [Häh03]):

(1) Satisfiability is NP-complete and validity is co-NP-complete for the propositional Łukasiewicz logics $Ł_n$ and the propositional Gödel logics G_n; for the propositional Łukasiewicz logic Ł; for the propositional Gödel logic G_∞; and for the propositional product logic L_Π.

(2) Validity in the first-order Gödel logic is Σ_1-complete, validity in the first-order Łukasiewicz logic is Π_2-complete, and validity in the first-order product logic is Π_2-hard.

3.2 Propositional non-classical logics

Many-valued logics are special logics, characterized by one given algebra of truth values, with a relatively simple structure. However, many non-classical logics are usually defined by describing the properties of premise combination and entailment by means of logical calculi (e.g. Gentzen-style calculi, Hilbert-style calculi, natural deduction systems). Logics defined this way usually have a natural algebraic model, namely their Lindenbaum algebra, which can be constructed by identifying provably equivalent formulae. The equivalence classes of the theorems can be regarded as *designated elements.* Thus, most of the known propositional non-classical logics can be regarded as many-valued logics with an infinite algebra of truth values, and a suitably defined set of designated elements. In many cases it is more convenient to identify *classes* of algebraic models for these logics. Below we briefly present some non-classical logics which have as algebraic models lattices or semilattices with additional operators (in particular: Boolean algebras with operators, Heyting algebras with operators, or (distributive) lattices or semilattices with operators). We briefly mention well-studied logics, such as modal logics, intuitionistic logic, fuzzy logics, relevant logics and other substructural logics such as BCC and related logics. We then present in some detail some newer results related to TBox reasoning in description logics.

3.2.1 Logics based on classes of distributive lattices with operators

Most of the well-studied non-classical logics fall into this class. We mention some well known examples: *Modal logics* are in general sound and complete with respect to classes of Boolean algebras with operators $\mathbf{B} = (B, \vee, \wedge, \neg, 0, 1, \Box, \Diamond)$, where $\Diamond$ is a join hemimorphism, $\Box$ is a meet hemimorphism, and for every $x \in B$, $\Box x = \neg \Diamond \neg x$. *Intuitionistic logic* is sound and complete with respect to the class of Heyting algebras. Various types of intuitionistic modal logics are sound and complete with respect to classes of Heyting algebras with operators. *Gödel's logic (or LC or Dummet's logic [Dum59])* has as class of algebraic models the class of linear Heyting algebras (Heyting algebras satisfying $a \Rightarrow b \vee b \Rightarrow a = 1$).

Checking whether a formula ϕ is a theorem in such a logic can usually be reduced to checking whether $\mathcal{A} \models \phi = 1$, where $\mathcal{A}$ is a class of algebraic models of the logic.

Another type of logics which fall into this class are the so-called *positive logics* (cf. also the so-called *binary logics* [Gol93] Ch.2, or the similar concept in [Dun95]). Positive logics [Gol93, Dun95] do not have the implication symbol as a logical connective. Their algebraic models are usually lattices with operators. In positive logics, logical consequence can only be expressed by using the provability relation $\vdash$. Checking whether $\phi_1 \vdash \phi_2$ can usually be reduced to checking whether $\mathcal{A} \models \phi_1 \leq \phi_2$, where $\mathcal{A}$ is a class of algebraic models of the logic.

3.2.2 Logics based on residuated (semi)lattices

Residuated distributive lattices occur in a natural way as algebraic models for fuzzy, relevant and substructural logics.

Many *fuzzy logics* are sound and complete with respect to classes of residuated distributive lattices: The basic fuzzy logic (BL), for instance, has as algebraic models the class of all linearly ordered BL-algebras (i.e. linearly ordered bounded lattices with two binary operators $\circ$ and $\rightarrow$, $(L, \vee, \wedge, 0, 1, \circ, \rightarrow)$, where $(L, \circ, 1)$ is a commutative semigroup with 1, $\circ$ is monotone in both arguments, and where for all $x, y, z \in L$, $x \circ z \leq y$ iff $z \leq (x \rightarrow y)$, and $x \wedge y = x \circ (x \rightarrow y)$ [Háj98]). The *Gödel logic* has as algebraic models the class of all linearly ordered Heyting algebras. The *Lukasiewicz logics* [Luk30] has as algebraic models the class of linearly ordered MV-algebras (BL-algebras in which the identity $x = ((x \rightarrow 0) \rightarrow 0)$ holds). The *product logic* has as algebraic models the class of all (linearly ordered) product algebras, i.e. BL-algebras that satisfy

$$(z \rightarrow 0) \rightarrow 0 \leq ((x \circ z \rightarrow y \circ z) \rightarrow (x \rightarrow y)) \quad \text{and} \quad x \cap (x \rightarrow 0) = 0.$$

The relevant logic RL introduced by Urquhart in [Urq96] has as class of algebraic models the class of relevant algebras (bounded distributive lattices $(L, \vee, \wedge, 0, 1)$ with a lattice antimorphism $\neg$ and a binary

Constructor name	Syntax	Semantics
negation	$\neg C$	$D^{\mathcal{I}} \backslash C^{\mathcal{I}}$
conjunction	$C_1 \sqcap C_2$	$C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
disjunction	$C_1 \sqcup C_2$	$C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$
existential restriction	$\exists R.C$	$\{x \mid \exists y((x,y) \in R^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}})\}$
universal restriction	$\forall R.C$	$\{x \mid \forall y((x,y) \in R^{\mathcal{I}} \implies y \in C^{\mathcal{I}})\}$

Table 1: Constructors for $\mathcal{ALC}$

join hemimorphism $\circ$, with neutral element e, and residuation $\rightarrow$). Other examples are BCC and related logics [OK85], sound and complete with respect to classes of lattice-ordered residuated monoids.

In many of these logics, checking whether a formula ϕ is a theorem can be reduced to checking whether $\mathcal{A} \models \phi \geq e$, where e is a designated element in their algebraic models $\mathcal{A}$, usually the neutral element with respect to a monoid operation (see Anderson and Belnap [AB75] p.364, [Ono93], p.272).

3.2.3 Description logics

The main descriptive means of description logics are the *concept descriptions.* Concepts are defined with the help of a set of *concept constructors*, starting with a set N_C of *concept names* and a set N_R of *rôles.* The available constructors determine the expressive power of a description logic. For instance, in the description logic $\mathcal{ALC}$, the constructors used are negation ($\neg$), conjunction ($\sqcap$), disjunction ($\sqcup$), existential restriction ($\exists R$) and universal restriction ($\forall R$). A *terminology* (or TBox, for short) is a finite set of concept definitions of the form $A \equiv C$, where A is a concept name and C a concept description. (In description logics it is usually required that TBoxes do not contain multiple definitions.)

The semantics of description logics is defined in terms of interpretations $\mathcal{I} = (D^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $D^{\mathcal{I}}$ is a non-empty set, and the function $\cdot^{\mathcal{I}}$ maps each concept name $C \in N_C$ to a set $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and each rôle name $R \in N_R$ to a binary relation $R^{\mathcal{I}} \subseteq D^{\mathcal{I}} \times D^{\mathcal{I}}$. Table 1 shows the constructor names used in $\mathcal{ALC}$, together with their syntax and their semantics. The extension of $\cdot^{\mathcal{I}}$ to concept descriptions is inductively defined using the semantics of the constructors described in Table 1. An interpretation $\mathcal{I}$ is a model of the TBox $\mathcal{T}$ if it satisfies all the concept definitions in $\mathcal{T}$, i.e. $A^{\mathcal{I}} = C^{\mathcal{I}}$ for all definitions $A \equiv C$ in $\mathcal{T}$.

Definition 3.2 *Let $\mathcal{T}$ be a TBox, and C_1, C_2 two concept descriptions. C_1 is subsumed by C_2 with respect to $\mathcal{T}$ (for short, $C_1 \sqsubseteq_{\mathcal{T}} C_2$) if and only if $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$.*

In practical applications also description logics which are not closed under all Boolean connectives occur in a natural way. If we allow, for instance, only intersection and existential restriction as concept constructors, we obtain the description logic $\mathcal{EL}$, a logic used in terminological reasoning in medicine [Baa03]. If we allow only intersection and universal restriction as concept constructors, we obtain the description logic $\mathcal{FL}_0$.

