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Resolution-based decision procedures for the universal theory of some classes of distributive lattices with operators

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Abstract

We establish a link between the satisfiability of universal sentences with respect to classes of distributive lattices with operators and their satisfiability with respect to certain classes of relational structures. This justifies a method for structure-preserving translation to clause form of universal sentences in such classes of algebras. We show that refinements of resolution yield decision procedures for the universal theory of some such classes. In particular, we obtain exponential space and time decision procedures for the universal clause theory of (i) the class of all bounded distributive lattices with operators satisfying a set of (generalized) residuation conditions, and (ii) the class of all bounded distributive lattices with operators, and a doubly-exponential time decision procedure for the universal clause theory of the class of all Heyting algebras.
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1. Introduction

In this paper we give a method for automated theorem proving in the universal theory of certain classes of distributive lattices with operators. Our interest in such algebras is motivated by the fact that many existing non-classical logics are sound and complete with respect to classes of distributive lattices with operators. Moreover, uniform word problems in lattices also occur in more general contexts such as database dependency theory (Cosmadakis, 1985).

It is known (cf. e.g. Burris and Sankappanavar, 1981, p. 242) that the elementary theory of every non-trivial variety of lattices is undecidable. Thus, the elementary theory of the class DLat of distributive lattices is undecidable. The uniform word problem for distributive lattices is decidable (since $\mathsf{DLat} = ISP(2)$, where 2 is the 2-element lattice), and has been proved to be co-NP-hard by Hunt et al. (1987). Therefore, since

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DLat is closed under direct products, by a result of McKinsey (1943) it follows that the universal theory of DLat is decidable. Rewriting-based approaches to automated theorem proving for the universal Horn theory of distributive lattices seem not to be very efficient, due to the fact that the lattice operators are associative, commutative and idempotent. In Struth (1998) a calculus based on non-symmetric rewriting modulo associativity, commutativity, and idempotency for the elementary theory of finite distributive lattices is given. Besides the possibility of extending this calculus to various families of well-behaved operators on lattices, and the complexity results established for Tarskian set constraints by McAllester et al. (1996) and Mielniczuk and Pacholski (1998), we are not aware of any systematic study on automated theorem proving or decidability and complexity results for the universal (Horn) theory of any classes of distributive lattices with operators.

The main contributions of this paper are the following:

- We establish a link between satisfiability of universal sentences with respect to classes of distributive lattices with operators and satisfiability with respect to corresponding classes of relational structures.
- This is used for giving a structure-preserving translation to clause form of universal sentences in such classes of algebras.
- We show that refinements of resolution yield decision procedures for the universal clause theory of some such classes of algebras. In particular:
  - We show that ordered resolution with selection yields an exponential decision procedure for the universal clause theory of the class $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ of all distributive lattices with operators in Σ that satisfy a set Res of generalized residuation conditions.
  - If the set of residuation conditions is empty, this yields an exponential decision procedure for the universal clause theory of the class DLO_{Σ} of all distributive lattices with operators in Σ. As a by-product, the selection strategy we use allows us to infer that the universal theory of the class DLO_{Σ} is the restriction to the lattice signature of the universal theory of the class BAO_{Σ} of all Boolean algebras with operators in Σ. Similar arguments apply to the class BAO_{Σ}.
  - We use results from Ganzinger et al. (2001) to show that ordered chaining with selection yields a doubly exponential decision procedure for the universal clause theory of the class of Heyting algebras.

In the special case of Boolean algebras with operators, the complexity of the algorithm we describe in this paper agrees with the complexity results established for Tarskian set constraints by McAllester et al. (1996). Actually, the method we present reduces the problem of deciding satisfiability of the universal clause theory of the class BAO_{Σ} to the problem of checking the satisfiability of a family of Tarskian set constraints with relations in Σ, without function symbols and without recursion. The results in Section 5.2 can also be used for deciding satisfiability of Tarskian set constraints with relations in Σ, without function symbols and without recursion.

1.1. Idea

We illustrate the idea of the algorithm we propose on a simple example. Consider the simplification rule (which holds in every distributive lattice) that can be expressed by the following universal Horn formula:

$$\phi = \forall a\ \forall b\ \forall c((a \wedge c = b \wedge c \text{ and } a \vee c = b \vee c) \rightarrow a = b).$$

One possibility for proving that ϕ is true in every bounded distributive lattice is to show that it is a consequence, in equational logic, of the bounded lattice axioms to which the distributivity axiom is added. The run example for the formula ϕ_4 in Section 6.1, Table 3, shows that using equational reasoning may be quite inefficient, even for very simple formulae.

Instead, we use the fact that every bounded distributive lattice L is isomorphic to a sublattice of the lattice of all order filters of a preordered set X_L. As a consequence, we can infer that ϕ holds in every bounded distributive lattice if and only if for every preordered set $(X, \leq)$, ϕ holds for every assignment that replaces its variables with upwards-closed subsets of $(X, \leq)$, if $\vee$ is interpreted as set union and $\wedge$ as set intersection. The last condition is equivalent to the fact that the following family of set constraints is unsatisfiable:

$$\begin{cases} (\mathsf{Dom_s}) & & (X, \leq) \text{ preordered set} \\ (\mathsf{Her_s}) & & x \in I_e, x \leq y \rightarrow y \in I_e \qquad \text{for all } e \in ST(\phi) \\ (\mathsf{Ren_s}) & (\wedge) & I_{a\wedge c} = I_a \cap I_c \qquad I_{b\wedge c} = I_b \cap I_c \\ & (\vee) & I_{a\vee c} = I_a \cup I_c \qquad I_{b\vee c} = I_b \cup I_c \\ (\mathsf{P_s}) & & I_{a\wedge c} = I_{b\wedge c}, \qquad I_{a\vee c} = I_{b\vee c} \\ (\mathsf{N_s}) & & I_a \neq I_b \end{cases}$$

where $ST(\phi)$ is the set of all subterms occurring in ϕ, and for every $e \in ST(\phi)$, I_e is an encoding of e as a (upwards-closed) subset of X. By encoding every set I_e, $e \in ST(\phi)$, by a unary predicate P_e we can reduce the problem of testing the satisfiability of the family of set constraints above to the problem of testing the satisfiability of the following conjunction in first-order logic:

$$\begin{cases} (\mathsf{Dom}) & \forall x & (x \leq x) & \\ & \forall x, y, z & (x \leq y, y \leq z \ \rightarrow \ x \leq z) & \\ (\mathsf{Her}) & \forall x, y & (x \leq y, P_e(x) \ \rightarrow \ P_e(y)) & \text{for all } e \in ST(\phi) \\ (\mathsf{Ren}) & & & \\ \quad (\wedge) & \forall x & (P_{e_1\wedge e_2}(x) \leftrightarrow P_{e_1}(x) \wedge P_{e_2}(x)) & \text{for all } e_1 \wedge e_2 \in ST(\phi) \\ \quad (\vee) & \forall x & (P_{e_1\vee e_2}(x) \leftrightarrow P_{e_1}(x) \vee P_{e_2}(x)) & \text{for all } e_1 \vee e_2 \in ST(\phi) \\ (\mathsf{P}) & \forall x & (P_{a\wedge c}(x) \leftrightarrow P_{b\wedge c}(x)), & \\ & \forall x & (P_{a\vee c}(x) \leftrightarrow P_{b\vee c}(x)) & \\ (\mathsf{N}) & \exists c & (P_a(c) \ \vee \ P_b(c)) & \\ & & (\neg P_a(c) \ \vee \ \neg P_b(c)). & \end{cases}$$

We obtain a structure-preserving translation to first-order logic, and, ultimately, to clause form. The satisfiability of the set of clauses obtained this way can be checked for instance by ordered resolution with selection (cf. also Section 4.2). By using a selection function

which selects all negative literals that contain $\leq$ we can show that the conjunction above is unsatisfiable if and only if (Ren) $\cup$ (P) $\cup$ (N) is unsatisfiable. Furthermore, since no function symbols occur, (Ren) $\cup$ (P) $\cup$ (N) is unsatisfiable if and only if the set of its ground instances (where the variable x is replaced by the constant c) is unsatisfiable.

In this paper we show that similar ideas can be used for many classes of bounded distributive lattices with operators that preserve at least part of the lattice structure. Moreover, we show that refinements of resolution can successfully be used to obtain decision procedures for the universal clause theory of $\mathcal{V}$ for several choices of the class $\mathcal{V}$. (Details are given in Section 3 and several examples are presented in Section 5.)

We first studied this kind of relationship in the context of finitely-valued logics in Sofronie-Stokkermans (1997), and then extended the results to other classes of non-classical logics in Sofronie-Stokkermans (1999b, 2000b). This paper shows that the idea is much more general, and can be used for deciding the universal theory of certain (quasi)varieties of distributive lattices with operators. In particular, the method presented here subsumes in a natural way existing methods for translating modal logics to classical logic (cf. e.g. Ohlbach, 1993; Ohlbach et al., 2001), as well as methods for automated theorem proving in finitely-valued logics based on distributive lattices with operators (Sofronie-Stokkermans, 1997, 2001).