We now show that in the description logics $\mathcal{ALC}, \mathcal{EL}$ and $\mathcal{FL}_0$ deciding the subsumption problem $C_1 \subseteq_{\mathcal{T}} C_2$ can be reduced to deciding a uniform word problem with respect to the class of all Boolean algebras (resp. distributive lattices, or semilattices) with operators. For this, we give a translation of concept descriptions into terms in a signature naturally associated with the set of constructors. For every rôle name R, we introduce two unary function symbols, $f_{\exists R}$ and $f_{\forall R}$. The renaming function is inductively defined by:

- $\overline{C} = C$ for every concept name C,
- $\overline{\neg C} = \neg \overline{C}$,
- $\overline{C_1 \sqcap C_2} = \overline{C}_1 \wedge \overline{C}_2$, $\quad \overline{C_1 \sqcup C_2} = \overline{C}_1 \vee \overline{C}_2$,
- $\overline{\exists R.C} = f_{\exists R}(\overline{C})$, $\quad \overline{\forall R.C} = f_{\forall R}(\overline{C})$.

It is easy to see that there exists a one-to-one correspondence between interpretations of description logics, $\mathcal{I} = (D, \cdot^{\mathcal{I}})$ and Boolean algebras of sets $(\mathcal{P}(D), \cup, \cap, \neg, \emptyset, D, \{f_{\exists R}, f_{\forall R}\}_{R \in N_R})$, together with valuations for the $v : N_C \rightarrow \mathcal{P}(D)$, where the additional operations are defined, for every $U \subseteq D$, by:

$$f_{\exists R}(U) = \{x \mid \exists y((x,y) \in R \text{ and } y \in U)\} \qquad f_{\forall R}(U) = \{x \mid \forall y((x,y) \in R \implies y \in U)\}.$$

We define the following classes of algebras:

- BAO_{N_R}, the class of all Boolean algebras with operators $\mathbf{B} = (B, \vee, \wedge, \neg, 0, 1, \{f_{\exists R}, f_{\forall R}\}_{R \in N_R})$ where $f_{\exists R}$ is a join hemimorphism, $f_{\forall R}$ is a meet hemimorphism, and $f_{\forall R}(x) = \neg f_{\exists R}(\neg x)$ for every $x \in B$.

- $\mathsf{DLO}^{\forall}_{N_R}$, the class of all bounded distributive lattices with operators $\mathbf{L} = (L, \vee, \wedge, 0, 1, \{f_{\forall R}\}_{R \in N_R})$ such that $f_{\forall R}$ is a meet hemimorphism;

- $\mathsf{DLO}^{\exists}_{N_R}$, the class of all bounded distributive lattices with operators $\mathbf{L} = (L, \vee, \wedge, 0, 1, \{f_{\exists R}\}_{R \in N_R})$ such that $f_{\exists R}$ is a join hemimorphism;

- $\mathsf{SLO}^{\forall}_{N_R}$, the class of all bounded meet-semilattices with operators $\mathbf{S} = (S, \wedge, 1, \{f_{\forall R}\}_{R \in N_R})$ such that $f_{\forall R}$ is a meet hemimorphism;

- $\mathsf{SLO}^{\exists}_{N_R}$, the class of all bounded meet-semilattices with operators $\mathbf{S} = (S, \wedge, 0, 1, \{f_{\exists R}\}_{R \in N_R})$ such that $f_{\exists R}$ is monotone and $f_{\exists R}(0) = 0$.

Theorem 3.3 *(1) For all concept descriptions C_1, C_2 and every TBox $\mathcal{T}$, $C_1 \sqsubseteq_{\mathcal{T}} C_2$ iff $\mathsf{BAO}_{N_R} \models (\bigwedge_{A \equiv C \in \mathcal{T}} A = \overline{C}) \to \overline{C_1} \leq \overline{C_2}$.*

(2) Assume that the only constructors are intersection and existential restriction. For all concept descriptions C_1, C_2 and every TBox $\mathcal{T}$, $C_1 \sqsubseteq_{\mathcal{T}} C_2$ iff $\mathsf{SLO}^{\exists}_{N_R} \models (\bigwedge_{A \equiv C \in \mathcal{T}} A = \overline{C}) \to \overline{C_1} \leq \overline{C_2}$.

(3) Assume that the only constructors are intersection and universal restriction. For all concept descriptions C_1, C_2 and every TBox $\mathcal{T}$, $C_1 \sqsubseteq_{\mathcal{T}} C_2$ iff $\mathsf{SLO}^{\forall}_{N_R} \models (\bigwedge_{A \equiv C \in \mathcal{T}} A = \overline{C}) \to \overline{C_1} \leq \overline{C_2}$.

Proof: The proof is given in Appendix A. □

Theorem 3.4 *The uniform word problem for BAO_{N_R} is* EXPTIME*-complete. The uniform word problem for $\mathsf{SLO}^{\exists}_{N_R}$ is decidable in polynomial time.*

Proof: The proof is given in Appendix B. □

Corollary 3.5 *Concept subsumption with respect to TBoxes in $\mathcal{ALC}$ and $\mathcal{FL}_0$ can be tested in exponential time. Concept subsumption with respect to TBoxes in $\mathcal{EL}$ can be tested in polynomial time.*

The EXPTIME-completeness of concept subsumption in $\mathcal{ALC}$ is well-known. Recently, Kazakov and Nivelle proved that concept subsumption with respect to TBoxes in $\mathcal{FL}_0$ is PSPACE-complete. The polynomial time complexity of concept subsumption with respect to TBoxes in $\mathcal{EL}$ was first proved by Baader [Baa02]. Theorem 3.4 provides a much simpler proof of this fact, and shows, in addition, that the restriction imposed in [Baa02] that TBoxes do not contain multiple definitions is not really necessary for polynomial time decidability of concept subsumption in $\mathcal{EL}$.

4 Automated theorem proving

We present several approaches to automated theorem proving in non-classical logic based on translations to classical logic, which allows the use of (various refinements of) resolution. Because of space limitations, neither tableau nor proof-theoretic methods are discussed, although they often provide optimal time and space complexity bounds.

This section is structured as follows. We first show that various versions of many-valued resolution for finitely-valued logics can be reconstructed by using general saturation-based techniques for first-order theories of transitive relations. The inference systems which we obtain this way are much more restricted, in particular by ordering constraints and selection functions. We then consider other non-classical logics which are not finitely valued, but for which nevertheless such embeddings into classical logics are possible. We present, for instance, a translation to clause form for prenex first-order Gödel logics [BFC01] which allowed the use of saturation-based techniques for dense total orderings, and then focus on propositional logics based on distributive lattices with operators (possibly many-sorted). We show that resolution-based decision procedures can be obtained in many interesting cases.

Notations, conventions, general definitions. As usual, the symbols $\vee$ and $\neg$ denote disjunction and negation, respectively. Formal equality will be denoted by $\approx$, and atoms of the form $s \approx t$ are called equations. The symmetry of equality is built into the notation in that we do not distinguish between $s \approx t$ and $t \approx s$. Negative equations $\neg(s \approx t)$ are also written as $s \not\approx t$. Semantically, equality is a congruence.

Consequently, a formula is called equationally satisfied in an interpretation I whenever the formula is satisfied in I, and the interpretation of $\approx$ in I is a congruence over the given signature, satisfying the respective set of *congruence axioms Eq*.

Orderings on syntactic expressions play an important rôle in theorem proving. Any ordering on ground terms can be extended to ground literals, and then to ground clauses (by taking the multiset extension). We say that a literal L is maximal with respect to a clause C (denoted $L \succeq C$) if $L' \succ L$ for no literal L' in C; and that L is strictly maximal with respect to C (denoted $L \succ C$) if $L' \succeq L$ for no L' in C.

In order to avoid unnecessary complication in the presentation we will in this paper only deal with the propositional variants of the various inference systems. That is, unless explicitly stated otherwise, all expressions (terms, literals, formulas) are assumed to be *ground*, that is, to not contain any variables. As the various completeness results also hold for infinite sets of clauses, lifting can be done in the standard manner by viewing non-ground expressions to represent the set of their ground instances and by employing unification to avoid their explicit enumeration.

4.1 Resolution in finitely-valued logics

In [BF95], Baaz and Fermüller extended the resolution procedure to arbitrary finitely-valued logics. They describe methods for translation to a *many-valued clause form*[2], formulate a sound and complete many-valued resolution calculus, and show that the completeness of the calculus is preserved when applying simplification rules such as subsumption and deletion of certain types of tautologies. Many-valued resolution has also been extended to literals signed by *sets* of truth values in [Häh94b], also see [BFS99]. A special kind of signs (when the set A of truth values is ordered by a total order $\leq$) are *regular signs* [Häh94b, Häh96], i.e. signs of the form $\uparrow v_j := \{v \mid v \geq v_j\}$ or $\downarrow v_j := \{v \mid v \leq v_j\}$.

A notion of regular signs has also been introduced in the context of annotated logics [KL92, LMR98] when the set A of truth values is a complete lattice with respect to an order $\leq$, with greatest element $\top$ and least element $\bot$. In this context, a regular literal is a literal with a sign of the form $\uparrow v$ or $A \backslash \uparrow v$ (notation: $\sim\uparrow v$), where $v \in A$; a regular clause is a disjunction of regular literals. An inference system consisting of annotated resolution, annotated reduction and elimination was shown to be sound and refutationally complete [KL92, LMR98].