A preliminary version of this paper is Sofronie-Stokkermans (1999a). Here we consider more general operators, e.g. generalizations of various forms of non-classical implication. This allows us to also consider classes of distributive lattices satisfying generalized residuation conditions. Compared with Sofronie-Stokkermans (1999a), we give an improved complexity analysis for the universal clause theory of the class of all bounded distributive lattices with operators.

1.2. Advantages

The main advantage of the method we present here is that it *avoids the explicit use of the full algebraic structure of (distributive) lattices*. The lattice operators, being associative, commutative, and idempotent, are difficult to treat from a computational point of view. Instead, we use *sets* endowed with a reflexive and transitive relation and with additional functions and relations that correspond to the operators in the lattices in a standard way. The lattice operators are thus encoded into *logical conjunction and disjunction*, which can be handled by any automated theorem prover for first-order logic.

Another advantage of the method we propose is that known saturation-based techniques for theories of reflexive and transitive relations, such as ordered chaining with selection, can be used successfully for the problems obtained using the encoding to first-order logic discussed above. Decidability and complexity results can be obtained in many cases by using refinements of resolution, such as ordered resolution or ordered chaining.

Moreover, considerations concerning the structure of the sets of clauses generated with our method and the possible inferences between these clauses make certain algebraic properties of these classes visible.

1.3. Limitations

The applicability of our method depends on the possibility of finding the appropriate relational structures that can replace the algebraic structures in the automated theorem proving process. It is known from modal logic that such structures may not always exist. Another limitation is given by the fact that resolution is a semi-decision procedure for first-order logic, and it may be hard or impossible to obtain resolution-based decision procedures for the classes of clauses generated by the method we describe. However, we show that in many important and rather general cases the method is applicable and leads to decision procedures.

1.4. Related work

1.4.1. Representation theorems

The idea of using representation theorems for establishing a link between the algebraic and relational semantics of non-classical logics goes back to Jónsson and Tarski (1951, 1952), who for this purpose used an extension of Stone's representation theorem for Boolean algebras with operators. Representation theorems for partial orders, semilattices and lattices are used for giving Kripke-style semantics for many substructural logics by Dunn (1993). Our work is influenced by the results of Goldblatt (1989), who showed that the "modal case" can be seen as a simple illustration of more general results from universal algebra. Goldblatt (1989) gives an extension of the Priestley duality to join and meet hemimorphisms, which we extended in Sofronie-Stokkermans (2000a) to various classes of anti(hemi)morphisms.

This paper uses an extension of the representation theorems in Goldblatt (1989) and Sofronie-Stokkermans (2000a) to more general classes of operators. (The representation theorem is presented in Sofronie-Stokkermans, 2003.) We thus show that the use of representation theorems has applications which range far beyond the area of applications in modal logics.

1.4.2. Automated theorem proving

The ideas we present here were initially used to give a method for automated theorem proving in finitely-valued logics based on distributive lattices with operators (Sofronie-Stokkermans, 1997, 2001). There are relationships also with existing methods for translating modal logics to classical logic, and automated theorem proving for modal logics (cf. e.g. Ohlbach, 1993, and the literature cited there).

1.4.3. Set constraints

As already mentioned, there are some natural relationships between the work described here and the problem of checking satisfiability of Tarskian set constraints without function symbols and without recursion. These relationships are explained in Section 6.3.

1.5. Structure of the paper

The paper is structured as follows. In Section 2 we present some results on representation theorems for distributive lattices with operators. In Section 3 we study the link between satisfiability with respect to classes of distributive lattices with operators

and satisfiability with respect to certain relational structures. We show that in many cases the problem of checking satisfiability of a universal clause formula in a class $\mathcal{V}$ of distributive lattices with operators can be reduced to checking the satisfiability of a family of set constraints. This justifies a structure-preserving translation to clause form (in first-order logic, usually without equality). Section 4 contains some generalities on automated theorem proving. Based on this and on the results in Section 3, in Section 5 we give resolution-based decidability proofs for the classes: $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ of all distributive lattices with operators that satisfy a set Res of residuation conditions, DLO_{Σ} of all distributive lattices with operators, and the variety of Heyting algebras. Section 6 contains comparisons with other methods, and Section 7 conclusions and some plans for future work.

2. Representation of distributive lattices with operators

This section discusses representation theorems for distributive lattices with operators.

2.1. Distributive lattices with operators

We assume known notions such as partially-ordered set and order filter in a partially-ordered set. For further information cf. Davey and Priestley (1990).

A structure $(L, \vee, \wedge)$ consisting of a non-empty set L together with two binary operations $\vee$ and $\wedge$ on L is called *lattice* if $\vee$ and $\wedge$ are associative, commutative and idempotent and satisfy the absorption laws. A *distributive lattice* is a lattice that satisfies either of the distributive laws $(D_\wedge)$ or $(D_\vee)$, which are equivalent in a lattice.

$$(D_\wedge) \qquad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$(D_\vee) \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

A lattice $(L, \vee, \wedge)$ has a *first element* if there is an element $0 \in L$ such that $0 \wedge x = 0$ for every $x \in L$; it has a *last element* if there is an element $1 \in L$ such that $1 \wedge x = x$ for every $x \in L$. A lattice having both a first and a last element is called *bounded*. If $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is a bounded lattice we denote by $\mathbf{L}^d$ the order-dual of $\mathbf{L}$, i.e. the lattice $(L, \vee^d, \wedge^d, 0^d, 1^d)$, where for every $x, y \in L$, $x \vee^d y = x \wedge y$, $x \wedge^d y = x \vee y$; $0^d = 1$; and $1^d = 0$. A filter in a lattice $(L, \vee, \wedge)$ is a non-empty order filter closed under meets. A filter F is said to be *prime* if $F \neq L$ and for every $x, y \in L$, if $x \vee y \in F$ then $x \in F$ or $y \in F$. In what follows the set of prime filters of a lattice $\mathbf{L}$ will be denoted by $\mathcal{F}_p(\mathbf{L})$.

2.1.1. Operators on bounded lattices

Bounded lattices with additional operators occur often as algebraic models of non-classical logics. The operations $\vee$ and $\wedge$ model logical disjunction and conjunction; the additional operations are usually interpretations of other logical connectives such as the modal connectives for necessity ($\Box$) or possibility ($\Diamond$), or various types of implication. The operators that correspond to the modal connectives often commute with part of the lattice structure, e.g.

$$\Box(1) = 1, \qquad \Box(x \wedge y) = \Box(x) \wedge \Box(y); \tag{1}$$
$$\Diamond(0) = 0, \qquad \Diamond(x \vee y) = \Diamond(x) \vee \Diamond(y). \tag{2}$$

Definition. A *join hemimorphism*[1] on a bounded lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is a function $f : L^n \to L$ such that for every i, $1 \leq i \leq n$,

(1) $f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) = 0$,
(2) $f(a_1, \ldots, a_{i-1}, b_1 \vee b_2, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, b_1, a_{i+1}, \ldots, a_n) \vee f(a_1, \ldots, a_{i-1}, b_2, a_{i+1}, \ldots, a_n)$.

We want to make the classes of distributive lattices with operators we consider broad enough to also encompass those obtained by considering weakened negations and implications, which satisfy identities such as, for instance:

$$\sim 0 = 1, \qquad \sim(x \vee y) = \sim x \wedge \sim y, \tag{3}$$

$$\sim 1 = 0, \qquad \sim(x \wedge y) = \sim x \vee \sim y; \tag{4}$$

$$(0 \Rightarrow z) = 1, \qquad ((x \vee y) \Rightarrow z) = (x \Rightarrow z) \wedge (y \Rightarrow z), \tag{5}$$

$$(x \Rightarrow 1) = 1, \qquad (x \Rightarrow (y \wedge z)) = (x \Rightarrow y) \wedge (x \Rightarrow z). \tag{6}$$

This means that we need to allow the operators to be hemimorphisms in some arguments, but antihemimorphisms in other arguments. We now formally define operators that have such properties. Similar definitions appear e.g. in Dunn (1993); less general classes of operators, such as join and meet hemimorphisms are defined in Goldblatt (1989) and Sofronie-Stokkermans (2000a).

If $\mathbf{L}$ is a lattice we use the notation $\mathbf{L}^{+1} := \mathbf{L}$ and $\mathbf{L}^{-1} := \mathbf{L}^d$.

Definition. Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice and let $\varepsilon_1, \ldots, \varepsilon_n, \varepsilon \in \{-1, +1\}$. A function $f : L^n \to L$ has type $\varepsilon_1 \ldots \varepsilon_n \to \varepsilon$ if $f : \mathbf{L}^{\varepsilon_1} \times \cdots \times \mathbf{L}^{\varepsilon_n} \to \mathbf{L}^{\varepsilon}$ is a join hemimorphism.

Thus, a join hemimorphism on a bounded lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is an operation of type $+1 \cdots + 1 \to +1$. A *meet hemimorphism* on $\mathbf{L}$ is an operation of type $-1 \cdots - 1 \to -1$, i.e. a function $f : L^n \to L$ such that for every i, $1 \leq i \leq n$:

(1) $f(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n) = 1$,
(2) $f(a_1, \ldots, a_{i-1}, b_1 \wedge b_2, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, b_1, a_{i+1}, \ldots, a_n) \wedge f(a_1, \ldots, a_{i-1}, b_2, a_{i+1}, \ldots, a_n)$.

Example 1. (1) In every bounded distributive lattice, $\wedge$ is a join hemimorphism (consequence of $(D_\wedge)$ and the fact that, for every x, $x \wedge 0 = 0 \wedge x = 0$).
(2) Let $\Box$ and $\diamond$ be operators on a bounded lattice satisfying conditions (1) respectively (2). Then $\diamond$ has type $+1 \to +1$ (it is a join hemimorphism), and $\Box$ has type $-1 \to -1$ (it is a meet hemimorphism).
(3) A bounded lattice homomorphism has type $+1 \to +1$ and $-1 \to -1$.
(4) The unary operation $\sim$ satisfying (3) and (4) has both type $+1 \to -1$ and $-1 \to +1$ (it is a lattice antimorphism).
(5) The operation $\Rightarrow$ satisfying (5) and (6) has type $+1 - 1 \to -1$.
(6) The Boolean implication on a Boolean algebra has type $-1 + 1 \to +1$ and $+1 - 1 \to -1$.

[1] The term "hemimorphism" was introduced by Halmos (1955); the concept was used for the representation of the necessity operator of modal logic.

2.1.2. Heyting algebras

Let $(L, \vee, \wedge)$ be a lattice. For every $a, b \in L$, the *pseudocomplement of a relative to b* (denoted by $a \Rightarrow b$) is the largest element of $\{c \in L \mid a \wedge c \leq b\}$ (if any).

Definition. A *Heyting algebra* is an algebra $(A, 0, 1, \vee, \wedge, \Rightarrow, \neg)$ where $(A, 0, 1, \vee, \wedge)$ is a bounded distributive lattice, for all $a, b \in A$, $a \Rightarrow b$ is the relative pseudo-complement of a with respect to b, and for all $a \in A$, $\neg a := (a \Rightarrow 0)$.

The condition that for all $a, b \in A$, $a \Rightarrow b$ is the relative pseudocomplement of a with respect to b is also expressed by the condition

(H) $a \wedge b \leq c$ if and only if $a \leq b \Rightarrow c$.

2.1.3. General residuation conditions

In some non-classical logics, premise combination is modelled by a binary operation $\circ$ which is, in general, neither idempotent, nor commutative, nor associative. The link between implication and premise combination is often expressed by a residuation rule, that generalizes condition (H) in the definition of Heyting algebras.

Definition. Let $(L, \leq)$ be a partially-ordered set, and let $\circ, \Rightarrow$ be two binary operations on L. $\Rightarrow$ is the *left residuation*[2] of $\circ$ if condition (LR) holds, and the *right residuation* of $\circ$ if condition (RR) holds:

(LR) for all $a, b, c \in L$, $a \circ b \leq c$ if and only if $a \leq b \Rightarrow c$;

(RR) for all $a, b, c \in L$, $b \circ a \leq c$ if and only if $a \leq b \Rightarrow c$.

Let $\mathbf{L} = (L, \vee, \wedge, \circ, \Rightarrow)$ be an algebra with the property that $(L, \vee, \wedge)$ is a lattice, $\circ$ is a binary join hemimorphism, and $\Rightarrow$ is the left residuation of $\circ$. Then $\Rightarrow : \mathbf{L} \times \mathbf{L}^d \to \mathbf{L}^d$ is a join hemimorphism, i.e. $\Rightarrow$ has type $+1 - 1 \to -1$ (cf. e.g. Anderson and Belnap, 1975, p. 358 or Dunn, 1993). Similar conditions hold for right residuation. Condition (RR) can be generalized as follows (cf. also Dunn, 1993).

Definition. Let $\mathbf{L} = (L, \vee, \wedge)$ be a lattice and let f, g be n-ary operators on $\mathbf{L}$ such that f is of type $\varepsilon_1 \ldots \varepsilon_i \ldots \varepsilon_n \to +1$, and g is of type $\varepsilon_1 \ldots \varepsilon_i' \ldots \varepsilon_n \to -1$, with $\varepsilon_i = +1$ and $\varepsilon_i' = -1$. We say that g is an i-residuation[3] associated with f if for all $a_1, \ldots, a_n, a \in L$:

$$f(a_1, \ldots, a_n) \leq a \text{ if and only if } a_i \leq g(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n).$$

In what follows, we will always distinguish subsets of operators which have specific commutation properties with meets and joins, i.e. are lattice homomorphisms, or antimorphisms, or operators of type $\varepsilon_1 \ldots \varepsilon_n \to \varepsilon$. Thus, we will consider *bounded distributive lattices with operators in* $\Sigma = Lh \cup La \cup \bigcup_{\varepsilon_i, \varepsilon \in \{-1,+1\}} \Sigma_{\varepsilon_1 \ldots \varepsilon_n \to \varepsilon}$, i.e. algebras $(A, \vee, \wedge, 0, 1, \{\sigma_A\}_{\sigma \in \Sigma})$ with the property that $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and if $\sigma \in Lh$ then σ_A is a lattice homomorphism, if $\sigma \in La$ then σ_A is a lattice antimorphism, and if $\sigma \in \Sigma_{\varepsilon_1 \ldots \varepsilon_n \to \varepsilon}$ then σ_A is an operation of type $\varepsilon_1 \ldots \varepsilon_n \to \varepsilon$.

[2] Two left (resp. right) residuations of the same operator coincide.

[3] Two i-residuations associated with the same operator coincide.

We denote by D_{01} the class of all bounded distributive lattices; by DLO_Σ the class of all bounded distributive lattices with operators in Σ; by $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ the class of all bounded distributive lattices with operators in Σ satisfying a set Res of residuation conditions; by BAO_Σ the class of all Boolean algebras with operators in Σ; and by HAO_Σ the class of all Heyting algebras with operators in Σ.

2.2. Representation theorems

We present a simplified version of Priestley's representation theorem stating that every bounded distributive lattice is isomorphic to a lattice of sets. For the original representation theorem cf. Priestley (1970) and Davey and Priestley (1990).

Theorem 2.1 (Priestley, 1970). *Let* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *be a bounded distributive lattice, let* $D(\mathbf{L}) = (\mathcal{F}_p(\mathbf{L}), \subseteq)$ *be the partially-ordered set having as points the prime filters of* $\mathbf{L}$*, ordered by inclusion, and let* $\mathcal{H}(D(\mathbf{L}))$ *be the lattice of all upwards-closed subsets of* $D(\mathbf{L})$*. Then* $\eta_L : \mathbf{L} \to \mathcal{H}(D(\mathbf{L}))$*, defined for every* $x \in L$ *by* $\eta_L(x) = \{F \in \mathcal{F}_p(\mathbf{L}) \mid x \in F\}$ *is an injective bounded lattice homomorphism.*

In what follows we will refer to the space $D(\mathbf{L})$ as the (Priestley) dual of $\mathbf{L}$.

The Priestley representation theorem can be extended to Heyting algebras (Priestley, 1984; Goldblatt, 1989). Operators on a bounded distributive lattice or Heyting algebra $\mathbf{L}$ induce in a canonical way maps respectively relations on $D(\mathbf{L})$ as shown below (if $F \in \mathcal{F}_p(\mathbf{L})$, we use the notation $F^{+1} := F$ and $F^{-1} := L \backslash F$):

Operation of $\mathbf{L}$	Corresponding map/relation of $D(\mathbf{L})$
$h : \mathbf{L} \to \mathbf{L}$, homomorphism	$H_h : D(\mathbf{L}) \to D(\mathbf{L})$, $\quad H_h(F) = h^{-1}(F)$;
$k : \mathbf{L} \to \mathbf{L}$, antimorphism	$H_k : D(\mathbf{L}) \to D(\mathbf{L})$, $\quad H_k(F) = L \backslash k^{-1}(F)$
f of type $\varepsilon_1 \dots \varepsilon_n \to \varepsilon$	$R_f \subseteq D(\mathbf{L})^{n+1}$
$\varepsilon_1, \dots, \varepsilon_n, \varepsilon \in \{-1, +1\}$	$R_f(F_1, \dots, F_n, F)$ iff $f(F_1^{\varepsilon_1}, \dots, F_1^{\varepsilon_1}) \subseteq F^{\varepsilon}$

If $\mathbf{L} \in \mathsf{DLO}_\Sigma$, then we will denote by $D(\mathbf{L})$ the space $(\mathcal{F}_p(\mathbf{L}), \subseteq, \{R_f\}_{f \in \Sigma})$, where for every $f \in \Sigma$, R_f is defined as explained above.