Many-valued and regular literals can be expressed in classical logic in a natural way: $L \approx v$ corresponds to L^v; $L \geq v$ corresponds to $\uparrow v : L$, and so on. Formally, the encoding can be achieved by using a two-sorted language $\mathcal{L}_{ext}$ with sorts, ter (for terms) and for (for formulae of the many-valued logic). The set of function symbols in $\mathcal{L}_{ext}$ includes constants of sort for for all elements of A; the predicate symbols of the many-valued logic (viewed as function symbols with arguments of sort ter and result of sort for), the logical connectives of the many-valued logic as function symbols with arguments and result of sort for. We call the constants $v \in A$ *truth values*, and the terms of the form $R(t_1, \dots, t_n)$, with R a predicate symbol in the language of the many-valued logic under consideration, are called *predicate terms*. Truth values will be denoted by u, v, w, s, t, and predicate terms by L. Everywhere in what follows we assume that the set of truth values is $A = \{v_1, \dots, v_n\}$.

In what follows let $\succ$ be a noetherian ordering on ground literals.

4.1.1 Many-valued clauses

With every set of many-valued clauses Φ, consisting of literals L^v signed by truth values, we associate a set Φ_1 of first-order clauses by replacing every signed literal L^v in Φ by the equation $L \approx v$.

In what follows, literals of the form $L \approx v$ are called MV-literals, and clauses consisting of MV-literals are called MV-clauses.

In [GSS00] we proved that a set Φ of many-valued clauses is satisfiable if and only if $\Phi_1 \cup \Phi_A \cup \mathsf{Fin}$ is (classically) equationally satisfiable, where

$$\Phi_A = \{u \not\approx v \mid u, v \in A, u \neq v\}, \quad \mathsf{Fin} = \{s \approx v_1 \vee \dots \vee s \approx v_n \mid s \text{ a term of sort } \mathsf{for}\}$$

are sets of clauses which express that there are exactly n pairwise different (congruence classes of) truth values $v_1, \dots, v_n$ in any equality Herbrand interpretation satisfying $\Phi_A \cup \mathsf{Fin}$.

Satisfiability of $\Phi_1 \cup \Phi_A \cup \mathsf{Fin}$ can for instance be checked by using superposition [BG94]. When applied to sets of MV-clauses, the superposition calculus specializes to the following calculus, SMV:

[2] Many-valued literals L^v are atomic formulae superscripted by truth values; many-valued clauses are disjunctions of many-valued literals.

Positive MV-superposition. From $L \approx t \vee C$ and $L \approx v \vee D$ derive $C \vee D$ provided that $t \neq v$ and (i) $L \approx t \succ C$; (ii) $L \approx v \succ D$; (iii) $L \approx v \succ L \approx t$.

Ordered factoring. From $L \approx t \vee L \approx t \vee C$ derive $L \approx t \vee C$ provided that $L \approx t$ is maximal with respect to C.

Theorem 4.1 ([GSS00]) *Let N be a set of MV-clauses such that $N \backslash \mathsf{Fin}$ is saturated up to $Eq \cup \Phi_A$-redundancy with respect to SMV. Then either N contains the empty clause or else $N \cup \mathsf{Fin} \cup \Phi_A$ is equationally satisfiable.*

As superposition into subterms is not possible for MV-clauses, $\succ$ needs not be a reduction ordering on terms.

In conclusion, the calculus SMV is an order-refinement of the many-valued resolution method of Baaz and Fermüller. Its compatibility with $Eq \cup \Phi_A$-redundancy, and with the simplification techniques which redundancy justifies follows from Theorem 4.1.

4.1.2 Annotated and regular clauses

Let $(A, \leq_A)$ be a finite partially ordered set, and $\mathsf{Min}(A)$ the set of minimal elements in A. Let Φ be a set of regular clauses, i.e. clauses containing only literals of the form $\uparrow v{:}L$ or $\sim\uparrow v{:}L$, where $v \in A$. The *encoding of Φ in first-order logic*, Φ_1, is the set of clauses obtained from Φ by replacing $\uparrow v{:}L$ by $v \leq L$ and $\sim\uparrow v{:}L$ by $v \not\leq L$, where $v \not\leq L$ is an abbreviation for $\neg(v{\leq}L)$. Consider the following additional sets of clauses:

$$\begin{array}{rcl}
\Phi_A & = & \{u \trianglelefteq v \mid u, v \in A, u \trianglelefteq_A v, \trianglelefteq \in \{\leq, \not\leq, \not\approx\}\} \\
\mathsf{Sup} & = & \{(u \not\leq s) \vee (v \not\leq s) \vee (\mathsf{sup}(u,v) \leq s) \mid \mathsf{sup}(u,v) \text{ exists in } A, s \text{ a term of sort } \mathsf{for}\} \\
\mathsf{Min} & = & \{ \bigvee_{m \in \mathsf{Min}(A)} (m{\leq}s) \mid s \text{ a term of sort } \mathsf{for}\}.
\end{array}$$

In the following we will only consider clauses with inequalities $s{\leq}t$ as atoms. Equalities $s \approx t$ will be used on the meta-level as an abbreviation for conjunctions $(s{\leq}t) \wedge (t{\leq}s)$. Fin will again denote the set of clauses (represented by) $\{s \approx v_1 \vee \ldots \vee s \approx v_n \mid s \text{ a term of sort } \mathsf{for}\}$.

By Tr we denote the transitivity axiom for $\leq$: $(x \leq y) \wedge (y \leq z) \rightarrow (x \leq z)$. By a *transitivity interpretation* we mean a model of Tr. We say that a set of clauses N is *Tr-satisfiable* if there exists a transitivity interpretation I that satisfies N. Otherwise N is *Tr-unsatisfiable.*
In [GSS00] we proved that if Φ be a set of regular clauses then:

(1) If $(A, \leq_A)$ is a partially ordered set, then Φ is satisfiable iff $\Phi_1 \cup \Phi_A \cup \mathsf{Fin}$ is (classically) Tr-satisfiable.

(2) If $(A, \leq_A)$ is a sup-semilattice, then Φ is satisfiable iff $\Phi_1 \cup \Phi_A \cup \mathsf{Sup} \cup \mathsf{Min}$ is (classically) Tr-satisfiable.

(3) If $(A, \leq_A)$ is a totally-ordered set with minimal element $\bot$ then Φ is satisfiable iff $\Phi_1 \cup \Phi_A \cup \{\bot{\leq}s \mid s \text{ a term of sort } \mathsf{for}\}$ is (classically) Tr-satisfiable.

A *$\leq$-literal* is a literal of the form $v \leq L$ or $v \not\leq L$, where L is a predicate term and v is a truth value. A $\leq$-clause is a disjunction of $\leq$-literals. When applied to $\leq$-clauses, the chaining calculus of Bachmair and Ganzinger [BG98] specializes to the following calculus, CS:

Negative chaining for $\leq$-clauses. From $(u \leq L) \vee C$ and $(v \not\leq L) \vee D$ derive $C \vee D$ provided that $v \leq_A u$ and (i) holds.

Sup-reduction. From $(u \leq L) \vee C$ and $(v \leq L) \vee D$, where u and v are incomparable, derive $(\mathsf{sup}(u,v) \leq L) \vee C \vee D$ provided that (ii) holds.

Ordered (positive) factoring. From $B \vee B \vee C$ derive $B \vee C$ provided that B is maximal with respect to C.

The restrictions are: (i) $(u \leq L){\succ}C$ and $(v \not\leq L){\succeq}D$; (ii) $(u \leq L){\succ}C$ and $(v \leq L){\succ}D$.

Theorem 4.2 ([GSS00]) *Let Φ be a set of regular clauses over a finite set A truth values, and let Φ_1 be the encoding of Φ in first-order logic.*

(i) If A is a sup-semilattice with minimal elements $\mathsf{Min}(A)$ then Φ is unsatisfiable if and only if the empty clause can be derived from $\Phi_1 \cup \mathsf{Min}$ by a finite number of applications of inference rules in CS.

(ii) Assume that A is a complete lattice with minimal element $\bot$. Let Φ_2 be obtained from Φ_1 by removing all literals of the form $\bot \not\leq L$ and all clauses containing a literal of the form $\bot \leq L$. Then Φ is unsatisfiable if and only if there exists a derivation in CS of the empty clause from Φ_2.

As chaining into subterms is not possible for $\leq$-clauses, $\succ$ needs not be a reduction ordering on terms. Thus, the calculus CS is an order-refinement of the annotated resolution calculus in [LMR98]. If $(A, \leq_A)$ is a totally ordered set then sup-reduction never applies. Let CT be the inference system consisting of all inference rules in CS except sup-reduction. The refutational completeness of the CT calculus in the case when $(A, \leq_A)$ is a totally ordered set is a direct consequence of Theorem 4.2.