Proposition 2.1. *Let* $\mathbf{L}$ *be a bounded distributive lattice. Then:*

(1) *If* h *is a lattice homomorphism then* H_f *preserves the order* $\subseteq$*;*
if k *is a lattice antimorphism then* H_k *reverses* $\subseteq$*.*
(2) *Let* $f : L^n \to L$ *be an operation of type* $\varepsilon_1 \dots \varepsilon_n \to \varepsilon$*.*
If $\varepsilon = +1$ *then* R_f *is increasing, and if* $\varepsilon = -1$ *then* R_f *is decreasing*[4]*.*
(3) *Let* f *and* g *be operators on* $\mathbf{L}$*, such that* g *is an* i*-residuation of* f*. Then, for every* $F_1, \dots, F_n, F \in \mathcal{F}_p(\mathbf{L})$*,*
$R_f(F_1, \dots, F_{i-1}, F_i, \dots, F_{i+1}, F_n, F)$ *iff* $R_g(F_1, \dots, F_{i-1}, F, F_{i+1}, \dots, F_n, F_i)$*.*

Proof. (1) and (2) are easy consequences of the definitions. (3) Assume that $R_f(F_1, \dots, F_{i-1}, F_i, \dots, F_{i+1}, F_n, F)$, i.e. $f(F_1^{\varepsilon_1}, \dots, F_i, \dots, F_n^{\varepsilon_n}) \subseteq F$. We show that

[4] A relation $R \subseteq X^{n+1}$ is increasing if for every $\overline{x} \in X^n$ (if $R(\overline{x}, y)$ and $y \leq y'$ then $R(\overline{x}, y')$). R is decreasing if for every $\overline{x} \in X^n$ (if $R(\overline{x}, y)$ and $y' \leq y$ then $R(\overline{x}, y')$).

$R_g(F_1, \ldots, F_{i-1}, F, F_{i+1}, \ldots, F_n, F_i)$, i.e. $g(F_1^{\varepsilon_1}, \ldots, L\backslash F, \ldots, F_n^{\varepsilon_n}) \subseteq L\backslash F_i$. Assume, in order to derive a contradiction, that there exist $a_j \in F_j^{\varepsilon_j}$, $j \neq i$, and $a \notin F$ with $a_i = g(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \in F_i$. Then $f(a_1, \ldots, a_i, \ldots, a_n) \leq a$, hence, as $f(F_1^{\varepsilon_1}, \ldots, F_i, \ldots, F_n^{\varepsilon_n}) \subseteq F$, $a \in F$, which is a contradiction. The other implication can be proved analogously. □

Example 2. If $\diamond : L \to L$ has type $+1 \to +1$ then, for all $F_1, F_2 \in \mathcal{F}_p(\mathbf{L})$, $R_\diamond(F_1, F_2)$ if and only if $\diamond(F_1) \subseteq F_2$. If $\Box$ has type $-1 \to -1$ then, for all $F_1, F_2 \in \mathcal{F}_p(\mathbf{L})$, $R_\Box(F_1, F_2)$ if and only if $\Box(L\backslash F_1) \subseteq L\backslash F_2$ (i.e. if and only if for every $x \in L$, $\Box(x) \in F_2$ implies $x \in F_1$).

Proposition 2.1 justifies the definition of *RT Σ-relational structures*.

Definition. An *RT Σ-relational structure* $(X, \leq, \{R_X\}_{R\in\Sigma})$ is a set endowed with a reflexive and transitive relation $\leq$ and with additional maps and relations indexed by Σ, where

- for every $H \in Lh$, $H_X : X \to X$ is a $\leq$-preserving map;
  for every $K \in La$, $K_X : X \to X$ is a $\leq$-reversing map;
- for every $R \in \Sigma_{\varepsilon_1\ldots\varepsilon_n\to+1}$, $R_X \subseteq X^{n+1}$ is increasing;
  for every $R \in \Sigma_{\varepsilon_1\ldots\varepsilon_n\to-1}$, $R_X \subseteq X^{n+1}$ is decreasing.

For every *RT* Σ-relational structure $\mathbf{X} = (X, \leq, \{R_X\}_{R\in\Sigma})$ we denote by $\mathcal{H}(\mathbf{X})$ the set of hereditary (i.e. upwards-closed with respect to $\leq$) subsets of X. $(\mathcal{H}(\mathbf{X}), \cup, \cap, \emptyset, X)$ is a bounded distributive lattice. A relative pseudocomplementation $\Rightarrow$ and a pseudocomplementation $\neg$ can be defined on $\mathcal{H}(\mathbf{X})$ by

$$U \Rightarrow V = \{x \mid \forall y((x \leq y \wedge y \in U) \to y \in V)\}, \qquad \neg U = U \Rightarrow \emptyset. \tag{7}$$

Additional operators on $\mathcal{H}(\mathbf{X})$ can be defined starting from the maps and relations of $\mathbf{X}$ as follows:

Operation/relation of $\mathbf{X}$	Corresponding operator of $\mathcal{H}(\mathbf{X})$
$H : X \to X$, $\leq$-preserving	$h_H : \mathcal{H}(\mathbf{X}) \to \mathcal{H}(\mathbf{X})$, $h_H(U) = H^{-1}(U)$
$K : X \to X$, $\leq$-reversing	$h_K : \mathcal{H}(\mathbf{X}) \to \mathcal{H}(\mathbf{X})$, $h_K(U) = X\backslash K^{-1}(U)$
$R \subseteq X^{n+1}$, increasing	$f_R : \mathcal{H}(\mathbf{X})^n \to \mathcal{H}(\mathbf{X})$
$\varepsilon_1, \ldots, \varepsilon_n \in \{-1, +1\}$	$f_R(U_1, \ldots, U_n) = R^{-1}(U_1^{\varepsilon_1}, \ldots, U_n^{\varepsilon_n})$
$R \subseteq X^{n+1}$, decreasing	$f_R : \mathcal{H}(\mathbf{X})^n \to \mathcal{H}(\mathbf{X})$
$\varepsilon_1, \ldots, \varepsilon_n \in \{-1, +1\}$	$f_R(U_1, \ldots, U_n) = X\backslash R^{-1}(U_1^{\varepsilon_1}, \ldots, U_n^{\varepsilon_n})$

where if $U, U_1, \ldots, U_n \subseteq X$ then $U^{+1} = U$, $U^{-1} = X\backslash U$, $F^{-1}(U) = \{x \mid F(x) \in U\}$ if $F \in \{H, K\}$, and $R^{-1}(U_1, \ldots, U_n) = \{x \mid \exists x_1 \ldots x_n(x_1 \in U_1, \ldots, x_n \in U_n, R(x_1, \ldots, x_n, x))\}$.

Proposition 2.2. *Let* $\mathbf{X}$ *be an RT Σ-relational structure. Then:*

(1) *for every $H \in Lh$, h_H is a lattice homomorphism;*
  for every $K \in La$, k_K is a lattice antimorphism;
(2) *for every $R \in \Sigma_{\varepsilon_1\ldots\varepsilon_n\to\varepsilon}$, f_R is an operation of type $\varepsilon_1 \ldots \varepsilon_n \to \varepsilon$;*

(3) *if* $R \in \Sigma_{\varepsilon_1 \ldots \varepsilon_i \ldots \varepsilon_n \to +1}$, *with* $\varepsilon_i = +1$, $Q \in \Sigma_{\varepsilon_1 \ldots \varepsilon_i' \ldots \varepsilon_n \to -1}$, *with* $\varepsilon_i' = -1$, *and*

$$\forall x_1, \ldots, x_n, x \in X,\ R(x_1, \ldots, x_i, \ldots, x_n, x) \text{ iff } Q(x_1, \ldots, x, \ldots, x_n, x_i),$$

then f_Q *is the* i*-residuation of* f_R.

Proof. (1) and (2) are easy consequences of the definitions. (3) Assume that, for all $x_1, \ldots, x_n, x \in X$, $(R(x_1, \ldots, x_i, \ldots, x_n, x)$ iff $Q(x_1, \ldots, x, \ldots, x_n, x_i))$, and $f_R(U_1, \ldots, U_n) \subseteq U$. Let $x \in U_i$. Assume, in order to derive a contradiction, that $x \notin f_Q(U_1, \ldots, U, \ldots, U_n)$, i.e. (using the definition of f_Q), $x \in Q^{-1}(U_1^{\varepsilon_1}, \ldots, X\backslash U, \ldots, U_n^{\varepsilon_n})$. Then there exist $x_j \in U_j^{\varepsilon_j}$, $j \neq i$, and $x \notin U$, such that $Q(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n, x_i)$. But then $R(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n, x)$, with $x_i \in U_i$, hence $x \in f_R(U_1, \ldots, U_n) \subseteq U$, which is a contradiction. The other implication can be proved analogously. □

Example 3. Let $R \subseteq X \times X$. If R is increasing then $f_R : \mathcal{H}(\mathbf{X}) \to \mathcal{H}(\mathbf{X})$ defined by $f_R(U) = R^{-1}(U) = \{x \in X \mid \exists y(y \in U \text{ and } R(y, x))\}$ is a join hemimorphism[5]. If R is decreasing then $f_R : \mathcal{H}(\mathbf{X}) \to \mathcal{H}(\mathbf{X})$ defined by $f_R(U) = X\backslash R^{-1}(X\backslash U) = \{x \in X \mid \forall y R(y, x) \to y \in U\}$ is an operator of type $-1 \to -1$.

The class of $RT\,\Sigma$-relational structures will be denoted by RT_Σ. For every $\mathbf{X} \in RT_\Sigma$, the algebra associated with X will be denoted $\mathcal{H}(\mathbf{X})$.

Theorem 2.2 (Sofronie-Stokkermans, 2003). *For every* $\mathbf{A} \in \mathsf{DLO}_\Sigma$, *the space* $D(\mathbf{A}) \in RT_\Sigma$, *and* $\eta_A : \mathbf{A} \to \mathcal{H}(D(\mathbf{A}))$ *defined by* $\eta_A(a) = \{F \in \mathcal{F}_p(\mathbf{A}) \mid a \in F\}$ *is an injective homomorphism between algebras in* DLO_Σ.

Similar representation theorems (but for less general classes of operators) appear already in Goldblatt (1989) and Sofronie-Stokkermans (2000a). Analogous results hold for the classes HAO_Σ, and $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$.

3. Subclasses of DLO_Σ and HAO_Σ and their universal theory

This section contains the main result of the paper. We show that the representation theorems discussed before allow us, under certain conditions, to avoid the explicit use of the full algebraic structure of distributive lattices when deciding truth of universal formulae. Instead we use lattices of sets over structures in RT_Σ. This justifies a structure-preserving translation to clause form.

Every universal sentence is equivalent to a conjunction of clauses. Therefore, in what follows we restrict attention to formulae of type

$$\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \to \bigvee_{j=1}^{m} t_{j1} = t_{j2}. \tag{8}$$

[5] The relations we define have the arguments reversed compared to the relations on Kripke models used in modal logic. This is only a syntactical difference.

3.1. A link between algebraic and relational models

We establish a link between truth of universal sentences in classes of distributive lattices with operators and truth in classes of $RT\Sigma$-relational structures. We will consider subclasses $\mathcal{V}$ of DLO_Σ (possibly with a Heyting algebra structure) for which there exists a subclass $\mathcal{K}$ of RT_Σ such that the correspondence

$$\mathsf{DLO}_\Sigma \underset{\mathcal{H}}{\overset{D}{\rightleftarrows}} RT_\Sigma$$

can be restricted to a correspondence $\mathcal{V} \underset{\mathcal{H}}{\overset{D}{\rightleftarrows}} \mathcal{K}$.

This natural requirement can be expressed by the condition:

(K) There exists $\mathcal{K} \subseteq RT_\Sigma$ such that (i) for every $\mathbf{A} \in \mathcal{V}$, $D(\mathbf{A}) \in \mathcal{K}$; (ii) for every $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \in \mathcal{V}$.

The most natural choice for $\mathcal{K}$ would be $\mathcal{K}_D = \{D(\mathbf{L}) \mid \mathbf{L} \in \mathcal{V}\}$, the class of all Priestley duals of algebras in $\mathcal{V}$. For $\mathcal{K}_D$, (K)(i) obviously holds, and if $\mathcal{V}$ is closed under canonical extensions[6], (K)(ii) also holds. However, it may be difficult to give a simple description of $\mathcal{K}_D$. For automated theorem proving it is important to find subclasses of RT_Σ with good theoretical and logic properties, for instance subclasses which are *first-order definable*. This is not always possible, as examples from modal logic show. However such classes can often be obtained by abstracting properties of the Priestley duals of algebras in $\mathcal{V}$.

Lemma 3.1. *Condition (K) holds in the following cases:*

1. $\mathcal{V} = \mathsf{D}_{01}$, *with* $\mathcal{K} = RT$, *the class of all preordered sets.*
   $\mathcal{V} = \mathsf{DLO}_\Sigma$, *with* $\mathcal{K} = RT_\Sigma$.
2. $\mathcal{V} = \mathsf{RDLO}_{\Sigma,\mathsf{Res}}$, *with* $\mathcal{K} = RT_{\Sigma,\mathsf{Res}}$, *the family of all relational structures in* RT_Σ *that satisfy* $C(\mathsf{Res})$, *where*

$$C(\mathsf{Res}) = \{R_f(x_1, \ldots, x_i, \ldots, x_n, x) \text{ iff } R_g(x_1, \ldots, x, \ldots, x_n, x_i) \mid \text{“}g \text{ is a } i\text{-residuation associated with } f\text{”} \in \mathsf{Res}\}.$$

3. $\mathcal{V} = \mathsf{Bool}$, *the variety of all Boolean algebras, with* $\mathcal{K}$ *being the class of all sets (with the discrete order, i.e.* $x \leq y$ *if and only if* $x = y$*).*
   $\mathcal{V} = \mathsf{BAO}_\Sigma$, *with* $\mathcal{K}$ *being the subclass of those structures in* RT_Σ *for which the ordering is discrete.*
4. $\mathcal{V} = \mathsf{H}$, *the variety of all Heyting algebras, with* $\mathcal{K} = RT$.
   $\mathcal{V} = \mathsf{HAO}_\Sigma$, *with* $\mathcal{K} = RT_\Sigma$.

Proof. The maps $D : \mathcal{V} \to \mathcal{K}$ and $\mathcal{H} : \mathcal{K} \to \mathcal{V}$ are the restrictions of the maps $D : \mathsf{DLO}_\Sigma \to RT_\Sigma$ and $\mathcal{H} : RT_\Sigma \to \mathsf{DLO}_\Sigma$ defined in Section 2.2. (K)(i) is in all cases a consequence of Proposition 2.1. (K)(ii) follows as a consequence of Proposition 2.2 for the

[6] The canonical extension of $\mathbf{L} \in \mathsf{DLO}_\Sigma$ is $\mathcal{H}(D(\mathbf{L}))$. Criteria that guarantee that a class of distributive lattices (Heyting algebras, or Boolean algebras) with operators is closed under canonical extensions are given e.g. in Goldblatt (1989), Gehrke and Jónsson (1994), Sofronie-Stokkermans (2000a) and Gehrke and Jónsson (2000).

cases 1, 2, 3. In case 4 we in addition use the fact that the operation $\Rightarrow$ defined on $\mathcal{H}(X)$ as in (7) is a relative pseudocomplementation. □

Proposition 3.1. *Assume that $\mathcal{V}$ satisfies condition (K). Then for every $\phi = \forall x_1, \ldots, x_k(\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^m t_{j1} = t_{j2})$,*

$\mathcal{V} \models \phi$ *if and only if for every* $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \models \phi$.

Proof. The direct implication follows from the fact that, by (K)(ii), for every $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \in \mathcal{V}$; the inverse implication follows from the fact that, by (K)(i), for every $\mathbf{A} \in \mathcal{V}$, the corresponding $RT\Sigma$-relational structure, $D(\mathbf{A})$, is in $\mathcal{K}$, and that, by Theorem 2.2, $\mathbf{A}$ is isomorphic to a subalgebra of $\mathcal{H}(D(\mathbf{A}))$. □

3.2. Embedding into first-order logic

The link between truth in algebraic and in relational models can be further used for obtaining an efficient method for automated theorem proving in the universal fragment of certain classes of distributive lattices with operators. We show that, if a subclass $\mathcal{V}$ of DLO_Σ or HAO_Σ satisfies condition (K) for some first-order definable subclass $\mathcal{K}$ of RT_Σ, then the problem of checking whether a formula of type (8) holds in $\mathcal{V}$ can be reduced to the problem of checking the satisfiability of a set of clauses.

Notation. In what follows, $\Sigma = Lh \cup La \cup \bigcup_{\varepsilon_i \in \{+1,-1\}} \Sigma_{\varepsilon_1 \ldots \varepsilon_n \rightarrow \varepsilon}$. If not explicitly specified otherwise, we use the following conventions: h will denote an operation symbol in Lh, k one in La, and f one in $\Sigma \backslash (Lh \cup La)$. Similarly, for structures in RT_Σ, we will denote by H (resp. K) the map corresponding to operation symbols in Lh (resp. La), and R the relation corresponding to a symbol in $\Sigma \backslash (Lh \cup La)$. All symbols h, k, f, H, K, R occurring in Propositions 3.2 and 3.3, are in the classes corresponding to the labelling in (Ren), and the symbols in the pairs (h, H), (k, K), and (f, R) correspond to the same element of Σ.

3.2.1. Satisfiability and set constraints

Let $\mathcal{K} \subseteq RT_\Sigma$; let $\phi = \forall x_1, \ldots, x_k(\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^m t_{j1} = t_{j2})$ be a formula in the language of DLO_Σ, BAO_Σ or HAO_Σ, and let $ST(\phi)$ be the set of all subterms of s_{il} and t_{jp}, $1 \leq i \leq n$, $1 \leq j \leq m, l, p \in \{1, 2\}$ (including all variables of ϕ, $\mathsf{Var}(\phi)$ and s_{il}, t_{jp} themselves). We show that checking if for every $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \models \phi$ can be reduced to checking the unsatisfiability of a family of set constraints.

Proposition 3.2. *Let $\mathcal{K} \subseteq RT_\Sigma$. The following are equivalent:*

(1) *For every* $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \models \phi$.