Since first-order many-valued logics are undecidable, in general we cannot hope to obtain decision procedures based on the calculi above. It can however be seen that, in the propositional case, they yield exponential time decision procedures in the length of the input.

4.2 Resolution for infinitely-valued logics

The method for translation to clause form for finitely valued logics of Baaz and Fermüller cannot be applied in the case when the set of truth values is infinite, nor for logics whose semantics is given in terms of a *class* of algebras.

There have been several attempts for giving methods for automated theorem proving in infinitely-valued logics. Some approaches rely on CNF translations which, for instance, allow reductions to mixed integer programming (MIP) in the case of infinitely-valued propositional Łukasiewicz logic and Gödel logics [Häh94a, Häh97]. (The connectives of the product logic, however, lead outside MIP, and into non-linear programming.) Other approaches reduce the problem of checking validity in infinitely-valued logics to checking validity in a suitable finitely-valued logic. For instance, Aguzzoli and Ciabattoni [AC00] proved that a formula ϕ is valid in the infinitely valued (propositional) Lukasiewicz logic Ł if and only if it is valid in a suitable m-valued Lukasiewicz logic Ł$_m$, where m only depends on the length of the formula to be proved (in fact, $m = 2^{\mathsf{lenght}(\phi)} + 1$). Thus, in this case, the methods discussed in Section 4.1 may still be used, but could be highly inefficient, since the size of the algebra Ł$_m$ is exponential in the length of the formula ϕ. A resolution-like calculus for the infinitely-valued sentential calculus of Łukasiewicz based on a different representation of clauses is given by Mundici and Olivetti in [MO98].

The particular properties of certain fragments of non-classical (first-order) logics allow to obtain embeddings into classical logic. The advantage in such situations is that refinements of classical resolution can be used directly, without any sophisticated encodings.

An embedding of the prenex fragment of first-orded Gödel logic into the first-order theory of dense total orderings with endpoints is presented in [BFC01]. Semantically, first-order Gödel logic G_∞ (sometimes called intuitionistic fuzzy logic, or Dummet's LC) is viewed as an infinitely-valued logic, with the real interval $[0, 1]$ as set of truth values, and the semantics of the quantifiers given by supremum (for $\exists$) and infimum (for $\forall$). [BFC01] study the logic G^Δ_∞, obtained by extending G_∞ with projection modalities ∇, Δ, interpreted by the maps $\nabla, \Delta : [0, 1] \to \{0, 1\}$, $\nabla(x) = 1$ iff $x = 0$ and $\Delta(x) = 1$ iff $x = 1$.

Theorem 4.3 ([BFC01]) *For each prenex formula $Q_1y_1 \dots Q_ny_n\phi(y_1, \dots, y_n)$ of G^Δ_∞ there exists a set $CF^d(\exists\overline{x}\phi(\overline{x}))$ of order clauses[3] (which can be computed in linear time), such that $Q_1y_1 \dots Q_ny_n\phi(y_1, \dots, y_n)$ is valid in G^Δ_∞ if and only if $CF^d(\exists\overline{x}\phi(\overline{x}))$ is unsatisfiable with respect to the theory of dense total orderings with endpoints.*

The embedding is, up to a certain extent, similar to that described in the previous section in the case of finitely-valued logics based on partially ordered, or totally ordered sets. A chaining calculus for dense total orderings with endpoints [BG98] is then used for efficient deduction with such sets of clauses. However, since G^Δ_∞ is undecidable, one cannot hope to use chaining for dense total orderings with endpoints as a decision procedure in this case. In order to be able to use resolution as a decision procedure, in what follows we focus on propositional non-classical logics.

[3] Order clauses are classical clauses with predicate symbols $<$ and $\leq$ interpreted as total dense orderings (strict and reflexive, respectively).

4.3 Resolution-based decision procedures for modal logics

In the attempt of understanding why so many modal logics are decidable many authors noticed that the definition of the Kripke-style semantics justifies an embedding into (decidable fragments of) classical logic. For instance, in [AvBN98] Andréka, Van Benthem and Németi introduced the so-called *guarded fragment* (GF) of classical logic, which abstracts many of the properties of formulae obtained from the structure-preserving translation to clause form for many modal logics. The main advantage of the embedding into first-order logic is that it is very suitable to use for automated theorem proving, since proof techniques developed for classical logic can be used for free. Refinements of resolution such as ordered resolution, the use of selection functions, and specially devised calculi to deal with equivalence (or congruence) relations, or with transitive relations proved to be extremely useful in this context.

For instance, ordered resolution was used as a decision procedure for modal logics such as K in [Ohl93, Sch99], ordered chaining with selection was used to obtain (doubly exponential[4]) decision procedures for the relational translation of propositional modal logics with modal operators satisfying the axioms D, T or $\mathsf{4}$ in [GHMS01]. A doubly-exponential decision procedure for the guarded fragment with equality, that uses superposition, was given in [GdN99].

The embedding into classical logic for modal logics mentioned above is a special instance of a more general result, which we present in the next section.

4.4 Resolution and uniform word problems in DLO

We show that links between truth of universal (Horn) sentences in classes of distributive lattices with operators and truth in classes of suitably chosen relational models (justified by representation theorems) can be used for obtaining embeddings into decidable fragments of first-order logic (without equality). This type of embeddings in many cases yield optimal resolution-based decision procedures.

These results justify, in particular, existing embeddings into classical logic for many-valued logics over finite distributive lattices with operators [SS01], but also for various classes of modal logics.

4.4.1 Algebraic and relational models

We establish a link between truth of universal sentences in classes of S-sorted distributive lattices with operators and truth in classes of S-sorted relational structures.

Definition 4.4 *An S-*sorted *RT Σ-*relational structure *$(\{(X_s, \leq)\}_{s \in S}, \{R_X\}_{R \in \Sigma})$ is an S-sorted family of sets, each endowed with a reflexive and transitive relation $\leq$ and with additional maps and relations indexed by Σ, where, if $\varepsilon_1, \dots, \varepsilon_n, \varepsilon \in \{-1, +1\}$, $s_1, \dots, s_n, s \in S$ then: if $R \in \Sigma$ is of type $s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n} \to s^{\varepsilon}$, $R_X \subseteq \prod_{i=1}^n X_{s_i} \times X_s$ is increasing if $\varepsilon = +1$ and decreasing if $\varepsilon = -1$.*

We denote by DLO_Σ^S, BAO_Σ^S, and HAO_Σ^S the class of all S-sorted bounded distributive lattices, Boolean algebras, and resp. Heyting algebras, with operators in Σ, and by RT_Σ^S the class of all S-sorted RT Σ-relational structures.

If $\mathbf{L} \in \mathsf{DLO}_\Sigma^S$, let $D(\mathbf{L}) = (\{(\mathcal{F}_p(\mathbf{L}_s), \subseteq)\}_{s \in S}, \{R_f\}_{f \in \Sigma})$, where if $f : \prod_{i=1}^n \mathbf{L}_{s_i}^{\varepsilon_i} \to \mathbf{L}_s^{\varepsilon}$ is a join hemimorphism, where $\varepsilon_1, \dots, \varepsilon_n, \varepsilon \in \{-1, +1\}$, then we define

$$R_f(F_1, \dots, F_n, F) \text{ if and only if } f(F_1^{\varepsilon_1}, \dots, F_n^{\varepsilon_n}) \subseteq F^{\varepsilon},$$

where $F^{+1} := F$ and F^{-1} is the complement of F. Conversely, for every $\mathbf{X} \in RT_\Sigma^S$, let $\mathcal{O}(\mathbf{X})$ be the many sorted algebra $(\{(\mathcal{O}(X_s), \cup, \cap, \emptyset, X_s)\}_{s \in S}, \{f_R\}_{R \in \Sigma})$, where, for every $s \in S$, $(\mathcal{O}(X_s), \cup, \cap, \emptyset, X_s)$ is the bounded distributive lattice of all upwards-closed subsets of X_s, and if $R \subseteq \prod_{i=1}^n X_{s_i} \times X_s$ is of type $s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n} \to s^{\varepsilon}$ then $f_R : \prod_{i=1}^n \mathcal{O}(X_{s_i}) \to \mathcal{O}(X_s)$ is defined, for every $(U_1, \dots U_n) \in \prod_{i=1}^n \mathcal{O}(X_{s_i})$ by

$$f_R(U_1, \dots, U_n) = \left(R^{-1}(U_1^{\varepsilon_1}, \dots, U_n^{\varepsilon_1})\right)^{\varepsilon},$$

where $R^{-1}(U_1, \dots, U_n) = \{x \mid \exists x_1 \dots x_n (x_1 \in U_1, \dots, x_n \in U_n, R(x_1, \dots, x_n, x))\}$, and $U^{+1} := U$ and U^{-1} is the complement of U.