(2) *For every* $\mathbf{X} = (X, \leq, \{R\}_{R \in \Sigma}) \in RT_\Sigma$ *and every family of subsets of X indexed by all subterms of ϕ, $\{I_e \subseteq X \mid e \in ST(\phi)\}$, if:*

$$
\begin{cases}
(\mathsf{Dom}_s) & \mathbf{X} \in \mathcal{K}, & & & & \\
(\mathsf{Her}_s) & \forall e \in ST(\phi), & I_e \in \mathcal{H}(\mathbf{X}), & & & \\
 & (1,0) & I_1 = X, & & I_0 = \emptyset, & \\
 & (\wedge) & I_{e_1 \wedge e_2} & = & I_{e_1} \cap I_{e_2}, & \\
 & (\vee) & I_{e_1 \vee e_2} & = & I_{e_1} \cup I_{e_2}, & \\
 & (Lh) & I_{h(e)} & = & H^{-1}(I_e), & \\
(\mathsf{Ren}_s) & (La) & I_{k(e)} & = & X \backslash K^{-1}(I_e), & \\
 & (\Sigma_{\varepsilon_1 \ldots \varepsilon_n \to \varepsilon}) & I_{f(e_1,\ldots,e_n)} & = & (R^{-1}(I_{e_1}^{\varepsilon_1}, \ldots, I_{e_n}^{\varepsilon_n}))^{\varepsilon}, & \\
 & (\Rightarrow) & I_{e_1 \Rightarrow e_2} & = & I_{e_1} \Rightarrow I_{e_2}, & \\
 & (\neg) & I_{\neg e} & = & \neg I_e, & \\
(\mathsf{P}_s) & & I_{s_{i1}} & = & I_{s_{i2}} \qquad \text{for all } i = 1, \ldots, n, &
\end{cases}
$$

$$
\mathit{then} : (\mathsf{C}_s) \qquad \mathit{for\ some}\ j \in \{1, \ldots, m\} I_{t_{j1}} = I_{t_{j2}},
$$

where the rules in (Lh), (La) *and, respectively,* $(\Sigma_{\varepsilon_1 \ldots \varepsilon_n \to \varepsilon})$, *range over all terms in* $ST(\phi)$ *starting with an operator in Lh, La, and, respectively,* $\Sigma_{\varepsilon_1 \ldots \varepsilon_n \to \varepsilon}$. *(We used the abbreviations:* $I^{+1} := I$, $I^{-1} := X \backslash I$, $F^{-1}(U) := \{x \mid F(x) \in U\}$ *for* $F \in Lh \cup La$; *and* $R^{-1}(U_1, \ldots, U_n) := \{x \mid \exists x_1 \in U_1 \ldots \exists x_n \in U_n : R(x_1, \ldots, x_n, x)\}$.)

Proof. (1) $\Rightarrow$ (2). Let $\mathbf{X} = (X, \leq, \{R\}_{R \in \Sigma})$ and $I_e \subseteq X$, $e \in ST(\phi)$ satisfy the conditions $(\mathsf{Dom_s}) \cup (\mathsf{Her_s}) \cup (\mathsf{Ren_s}) \cup (\mathsf{P_s})$ in (2). By $(\mathsf{Dom_s})$, $\mathbf{X} \in \mathcal{K}$; $(\mathsf{Her_s})$ ensures that the map $h : \mathsf{Var}(\phi) \to \mathcal{P}(X)$, defined by $h(x) := I_x$ for each $x \in \mathsf{Var}(\phi)$, has as values upwards-closed sets. The conditions in $(\mathsf{Ren_s})$ and the definitions of the operations in $\mathcal{H}(\mathbf{X})$ imply that for every $e \in ST(\phi)$, $I_e = \overline{h}(e)$, where $\overline{h}$ is the unique homomorphism extending h to terms. Condition $(\mathsf{P_s})$ states that $\overline{h}(s_{i1}) = \overline{h}(s_{i2})$ for all $i = 1, \ldots, n$. Thus, if $(\mathsf{Dom_s}) \cup (\mathsf{Her_s}) \cup (\mathsf{Ren_s}) \cup (\mathsf{P_s})$ hold, then, by (1), $\overline{h}(t_{j1}) = \overline{h}(t_{j2})$ for some $j = 1, \ldots, m$, hence $(\mathsf{C_s})$ holds.

(2) $\Rightarrow$ (1). Let $\mathbf{X} = (X, \leq, \{R\}_{R \in \Sigma}) \in \mathcal{K}$, and let $h : \mathsf{Var}(\phi) \to \mathcal{H}(X)$ be such that $\overline{h}(s_{i1}) = \overline{h}(s_{i2})$ for all $i = 1, \ldots, n$. Then condition $(\mathsf{Dom_s})$ is satisfied and, if $I_e := \overline{h}(e)$ for every $e \in ST(\phi)$, then the conditions in $(\mathsf{Her_s}) \cup (\mathsf{Ren_s}) \cup (\mathsf{P_s})$ are fulfilled too. Hence, for some j, $I_{t_{j1}} = I_{t_{j2}}$, i.e. $\overline{h}(t_{j1}) = \overline{h}(t_{j2})$. □

3.2.2. Structure-preserving translation to clause form

If the class $\mathcal{K}$ is first-order definable, Proposition 3.2 justifies a structure-preserving translation of universal formulae to sets of clauses.

As before, let $\phi = \forall x_1, \ldots, x_k (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \to \bigvee_{j=1}^{m} t_{j1} = t_{j2})$, and let $ST(\phi)$ be the family of all subterms occurring in ϕ.

Proposition 3.3. *Let* $\mathcal{K}$ *be a subclass of* RT_Σ *which is definable by a finite set* C *of first-order sentences. Then the following are equivalent:*

(1) *For every* $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \models \phi$.

(2) *The conjunction of* $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N_1}) \cup \cdots \cup (\mathsf{N_m})$ *is unsatisfiable, where:*

$$
\begin{array}{lrcl}
(\mathsf{Dom}) & C, & & \\
(\mathsf{Her}) & \forall x, y(x \leq y \wedge P_e(x) & \rightarrow & P_e(y)), \\
(\mathsf{Ren}) & & & \\
(1) & \forall x\, P_1(x) & & \\
(0) & \forall x \neg P_0(x) & & \\
(\wedge) & \forall x(P_{e_1 \wedge e_2}(x) & \leftrightarrow & P_{e_1}(x) \wedge P_{e_2}(x)), \\
(\vee) & \forall x(P_{e_1 \vee e_2}(x) & \leftrightarrow & P_{e_1}(x) \vee P_{e_2}(x)), \\
(Lh) & \forall x(P_{h(e)}(x) & \leftrightarrow & P_e(H(x))), \\
(La) & \forall x(P_{k(e)}(x) & \leftrightarrow & \neg P_e(K(x))), \\
(\Sigma_{+1}) & \forall x(P_{f(e_1,\ldots,e_n)}(x) & \leftrightarrow & \exists x_1 \ldots x_n(\bigwedge_{i=1}^{n} P_{e_i}^{\varepsilon_i}(x_i) \wedge R(x_1, \ldots, x_n, x))), \\
(\Sigma_{-1}) & \forall x(P_{g(e_1,\ldots,e_n)}(x) & \leftrightarrow & \forall x_1 \ldots x_n(R(x_1, \ldots, x_n, x) \rightarrow \\
 & & & (\bigvee_{i=1}^{n} P_{e_i}^{-\varepsilon_i}(x_i)))), \\
(\Rightarrow) & \forall x(P_{e_1 \Rightarrow e_2}(x) & \leftrightarrow & \forall y(x \leq y \wedge P_{e_1}(y) \rightarrow P_{e_2}(y))), \\
(\neg) & \forall x(P_{\neg e}(x) & \leftrightarrow & \forall y(x \leq y \rightarrow \neg P_e(y))), \\
(\mathsf{P}) & \forall x \bigwedge_{i=1}^{n}(P_{s_{i1}}(x) & \leftrightarrow & P_{s_{i2}}(x)), \\
(\mathsf{N}_1) & \exists x_1(P_{t_{11}}(x_1) & \nleftrightarrow & P_{t_{12}}(x_1)), \\
\ldots & & \ldots & \\
(\mathsf{N}_m) & \exists x_m(P_{t_{m1}}(x_m) & \nleftrightarrow & P_{t_{m2}}(x_m)),
\end{array}
$$

where the unary predicates P_e are indexed by elements in $ST(\phi)$; the formulae in Σ_{+1} range over all operators $f \in \Sigma_{\varepsilon_1 \ldots \varepsilon_n \rightarrow +1}$, and those in Σ_{-1} range over all operators $g \in \Sigma_{\varepsilon_1 \ldots \varepsilon_n \rightarrow -1}$. For every $e \in ST(\phi)$, we use the abbreviations: $P_e^{+1}(x) := P_e(x)$, and $P_e^{-1}(x) := \neg P_e(x)$.

Proof. (1) $\Rightarrow$ (2). Assume that $\mathcal{H}(\mathbf{X}) \models \phi$ for every $\mathbf{X} \in \mathcal{K}$, and that the conjunction of formulae in $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N}_1) \cup \cdots \cup (\mathsf{N}_m)$ has a model $(X, \leq, \{R\}_{R \in \Sigma}, \{P_e\}_{e \in ST(\phi)})$. By condition Dom, $\mathbf{X} = (X, \leq, \{R\}_{R \in \Sigma}) \in \mathcal{K}$. For every $e \in ST(\phi)$ let $I_e := \{x \in X \mid P_e(x)\}$.

Then $(\mathsf{Dom}_s) \cup (\mathsf{Her}_s) \cup (\mathsf{Ren}_s) \cup (\mathsf{P}_s)$ as defined in Proposition 3.2(2) hold for $\mathbf{X}$, hence, by Proposition 3.2, condition (C_s) holds. As the conjunction $(\mathsf{N}_1) \wedge \cdots \wedge (\mathsf{N}_m)$ corresponds to the negation of (C_s), this is a contradiction.