Similar correspondences can be established for (possibly many-sorted) Boolean algebras with operators or Heyting algebras with operators. Note that in the case of Boolean algebras, the dual spaces are discretely ordered (i.e. $x \leq y$ iff $x = y$).

[4] In [GHMS01] it is actually showed that a single-exponential space representation can be obtained by splitting the clauses into their variable-disjoint regions and connecting them with the help of auxiliary monadic predicates.

4.4.2 Structure-preserving translation to clause form

We consider subclasses $\mathcal{V}$ of DLO^S_Σ, BAO^S_Σ or HAO^S_Σ that satisfy the following condition:

(K) There exists a $\mathcal{K} \subseteq RT^S_\Sigma$ such that (i) for every $\mathbf{A} \in \mathcal{V}$, $D(\mathbf{A}) \in \mathcal{K}$;
(ii) for every $\mathbf{X} \in \mathcal{K}$, $\mathcal{O}(\mathbf{X}) \in \mathcal{V}$.

In [SS03b] we showed that if $\mathcal{V}$ satisfies condition (K) then, for every formula $\phi = \forall x_1, \dots, x_k(\bigwedge_{i=1}^n s_{i1} = s_{i2} \to \bigvee_{j=1}^m t_{j1} = t_{j2})$, $\mathcal{V} \models \phi$ if and only if for every $\mathbf{X} \in \mathcal{K}, \mathcal{O}(\mathbf{X}) \models \phi$.

For automated theorem proving it is important to find subclasses of RT^S_Σ with good theoretical and logical properties, for instance subclasses which are *first-order definable.*

Theorem 4.5 *Condition (K) holds in the following cases:*

1. $\mathcal{V} = \mathsf{DLO}^S_\Sigma$ *the class of all S-sorted distributive lattices with operators in Σ; and* $\mathcal{K} = RT^S_\Sigma$.

2. $\mathcal{V} = \mathsf{RDLO}^S_{\Sigma,\mathsf{Res}}$ *the class of all S-sorted distributive lattices with operators in Σ satisfying the residuation conditions in* Res*; and*

   $\mathcal{K} = RT^S_{\Sigma,\mathsf{Res}}$ *the class of those spaces in RT^S_Σ which satisfy in addition:*
   $\{R_f(x_1, \dots, x_n, x) \leftrightarrow R_g(x_1, \dots, x, \dots, x_n, x_i) \mid$ "*g is the i-residuation of f*" $\in \mathsf{Res}\}$.

3. $\mathcal{V} = \mathsf{BAO}^S_\Sigma = \{(\{\mathbf{B}_s\}_{s\in S}, \{f\}_{f\in\Sigma}) \mid \mathbf{B}_s \in \mathsf{Bool}$ *for all* $s \in S; f : \prod_{i=1}^n \mathbf{B}_{s_i} \to \mathbf{B}_s$ *join hemimorphism, for every* $f \in \Sigma_{s_1\dots s_n \to s}\}$*; and*

   $\mathcal{K} = R^S_\Sigma$ *the subclass of RT^S_Σ consisting only of those S-sorted spaces in which all supports are discretely ordered.*

4. $\mathcal{V} = \mathsf{H}$*, the class of all Heyting algebras; and $\mathcal{K}$ the family of all preordered spaces.*

If $\mathbf{A} \in \mathsf{D}_{01}$ *is a fixed* finite *lattice and* $S = \{\mathsf{lat}, \mathsf{a}\}$*, then condition (K) holds in the following cases:*

5. $\mathcal{V} = \mathsf{DLO}^A_\Sigma = \{(\mathbf{L}, \mathbf{A}, \{f_L\}_{f\in\Sigma_L}, \{f_b\}_{f\in\Sigma_b}) \mid \mathbf{L} \in \mathsf{D}_{01};\ f_L : \mathbf{L}^k \to \mathbf{L}$ *join hemimorphism, for every* $f \in \Sigma_L$*, of type* $\mathsf{lat}^k \to \mathsf{lat}$; $f_b : \mathbf{L}^m \to \mathbf{A}$ *join hemimorphism for every* $f \in \Sigma_b$*, of type* $\mathsf{lat}^m \to \mathsf{a}\}$*; and*

   $\mathcal{K} = \{(X, D(\mathbf{A}), \{R_f\}_{f\in\Sigma_L}, \{R_g\}_{g\in\Sigma_b}) \mid (X, \{R_f\}_{f\in\Sigma_L}) \in RT_{\Sigma_L}$ *and* $R_g \subseteq X^m \times D(\mathbf{A})$ *increasing for all* $g \in \Sigma_b$ *of type* $\mathsf{lat}^m \to \mathsf{a}\}$.

6. $\mathcal{V} = \mathsf{RDLO}^A_{\Sigma,\mathsf{Res}}$ *the subclass of all algebras in* DLO^A_Σ *in which the operators in Σ_L satisfy the residuation conditions in* Res*; and*

   $\mathcal{K} = \{(X, D(\mathbf{A}), \{R_f\}_{f\in\Sigma_L}, \{R_g\}_{g\in\Sigma_b}) \mid (X, \{R_f\}_{f\in\Sigma_L}) \in RT_{\Sigma_L}$ *and* $R_g \subseteq X^m \times D(\mathbf{A})$ *increasing for all* $g \in \Sigma_b$ *of type* $\mathsf{lat}^m \to \mathsf{a}$ *and, in addition,* $R_f(x_1, \dots, x_n, x) \leftrightarrow R_g(x_1, \dots, x, \dots, x_n, x_i)$ *for all* $f, g \in \Sigma_L$ *such that the condition "g is the i-residuation of f" occurs in* $\mathsf{Res}\}$.

7. $\mathcal{V} = \mathsf{BAO}^A_\Sigma = \{(\mathbf{B}, \mathbf{A}, \{f_B\}_{f\in\Sigma_B}, \{f_b\}_{f\in\Sigma_b}) \mid \mathbf{B} \in \mathsf{Bool};\ f_B : \mathbf{B}^k \to \mathbf{B}$ *join hemimorphism, for every* $f \in \Sigma_B$*, of type* $\mathsf{lat}^k \to \mathsf{lat}$; $f_b : \mathbf{B}^m \to \mathbf{A}$ *join hemimorphism for every* $f \in \Sigma_b$*, of type* $\mathsf{lat}^m \to \mathsf{a}\}$*; and*

   $\mathcal{K} = \{(X, D(\mathbf{A}), \{R_f\}_{f\in\Sigma_L}, \{R_g\}_{g\in\Sigma_b}) \mid (X, \{R_f\}_{f\in\Sigma_L}) \in R_{\Sigma_L}$ *and* $R_g \subseteq X^m \times D(\mathbf{A})$ *increasing for all* $g \in \Sigma_b$ *of type* $\mathsf{lat}^m \to \mathsf{a}\}$.

Proof: The proof uses extensions of the Priestley representation theorem to various classes of distributive lattices with operators, and extensions of the Stone representation theorem to Boolean algebras with operators. For details we refer to [SS02, SS03a, SS03b]. □

If a subclass $\mathcal{V}$ of $\mathsf{DLO}^{\mathsf{S}}_\Sigma$, $\mathsf{BAO}^{\mathsf{S}}_\Sigma$ or $\mathsf{HAO}^{\mathsf{S}}_\Sigma$ satisfies condition (K) for some first-order definable subclass $\mathcal{K}$ of RT^S_Σ, then the problem of checking whether a formula

$$\phi = \forall x_1, \dots, x_k(\bigwedge_{i=1}^n s_{i1} = s_{i2} \to \bigvee_{j=1}^m t_{j1} = t_{j2})$$

holds in $\mathcal{V}$ can be reduced to the problem of checking the satisfiability of a set of clauses.

Let $ST(\phi)$ be the set of all subterms of s_{il} and t_{jp}, $1 \leq i \leq n, 1 \leq j \leq m, l, p \in \{1, 2\}$ (including the variables and s_{il}, t_{jp} themselves).

Theorem 4.6 ([SS02]) *Assume that $\mathcal{V}$ and $\mathcal{K}$ satisfy condition (K), where $\mathcal{K}$ is a class of RT Σ-structures definable by a finite set C of first-order sentences. The following are equivalent:*

(1) $\mathcal{V} \models \phi$.

(2) *The conjunction of* $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N_1}) \cup \cdots \cup (\mathsf{N_m})$
is unsatisfiable, where:

$$\begin{array}{llrcll}
(\mathsf{Dom}) & & C, & & & \\
 & & \leq \subseteq X_s \times X_s & \textit{is} & \textit{reflexive and transitive} & \textit{for every sort } s \in S, \\
 & & R_f \subseteq \prod_{i=1}^{n+1} X_{s_i} & \textit{is} & \textit{increasing} & \textit{for every } f \in \Sigma_{s_1 \dots s_n \to s_{n+1}}, \\
(\mathsf{Her}) & \forall x, y & (x \leq y \wedge P_e(x) & \to & P_e(y)) & \\
(\mathsf{Ren}) & & & & & \\
\quad (1) & \forall x & P_{1_s}(x) & & & \textit{for every sort } s \in S, \\
\quad (0) & \forall x & \neg P_{0_s}(x) & & & \textit{for every sort } s \in S, \\
\quad (\wedge) & \forall x & (P_{e_1 \wedge e_2}(x) & \leftrightarrow & P_{e_1}(x) \wedge P_{e_2}(x)) & \\
\quad (\vee) & \forall x & (P_{e_1 \vee e_2}(x) & \leftrightarrow & P_{e_1}(x) \vee P_{e_2}(x)) & \\
\quad (\Sigma) & \forall x & (P_{f(e_1, \dots, e_n)}(x)^{\varepsilon} & \leftrightarrow & \exists x_1 \dots x_n (\bigwedge_{i=1}^n P_{e_i}(x_i)^{\varepsilon_i} \wedge R_f(x_1, \dots, x_n, x))) & \\
(\mathsf{P}) & \forall x & (\bigwedge_{i=1}^n P_{s_{i1}}(x) & \leftrightarrow & P_{s_{i2}}(x)) & \\
(\mathsf{N_1}) & \exists x_1 & (P_{t_{11}}(x_1) & \not\leftrightarrow & P_{t_{12}}(x_1)) & \\
 & \dots & & \dots & & \\
(\mathsf{N}_m) & \exists x_m & (P_{t_{m1}}(x_m) & \not\leftrightarrow & P_{t_{m2}}(x_m)) &
\end{array}$$

where the unary predicates P_e are indexed by elements in $ST(\phi)$, and the formulae in Σ range over all operators $f \in \Sigma$ such that f is a join hemimorphism of type $s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n} \to s^{\varepsilon}$, where $\varepsilon_i, \varepsilon \in \{-1, +1\}$, and $L^{+1} := L$ and $L^{-1} := \neg L$.

In addition, in many situations polarity of subformulae can be used for using only one direction of the implications in (Ren). Similar ideas can be used for obtaining translations to clause form for formulae of the form $\bigwedge_{i=1}^n s_{i1} {\leq} s_{i2} \to \bigwedge_{j=1}^m t_{j1} {\leq} t_{j2}$. Then only the direct implications are necessary in (P) and (N).

If $\mathcal{V} = \mathsf{RDLO}^S_{\Sigma, \mathsf{Res}}$ or $\mathcal{V} = \mathsf{RDLO}^A_{\Sigma, \mathsf{Res}}$, where Res is a set of generalized residuation rules, then the set C of formulae contains the conditions:

$$R_f(x_1, \dots, x_i, \dots, x_n, x) \leftrightarrow R_g(x_1, \dots, x, \dots, x_n, x_i)$$

for all operators f, g such that "g is an i-residuation of f" $\in$ Res.

If $\mathcal{V} = \mathsf{H}$, the class of Heyting algebras, then the set C contains only the preorder axioms for the relation $\leq$.

4.4.3 Some decidability results

We now present some examples in which decidability results can be obtained.

Theorem 4.7 *Ordered resolution with selection decides, in time* exponential in the size of the input *if the arity of operators in Σ has an upper bound, and* exponential in the square of the size of the input *in general, the universal clause theory of* DLO^S_{Σ}*, and* RDLO^S_{Σ}.

Proof: (Idea) The results of [SS03b], Section 5.1 can easily be adapted to prove this theorem. As pointed out in [SS03b], the selection strategy we adopt for this purpose shows that in this case inferences with the clauses containing the $\leq$ symbol are not needed for refutational completeness. □

It can be seen that for uniform word problems which contain only conjunction and join hemimorphisms, resolution yields a polynomial time decision procedure[5].

Theorem 4.8 *Ordered resolution with selection decides, in time* exponential in the size of the input *if the arity of operators in Σ has an upper bound, and* exponential in the square of the size of the input *in general, the universal clause theory of* DLO^A_{Σ}*, and* RDLO^A_{Σ}*, where A is a finite distributive lattice.*

Proof: (Idea) We can show that inferences with the clauses containing the $\leq$ symbol applied to arguments of sort lat are not needed in the case of DLO^A_{Σ}. Since $D(A)$ is finite, the monotonicity and heredity rules

[5] Y. Kazakov, personal communication

for sort a, can be replaced with their instances with elements in $D(A)$. For instance the monotonicity and heredity rule can alternatively be expressed by:

$$R_f(x_1, \dots, x_n, a) \rightarrow R_f(x_1, \dots, x_n, b) \qquad \text{for all } a, b \in D(A), a \leq b \tag{9}$$

$$P_e(a) \rightarrow P_e(b) \qquad \text{for all } a, b \in D(A), a \leq b \tag{10}$$

We can now introduce $D(A)$ copies for every predicate symbol with last argument of sort a, e.g. by replacing, for every $a \in D(A)$, $R_f(x_1, \dots, x_n, a)$ with $R^a_f(x_1, \dots, x_n)$ and $P_e(a)$ with P^a_e. Arguments in [SS03b], Section 5.1 can now be applied and also in this case yield the desired complexity results. □

Similar arguments can be also used for (many sorted) Boolean algebras with operators, by considering, in addition, the renaming rules for Boolean negation. Thus, we obtain decision procedures with optimal time complexity.

Theorem 4.9 ([SS03b]) *For every formula $\phi = \forall x_1, \dots, x_k (\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^m t_{j1} = t_{j2})$, ordered chaining with eager condensation and selection decides (in at most doubly exponential time and exponential space with respect to the length of ϕ) whether $\mathsf{H} \models \phi$.*

4.4.4 A special case: Finitely-valued logics based on DLO.

As a special case, the results above can be applied to automated theorem proving in propositional many-valued logics based on finite distributive lattices with operators.

Let $\mathbf{A} = (A, \vee, \wedge, 0, 1, \{f_A\}_{f \in \Sigma})$ be a finite distributive lattice with operators, and let $D(\mathbf{A})$ be the Priestley dual of $\mathbf{A}$. Since $\mathbf{A}$ is finite, $D(\mathbf{A}) = (\{\uparrow j \mid j \in A, \text{ join irreducible}\}, \subseteq)$, and $\mathbf{A}$ is isomorphic to $\mathcal{O}(D(\mathbf{A}))$. In this case $\mathcal{V} = \{\mathbf{A}\}$ and $\mathcal{K} = \{D(\mathbf{A})\}$ satisfy condition (K). Let ϕ be the following formula in the signature of $\mathbf{A}$:

$$\phi = \forall x_1, \dots, x_k (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^{m} t_{j1} = t_{j2}).$$

Corollary 4.10 *Let $\mathbf{A} = (A, \vee, \wedge, 0, 1, \{f_A\}_{f \in \Sigma})$ be a finite distributive lattice with operators, where $A = \{a_1, \dots, a_n\}$. Let $D(A) = (\{\uparrow j_1, \dots, \uparrow j_k\}, \subseteq)$. The following are equivalent:*

(1) *$A \models \phi$.*

(2) *The conjunction of* $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N_1}) \cup \dots \cup (\mathsf{N_m})$ *is unsatisfiable, where:*

$$\begin{array}{llrcl}
(\mathsf{Dom}) & \forall x & x = \uparrow j_1 \vee \cdots \vee x = \uparrow j_k & & \\
 & & \uparrow j_i \leq \uparrow j_k & & \textit{whenever } j_k \leq j_i \textit{ in } A \\
 & & R_f(\uparrow j_{i_1}, \dots, \uparrow j_{i_l}, \uparrow j_{i_{l+1}}) & & \textit{whenever } R_f(\uparrow j_{i_1}, \dots, \uparrow j_{i_l}, \uparrow j_{i_{l+1}}) \textit{ holds in } D(A) \\
(\mathsf{Her}) & \forall x, y & (x \leq y \wedge P_e(x) & \rightarrow & P_e(y)) \\
(\mathsf{Ren}) & & & & \\
\quad (1) & \forall x & P_1(x) & & \\
\quad (0) & \forall x & \neg P_0(x) & & \\
\quad (\wedge) & \forall x & (P_{e_1 \wedge e_2}(x) & \leftrightarrow & P_{e_1}(x) \wedge P_{e_2}(x)) \\
\quad (\vee) & \forall x & (P_{e_1 \vee e_2}(x) & \leftrightarrow & P_{e_1}(x) \vee P_{e_2}(x)) \\
\quad (\Sigma) & \forall x & (P_{f(e_1, \dots, e_n)}(x)^{\varepsilon} & \leftrightarrow & \exists x_1 \dots x_n (\bigwedge_{i=1}^n P_{e_i}(x_i)^{\varepsilon_i} \wedge R_f(x_1, \dots, x_n, x))) \\
(\mathsf{P}) & \forall x & (\bigwedge_{i=1}^n P_{s_{i1}}(x) & \leftrightarrow & P_{s_{i2}}(x)) \\
(\mathsf{N_1}) & \exists x_1 & (P_{t_{11}}(x_1) & \not\leftrightarrow & P_{t_{12}}(x_1)) \\
 & \dots & & \dots & \\
(\mathsf{N}_m) & \exists x_m & (P_{t_{m1}}(x_m) & \not\leftrightarrow & P_{t_{m2}}(x_m))
\end{array}$$