(2) $\Rightarrow$ (1). Assume that the conjunction of all formulae in $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N}_1) \cup \cdots \cup (\mathsf{N}_m)$ is unsatisfiable. In order to show that for every $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \models \phi$ we use Proposition 3.2. Let $\mathbf{X} = (X, \leq, \{R\}_{R \in \Sigma}) \in \mathcal{K}$, and $\{I_e \mid e \in ST(\phi)\}$ be an arbitrary family of subsets of X such that $(\mathsf{Dom}_s) \cup (\mathsf{Her}_s) \cup (\mathsf{Ren}_s) \cup (\mathsf{P}_s)$ hold. This means that $(X, \leq, \{R\}_{R \in \Sigma}, \{I_e\}_{e \in ST(\phi)})$ is a model of $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P})$, hence it cannot be also a model of $(\mathsf{N}_1) \cup \cdots \cup (\mathsf{N}_m)$. This shows that (C_s) holds. By Proposition 3.2 it follows that for every $\mathbf{X} \in \mathcal{K}$, $\mathcal{H}(\mathbf{X}) \models \phi$. □

Theorem 3.1. *Assume that $\mathcal{V}$ and $\mathcal{K}$ satisfy condition (K), where $\mathcal{K}$ is a class of $RT\Sigma$-structures definable by a finite set C of first-order sentences. The following are equivalent:*

(1) $\mathcal{V} \models \phi$.
(2) *The conjunction of* $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N}_1) \cup \cdots \cup (\mathsf{N}_m)$ *(as defined in Proposition 3.3) is unsatisfiable.*

Proof. Direct consequence of Propositions 3.1–3.3. □

3.3. Examples

We specialize Theorem 3.1 to the classes DLO_Σ, $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ and HAO_Σ.

By Lemma 3.1, the class DLO_Σ satisfies condition (K), with $\mathcal{K} = RT_\Sigma$. The class RT_Σ can be described by a set $RT \cup C_\Sigma$ of formulae, where

- RT expresses the reflexivity and transitivity of $\leq$,
- C_Σ expresses the fact that in every structure in RT_Σ the functions in Lh preserve $\leq$, those in La reverse $\leq$, the relations in $\Sigma_{\varepsilon_1\ldots\varepsilon_n\to+1}$ are increasing and those in $\Sigma_{\varepsilon_1\ldots\varepsilon_n\to-1}$ are decreasing:

$$\begin{array}{ll}
\forall x, y(x \leq y \to H(x) \leq H(y)) & \text{if } H \in Lh,\\
\forall x, y(x \leq y \to K(y) \leq K(x)) & \text{if } K \in La,\\
\forall x_1 \ldots x_n, x, y(x \leq y \wedge R(x_1, \ldots, x_n, x) \to R(x_1, \ldots, x_n, y)) & \text{if } R \in \Sigma_{\varepsilon_1\ldots\varepsilon_n\to+1},\\
\forall x_1 \ldots x_n, x, y(y \leq x \wedge R(x_1, \ldots, x_n, x) \to Q(x_1, \ldots, x_n, y)) & \text{if } R \in \Sigma_{\varepsilon_1\ldots\varepsilon_n\to-1}.
\end{array}$$

The set of clauses $\mathcal{C}_\Sigma(\phi) = RT \cup C_\Sigma \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ generated by translating the conjunction in Proposition 3.3 to clause form is indicated in Table 1.

Corollary 3.1. $\mathsf{DLO}_\Sigma \models \phi$ *if and only if* $\mathcal{C}_\Sigma(\phi)$ *is unsatisfiable.*

By Lemma 3.1, the class $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ satisfies the condition (K), where $\mathcal{K} = RT_{\Sigma,\mathsf{Res}}$, the family of all relational structures in RT_Σ that satisfy the set of clauses $\mathcal{C}_{\mathsf{Res}} = \bigcup_{(f,g)\in\mathsf{Res}} \mathsf{Res}(f, g)$, where for every (f, g) such that the condition that g is the i-residuation associated with f is in Res, $\mathsf{Res}(f, g)$ is the clause form of the formula:

$$\forall x_1, \ldots, x_n, x \qquad R_f(x_1, \ldots, x_i, \ldots, x_n, x) \leftrightarrow R_g(x_1, \ldots, x, \ldots, x_n, x_i).$$

Corollary 3.2. $\mathsf{RDLO}_{\Sigma,\,\mathsf{Res}} \models \phi$ *if and only if* $\mathcal{C}_{\mathsf{RDLO}(\Sigma,\mathsf{Res})}(\phi) := \mathcal{C}_\Sigma(\phi) \cup \mathcal{C}_{\mathsf{Res}}$ *is unsatisfiable.*

By Lemma 3.1, HAO_Σ fulfils condition (K) with $\mathcal{K} = RT_\Sigma$, i.e.

(i) for every $A \in \mathsf{HAO}_\Sigma$, $D(A) \in RT_\Sigma$; and
(ii) for every $X \in RT$, $(\mathcal{H}(X), \cup, \cap, \emptyset, X, \{f_R\}_{R\in\Sigma}, \Rightarrow, \neg) \in \mathsf{HAO}_\Sigma$,

where $\neg$ and $\Rightarrow$ are defined for every $U, V \in \mathcal{H}(X)$ by:

$$U \Rightarrow V = \{x \in X \mid \forall y(x \leq y \text{ and } y \in U) \text{ imply } y \in V\};$$
$$\neg U = U \Rightarrow \emptyset.$$

Let $\mathcal{C}_{\Rightarrow,\neg}(\phi)$ be the clause forms of the rules in $\mathsf{Ren}(\neg, \Rightarrow)(\phi)$, namely:

(Ren) ($\neg$)	$\{\neg P_{\neg e}(x), \neg x \leq y, \neg P_e(y)\}$, $\{x \leq c^{\neg e}(x), P_{\neg e}(x)\}$, $\quad \{P_e(c^{\neg e}(x)), P_{\neg e}(x)\}$
(Ren)($\Rightarrow$)	$\{\neg P_{e_1\Rightarrow e_2}(x), \neg x \leq y, \neg P_{e_1}(y), P_{e_2}(y)\}$ $\{x \leq c^{e_1\Rightarrow e_2}(x), P_{e_1\Rightarrow e_2}(x)\}$, $\{P_{e_1}(c^{e_1\Rightarrow e_2}(x)), P_{e_1\Rightarrow e_2}(x)\}$, $\quad \{\neg P_{e_2}(c^{e_1\Rightarrow e_2}(x)), P_{e_1\Rightarrow e_2}(x)\}$

where $\neg e, e_1 \Rightarrow e_2 \in ST(\phi)$ and $c^{\neg e}, c^{e_1\Rightarrow e_2}$ are Skolem functions.

Table 1
The set of clauses in $C_{\Sigma}(\phi)$

(Dom)	Clause form of the formulae in C_{Σ},
(R)	$\{x \leq x\}$ reflexivity
(T)	$\{\neg x \leq y, \neg y \leq z, x \leq z\}$ transitivity
(Her)	$\{\neg x \leq y, \neg P_e(x), P_e(y)\}$,
(Ren)	
(1)	$\{P_1(x)\}$,
(0)	$\{\neg P_0(x)\}$,
$(\wedge)$	$\{\neg P_{e_1 \wedge e_2}(x), P_{e_i}(x)\}, i = 1, 2,$
	$\{\neg P_{e_1}(x), \neg P_{e_2}(x), P_{e_1 \wedge e_2}(x)\}$,
$(\vee)$	$\{\neg P_{e_1 \vee e_2}(x), P_{e_1}(x), P_{e_2}(x)\}$,
	$\{\neg P_{e_1}(x), P_{e_1 \vee e_2}(x)\}, i = 1, 2,$
(Lh)	$\{\neg P_{h(e)}(x), P_e(H(x))\}$,
	$\{P_{h(e)}(x), \neg P_e(H(x))\}$,
(La)	$\{P_{k(e)}(x), P_e(K(x))\}$,
	$\{\neg P_{k(e)}(x), \neg P_e(K(x))\}$,
$(\Sigma_{\varepsilon_1 \ldots \varepsilon_n \to 1})$	
(1)	$\{\neg P_{f(e_1,\ldots,e_n)}(x), P_{e_i}^{\varepsilon_i}(c_i^{f(e_1,\ldots,e_n)}(x))\}, i = 1, \ldots, n,$
(2)	$\{\neg P_{f(e_1,\ldots,e_n)}(x), R(c_1^{f(e_1,\ldots,e_n)}(x), \ldots, c_n^{f(e_1,\ldots,e_n)}(x), x)\}$,
(3)	$\{P_{f(e_1,\ldots,e_n)}(x), \neg P_{e_1}^{\varepsilon_1}(y_1), \ldots, \neg P_{e_n}^{\varepsilon_n}(y_n), \neg R(y_1, \ldots, y_n, x)\}$,
$(\Sigma_{\varepsilon_1 \ldots \varepsilon_n \to -1})$	
(1)	$\{P_{f(e_1,\ldots,e_n)}(x), P_{e_i}^{\varepsilon_1}(c_i^{f(e_1,\ldots,e_n)}(x))\}, i = 1, \ldots, n,$
(2)	$\{P_{f(e_1,\ldots,e_n)}(x), R(c_1^{f(e_1,\ldots,e_n)}(x), \ldots, c_n^{f(e_1,\ldots,e_n)}(x), x)\}$,
(3)	$\{\neg P_{f(e_1,\ldots,e_n)}(x), \neg R(y_1, \ldots, y_n, x), \neg P_{e_1}^{\varepsilon_1}(y_1), \ldots, \neg P_{e_n}^{\varepsilon_n}(y_n)\}$,
(P)	$\{\neg P_{s_{i1}}(x), P_{s_{i2}}(x)\}, \{P_{s_{i1}}(x), \neg P_{s_{i2}}(x)\}, i = 1, \ldots, n,$
(N)	$\{P_{t_{j1}}(c_j), P_{t_{j2}}(c_j)\}, \{\neg P_{t_{j1}}(c_j), \neg P_{t_{j2}}(c_j)\}, j = 1, \ldots, m.$