where the unary predicates P_e are indexed by elements in $ST(\phi)$, and the formulae in Σ range over all operators $f \in \Sigma$ such that f is a join hemimorphism of type $\varepsilon_1 \dots \varepsilon_n \rightarrow \varepsilon$, where $\varepsilon_i, \varepsilon \in \{-1, +1\}$, and $L^{+1} := L$ and $L^{-1} := \neg L$.

It is easy to see that the conjunction above is unsatisfiable if and only if the set of all its ground instances, where the variables are instantiated with elements in $D(A)$ is satisfiable. We thus recover some of the results on automated theorem proving in many-valued logics having as algebra of truth values a distributive lattice with operators in [SS01]. (The labeled literals of the form $\boxed{\uparrow j}\ P_e$ used in [SS01] correspond to ground literals of the form $P_e(\uparrow j)$ in the present setting.)

For an extension to automated theorem proving in first-order many-valued logics based on distributive lattices with operators we refer to [SS01].

5 Conclusions

The main goal of this paper was to show that, in many situations, efficient methods for automated theorem proving can be obtained if we can find suitable embeddings into first-order classical logic. We illustrated the ideas by means of various examples, ranging from many-valued logics to description logics.

In the case of many-valued logics, such embeddings into classical logic allow to reconstruct known completeness results for existing methods for automated theorems proving. Apart from this, the inference systems we obtain are much more restricted, in particular with ordering constraints and selection functions. In addition, general results in first-order logic for simplification and for eliminating redundancies can be then used for free in the derived calculi. In many cases, the complexity of the decision procedures obtained this way is optimal. Both in many-valued logics and in more general logics, such as modal logic, intuitionistic logic and generalizations thereof, such embeddings into classical logic allow to use existing efficient theorem provers for first-order logic; there is no need to devise specialized theorem provers for particular non-classical logics.

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A Proof of Theorem 3.3

Proposition A.1 *For all concept descriptions C_1, C_2, and every TBox $\mathcal{T}$, $C_1 \sqsubseteq_{\mathcal{T}} C_2$ if and only if* $\mathsf{BAO}_{N_R} \models (\bigwedge_{A \equiv C \in \mathcal{T}} A = \overline{C}) \to \overline{C_1} \leq \overline{C_2}$.

Proof: This follows from the fact that every algebra in BAO_{N_R} homomorphically embeds into a Boolean algebra of sets. □

Lemma A.2 *Every semilattice* $\mathbf{S} \in \mathsf{SLO}^{\exists}_{N_R}$ *embeds into a lattice in* $\mathsf{DLO}^{\exists}_{N_R}$.

Proof: Let $\mathbf{S} = (S, \wedge, 0, 1, \{f_S\}_{f \in \Sigma})$ be a semilattice with 0, 1, and with monotone operators in Σ. Let $\mathcal{OI}^*(\mathbf{S}) = (\mathcal{OI}^*(\mathbf{S}), \cap, \cup, \{0\}, S, \{\overline{f}_S\}_{f \in \Sigma})$ be the lattice of all non-empty order-ideals of S, where join is set union, meet is set intersection, and the additional operators in Σ are defined, for every non-empty order ideal of S, U, by $\overline{f}_S(U) = \downarrow f_S(U)$.

It is easy to see that for every $f \in \Sigma$, $\overline{f}_S(\{0\}) = \downarrow f_S(\{0\}) = \{0\}$ and, for every $U_1, U_2 \in \mathcal{OI}^*(\mathbf{S})$, $\overline{f}_S(U_1 \cup U_2) = \overline{f}_S(U_1) \cup \overline{f}_S(U_2)$. Obviously, $(\mathcal{OI}^*(\mathbf{S}), \cap, \cup, \{0\}, S)$ is a bounded distributive lattice. Thus, $\mathcal{OI}(\mathbf{S}) \in \mathsf{DLO}^{\exists}_{N_R}$.

Let $\eta : \mathbf{S} \to \mathcal{OI}^*(\mathbf{S})$ defined by $\eta(x) := \downarrow x$. Obviously, $\eta(0) = \{0\}, \eta(1) = S$ and $\eta(x \wedge y) = \downarrow(x \wedge y) = \downarrow x \cap \downarrow y$. We show that $\eta(f_S(x)) = \downarrow f_S(x) = \overline{f}_S(\downarrow x)$. If $y \in \downarrow f_S(x)$ then $y \leq f_S(x)$, so $y \in \downarrow\{f_S(\downarrow x)$. Conversely, if $y \in \downarrow\{f_S(\downarrow x)$ then $y \leq f_S(z)$ for some $z \leq x$, hence, by the monotonicity of f_S, $y \leq f_S(x)$.) Thus, η is a homomorphism with respect to the whole signature of $\mathbf{S}$. □

Lemma A.3 *Every semilattice* $\mathbf{S} \in \mathsf{SLO}^{\forall}_{N_R}$ *embeds into a lattice in* $\mathsf{DLO}^{\forall}_{N_R}$.

Proof: Let $\mathbf{S} = (S, \wedge, 1, \{f_S\}_{f \in \Sigma})$ be a semilattice with 1, with operators in Σ such that f_S a meet homomorphism for every $f \in \Sigma$. Let $\mathcal{OI}(\mathbf{S}) = (\mathcal{OI}(\mathbf{S}), \cap, \cup, \emptyset, S, \{\overline{f}_S\}_{f \in \Sigma})$ be the lattice of all order-ideals of S, where join is set union, meet is set intersection, and the additional operators in Σ are defined, for every non-empty order ideal of S, U, by $\overline{f}_S(U) = \downarrow f_S(U)$.

It is easy to see that for every $f \in \Sigma$, $\overline{f}_S(S) = \downarrow f_S(S) = S$ (since $f_S(1) = 1$). We show that if f is a meet hemimorphism then for every $U_1, U_2 \in \mathcal{OI}(\mathbf{S})$, $\overline{f}_S(U_1 \cap U_2) = \overline{f}_S(U_1) \cap \overline{f}_S(U_2)$. The direct inclusion is obvious. In order to prove the converse inclusion, let $x \in \overline{f}_S(U_1) \cap \overline{f}_S(U_2)$. Then there exist $y_1 \in U_1$ and $y_2 \in U_2$ such that $x \leq f_S(y_1)$ and $x \leq f_S(y_2)$ Then $x \leq f_S(y_1) \wedge f_S(y_2) = f_S(y_1 \wedge y_2)$ (since f_S is a meet hemimorphism). Let $y = y_1 \wedge y_2$. Then $y \leq y_i$ for $i = 1, 2$, so $y \in U_1 \cap U_2$. This shows that $x \leq f_S(y)$, with $y \in U_1 \cap U_2$, so $x \in \overline{f}_S(U_1 \cap U_2)$.

The fact that $\eta : \mathbf{S} \to \mathcal{OI}(\mathbf{S})$ defined by $\eta(x) := \downarrow x$ is a homomorphism with respect to the whole signature of $\mathbf{S}$ can be proved as before. □

Lemma A.4 *Every bounded distributive lattice with join (meet) hemimorphisms in Σ homomorphically embeds into a Boolean algebra with join (meet) hemimorphisms in Σ.*

Proof: Consequence of results of Priestley duality for distributive lattices and Stone duality for Boolean algebras. □

Proposition A.5 *Assume that the only concept constructors are intersection and existential restriction. For all concept descriptions C_1, C_2, and every TBox $\mathcal{T}$, $C_1 \sqsubseteq_{\mathcal{T}} C_2$ if and only if* $\mathsf{SLO}^{\exists}_{N_R} \models (\bigwedge_{A \equiv C \in \mathcal{T}} A = \overline{C}) \to \overline{C_1} \leq \overline{C_2}$.

Proof: Assume that $C_1 \sqsubseteq_{\mathcal{T}} C_2$. Then, by Proposition A.1, $\mathsf{BAO}_{N_R} \models (\bigwedge_{A \equiv C \in \mathcal{T}} A = \overline{C}) \to \overline{C_1} \leq \overline{C_2}$. Let $\mathbf{S} = (S, \wedge, 0, 1, \{f_{\exists R}\}_{R \in N_R}) \in SLO^{\exists}_{N_R}$. Then there exists an injective bounded semilattice homomorphism $\eta : \mathbf{S} \to \mathcal{OI}^*(\mathbf{S})$, and a lattice homomorphic embedding h of $\mathcal{OI}^*(\mathbf{S})$ into a Boolean algebra with operators $\mathbf{B}$. Let $v : N_C \to \mathbf{S}$ be an arbitrary valuation in $\mathbf{S}$ such that $v(A) = \overline{v}(C)$ for every $A \equiv C \in \mathcal{T}$. Then $h \circ \eta \circ v : N_C \to \mathbf{B}$ is an assignment into an algebra in BAO_{N_R} with $h(\eta(v(A))) = \overline{h \circ \eta \circ v}(C)$ for every $A \equiv C \in \mathcal{T}$. So, $h(\eta(\overline{v}(C_1 \wedge C_2))) = \overline{h \circ \eta \circ v}(C_1 \wedge C_2) = \overline{h \circ \eta \circ v}(C_1) = h(\eta(\overline{v}(C_1)))$, so, by the injectivity of $h \circ \eta$, $\overline{v}(C_1 \wedge C_2) = \overline{v}(C_1)$.

The converse follows immediately from the fact that the reduct of every algebra in BAO_{N_R} to the signature of the problem is in $SLO^{\exists}_{N_R}$. □

Proposition A.6 *Assume that the only concept constructors are intersection and universal restriction. For all concept descriptions C_1, C_2, and every TBox $\mathcal{T}$, $C_1 \sqsubseteq_{\mathcal{T}} C_2$ if and only if* $\mathsf{SLO}^{\forall}_{N_R} \models (\bigwedge_{A \equiv C \in \mathcal{T}} A = \overline{C}) \to \overline{C_1} \leq \overline{C_2}$.

Proof: Assume that $C_1 \sqsubseteq_{\mathcal{T}} C_2$. Then, by Proposition A.1, $\mathsf{BAO}_{N_R} \models (\bigwedge_{A \equiv C \in \mathcal{T}} A = \overline{C}) \to \overline{C_1} \leq \overline{C_2}$. Let $\mathbf{S} = (S, \wedge, 1, \{f_{\forall R}\}_{R \in N_R}) \in SLO^{\forall}_{N_R}$. Then there exists an injective bounded semilattice homomorphism $\eta : \mathbf{S} \to \mathcal{OI}(\mathbf{S})$, and a lattice homomorphic embedding h of $\mathcal{OI}(\mathbf{S})$ into a Boolean algebra with operators $\mathbf{B}$. Let $v : N_C \to \mathbf{S}$ be an arbitrary valuation in $\mathbf{S}$ such that $v(A) = \overline{v}(C)$ for every $A \equiv C \in \mathcal{T}$. Then $h \circ \eta \circ v : N_C \to \mathbf{B}$ is an assignment into an algebra in BAO_{N_R} with $h(\eta(v(A))) = \overline{h \circ \eta \circ v}(C)$ for every $A \equiv C \in \mathcal{T}$. So, $h(\eta(\overline{v}(C_1 \wedge C_2))) = \overline{h \circ \eta \circ v}(C_1 \wedge C_2) = \overline{h \circ \eta \circ v}(C_1) = h(\eta(\overline{v}(C_1)))$, so, by the injectivity of $h \circ \eta$, $\overline{v}(C_1 \wedge C_2) = \overline{v}(C_1)$.

The converse follows immediately from the fact that the reduct of every algebra in BAO_{N_R} to the signature of the problem is in $SLO^{\forall}_{N_R}$. □

B Proof of Theorem 3.4

Theorem B.1 *The uniform word problem for* BAO_{N_R} *is* EXPTIME*-complete.*

Proof: (Sketch) A resolution-based exponential time algorithm for the uniform word problem for BAO_{N_R} is obtained for instance in [SS03b]. EXPTIME-hardness (even for word problems which contain only conjunction and universal and existential restriction) can be proved using arguments similar to those used in [MGKW02], Theorem 1. □

Lemma B.2 *Every finite partial* $\mathsf{SLO}^{\exists}_{N_R}$*-algebra weakly embeds into a distributive lattice in* $\mathsf{DLO}^{\exists}_{N_R}$.

Proof: Let $\mathbf{P} = (P, \wedge, 0, 1, \{f_{\exists R}\}_{R \in N_R})$ be a partial semilattice. This means that:

(i) $\wedge$ is a partially defined binary operation,

(ii) for every $R \in N_R$, $f_{\exists R}$ is a partially defined unary operation,

(iii) $x \wedge x$ is defined in P for every $x \in P$;
$x \wedge y$ is defined in P iff $y \wedge x$ is defined in P and they are equal;
if $x \wedge y$ is defined in P and $x \wedge (y \wedge x)$ is defined in P then also $(x \wedge y) \wedge z$ is defined in P and $x \wedge (y \wedge x) = (x \wedge y) \wedge z$;

(iv) $f_{\exists R}(0)$ is defined in P and equal to 0 for every $R \in N_R$;
if $x \wedge y$ is defined in P and equals $x \wedge y = x$, and $f_{\exists R}(x)$ and $f_{\exists R}(y)$ are defined in P then $f_{\exists R}(x) \wedge f_{\exists R}(y)$ is defined in P and equals $f_{\exists R}(x)$, for every $R \in N_R$.

We can define a partial order on P by $x \leq y$ iff $x \wedge y$ is defined in P and equals x. Let $\mathcal{OI}(\mathbf{P}) := (\mathcal{OI}(P, \leq), \cap, \{0\}, S, \{\overline{f}_{\exists R}\}_{R \in N_R})$, where union is join, intersection is meet, and the additional operators are defined, for every order ideal of S, U, by $\overline{f}_{\exists R}(U) = \downarrow\{f_{\exists R}(x) \mid f_{\exists R}(x) \text{ defined in } P, x \in U\}$. It is easy to see that $\overline{f}_{\exists R}(\{0\}) = \downarrow\{f_{\exists R}(0)\} = \{0\}$; and $\overline{f}_{\exists R}$ is monotone for every $R \in N_R$.

Let $\eta : \mathbf{P} \to \mathcal{OI}(\mathbf{P})$ be defined by $\eta(x) = \downarrow x$. We show that η is a weak embedding, i.e. it is injective, and whenever $f_P(p_1, \dots, p_n)$ is defined in $\mathbf{P}$, $\eta(f_P(p_1, \dots, p_n)) = \overline{f}(\eta(p_1), \dots, \eta(p_n))$.

- η is obviously injective.

- It is easy to prove that $\eta(1) = S$, $\eta(0) = \{0\}$, and whenever $x \wedge y$ is defined in P, $\eta(x \wedge y) = \eta(x) \cap \eta(y)$.

- Assume that $f_{\exists R}(x)$ is defined in P. Then $\eta(f_{\exists R}(x)) = \downarrow f_{\exists R}(x)$. On the other hand, $\overline{f}_{\exists R}(\downarrow x) = \downarrow\{f_{\exists R}(y) \mid f_{\exists R}(y) \text{ defined in } P, y \leq x\}$. If $y \in \eta(f_{\exists R}(x))$ then $y \leq f_{\exists R}(x)$, so $y \in \overline{f}_{\exists R}(\downarrow x)$. If $y \in \overline{f}_{\exists R}(\downarrow x)$ then $y \leq f_{\exists R}(z)$ for some z such that $f_{\exists R}(z)$ is defined and $z \leq x$ (i.e. such that $z \wedge x$ is defined in P and equals x). But then $f_{\exists R}(z) \wedge f_{\exists R}(x)$ is defined in P and equal to $f_{\exists R}(z)$, so $y \leq f_{\exists R}(z) \leq f_{\exists R}(x)$. Hence $y \in \eta(f_{\exists R}(x))$. This shows that $\eta(f_{\exists R}(x)) = \overline{f}_{\exists R}(\eta(x))$. □

Proposition B.3 *The uniform word problem for* $\mathsf{SLO}^{\exists}_{N_R}$ *is decidable in polynomial time.*

Proof: By a result of Burris [Bur95], a quasivariety $\mathcal{K}$ has a polynomial time decidable word problem if every finite partial algebra which weakly satisfies the (quasi-)identities of $\mathcal{K}$ weakly embeds into a total algebra in $\mathcal{K}$. Lemma B.2 shows that this is the case for $\mathcal{K} = \mathsf{SLO}^{\exists}_{N_R}$. □