where – the predicate symbols are indexed by subterms in $ST(\phi)$,
– $c_i^{f(e_1,\ldots,e_n)}, c_i^{g(e_1,\ldots,e_n)}, i = 1, \ldots, n$ are Skolem functions obtained from the existential quantifiers introduced in the transformation of terms of the form $f(e_1, \ldots, e_n)$, and
– $c_1, \ldots, c_m$ are Skolem constants introduced by the existential quantifiers in $(\mathsf{N}_1), \ldots (\mathsf{N}_m)$ in the conjunction in Proposition 3.3. x

Corollary 3.3. *$\mathsf{HAO}_{\Sigma} \models \phi$ if and only if $\mathcal{C}_{\mathsf{HAO}_{\Sigma}}(\phi) := \mathcal{C}_{\Sigma}(\phi) \cup \mathcal{C}_{\Rightarrow,\neg}(\phi)$ is unsatisfiable.*

4. Automated theorem proving: generalities

This section contains the main notions and results on logic and automated theorem proving needed in this paper.

4.1. Clause logic, orderings

We consider (restricted versions of) first-order predicate logic with equality; more specifically first-order languages with function symbols, variables, and predicate symbols. Satisfiability and logical consequence are defined as usual.

An atomic formula (or atom) is an expression of the form $P(t_1, \ldots, t_n)$, where P is a predicate symbol of arity n and $t_1, \ldots, t_n$ are terms. Literals are atomic formulae or negations thereof. Clauses are disjunctions of literals $C = L_1 \vee L_2 \vee \cdots \vee L_n$; they can also be regarded as multisets, written $L_1, \ldots, L_n$ or $\{L_1, \ldots, L_n\}$. We write $E[s]$ to indicate that s is a subterm of E at some position.

Many theorem proving calculi employ orderings to obtain an approximate measure of the progress of a derivation towards a particular goal. In what follows we start with a total, well-founded ordering on ground literals. Any literal ordering $\succ$ can be extended to a clause ordering by taking the multiset extension. If $\succ$ is total (resp. well-founded) so is its multiset extension.

An ordering $\succ$ on ground expressions can be extended to non-ground expressions by defining: $E \succ E'$ if and only if $E\sigma \succ E'\sigma$ for all ground instances $E\sigma$ and $E'\sigma$. Thus, we have $E \not\succeq E'$ if $E'\sigma \succ E\sigma$ for some ground instances $E\sigma$ and $E'\sigma$. We say that a literal L is maximal in a clause C (denoted $L \succeq C$) if $L' \succ L$ for no literal L' in C; and that L is strictly maximal in C (denoted $L \succ C$) if $L' \succeq L$ for no $L' \neq L$ in C.

4.2. Ordered resolution with selection

Let $\succ$ be a total well-founded ordering on ground literals, and let S be an arbitrary selection function that assigns with every clause a multiset of negative selected literals. Let $R_S^{\succ}$ be the following inference system for ground clauses, consisting of ordered resolution and ordered factoring with selection function S:

Ordered resolution:

$$\frac{C \vee A \quad D \vee \neg A}{C \vee D}$$

where (i) A is strictly maximal in $C \vee A$, and C contains no selected atoms; (ii) the literal $\neg A$ is either selected by S in $D \vee \neg A$ or else $D \vee \neg A$ contains no selected literals and $\neg A$ is a maximal negative literal in A.

Ordered (positive) factoring:

$$\frac{C \vee A \vee A}{C \vee A}$$

where A is a positive atom which is maximal in C, and no atom in C is selected.

This calculus can be lifted to non-ground clauses by viewing non-ground expressions to represent the set of their ground instances and by employing unification to avoid the explicit enumeration of ground instances (Bachmair and Ganzinger, 2001).

5. Some decidability results

5.1. Example 1: residuated distributive lattices

We show that ordered resolution with selection provides a decision procedure for the universal clause theory of $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ which is exponential in the size of the input if the

arity of operators in Σ has an upper bound, and exponential in the square of the size of the input in general. The selection strategy we adopt for this purpose shows, as a by-product, that inferences with the clauses containing the $\leq$ symbol are not needed for refutational completeness in this case.

Let $\phi = \forall x_1, \ldots, x_k(\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^{m} t_{j1} = t_{j2})$. By Corollary 3.2, $\mathsf{RDLO}_{\Sigma,\mathsf{Res}} \models \phi$ if and only if $\mathcal{C}_{\mathsf{RDLO}(\Sigma,\mathsf{Res})}(\phi) = \mathcal{C}_{\Sigma}(\phi) \cup \mathcal{C}_{\mathsf{Res}}$ is unsatisfiable.

In the rest of this paper we refer to literals containing the predicate symbol $\leq$ as $\leq$-*literals*; to those containing a predicate symbol in $\Sigma\backslash(Lh \cup La)$ as *R-literals*; and to those containing a predicate symbol of the form P_e, $e \in ST(\phi)$, as *P-literals*.

Let $\succ$ be an ordering which is total and well founded on ground terms and has the property that for all terms u, t, $u[t] \succ t$ (such orderings can always be found cf. e.g. Bachmair and Ganzinger, 2001). Based on $\succ$, an ordering on literals (also denoted by $\succ$) is defined as follows. Let c be the complexity measure defined for every ground literal L by $c_L = (\mathsf{max}_L, \mathsf{pred}_L, p_L)$ where max_L is the maximal term occurring in L; pred_L is the predicate symbol occurring in L; and p_L is 1 if L is negative and 0 if L is positive. Assume that $\succ_P$ is a total order on the predicate symbols $\{P_e \mid e \in ST(\phi)\} \cup \{R \mid R \in \bigcup \Sigma_{\varepsilon_1 \ldots \varepsilon_n \rightarrow \varepsilon}$ and R occurs in $\phi\}$, with the property that $R \succ_P P_e$ for every $R \in \bigcup \Sigma_{\varepsilon_1 \ldots \varepsilon_n \rightarrow \varepsilon}$ and every $e \in ST(\phi)$. The complexity measure c induces a well-founded ordering $\succ_c$ on ground literals, defined by $L \succ_c L'$ if and only if $c_L > c_{L'}$ in the lexicographic combination of $\succ$, $\succ_P$, and $>$ (where $1 > 0$). Let $\succ$ be a total and well-founded extension of $\succ_c$. Let S be the selection function that selects (i) all occurrences of negative $\leq$-literals, and (ii) all occurrences of negative R-literals in clauses which do not contain any $\leq$-literal. The ordered resolution calculus based on the ordering $\succ$ and the selection function S will be denoted $\mathsf{R}_S^{\succ}$. We assume that the following simplification and redundancy elimination rules are applied eagerly: removing repeated literals in clauses, and removing clauses which contain both a literal and its negation.

In what follows, all the clause we consider are simplified with respect to these rules.

We now study all possible inferences in $\mathsf{R}_S^{\succ}$ between clauses in $\mathcal{C}_{\Sigma}(\phi) \cup \mathcal{C}_{\mathsf{Res}}$.

Lemma 5.1. *No inferences in* $\mathsf{R}_S^{\succ}$ *are possible between clauses in* (Dom) ∪ (Her) ∪ (R) *and clauses in* $\mathcal{C}_{\mathsf{Res}}$ ∪ (Ren) ∪ (P) ∪ (N).

Proof. All clauses in (Dom) ∪ (Her) contain a negative, and hence selected, $\leq$-literal. Therefore literals not containing $\leq$ cannot be resolved upon. The reflexivity axiom (R) consists of exactly one positive $\leq$-literal. This shows that no inferences are possible between clauses in (Dom) ∪ (Her) ∪ (R) and clauses in $\mathcal{C}_{\mathsf{Res}}$ ∪ (Ren) ∪ (P) ∪ (N), which contain no $\leq$-literals. □

Lemma 5.2. *All possible* $\mathsf{R}_S^{\succ}$*-resolvents between clauses in* (Dom) ∪ (Her) ∪ (R) *are either tautologies or instances of the reflexivity axiom.*

Proof. All clauses in (Dom) ∪ (Her) contain a negative, and hence selected, $\leq$-literal. Hence, no inferences between clauses in this class are possible. It can easily be checked that all inferences with the reflexivity axiom are tautologies or instances of the reflexivity axiom. □

Proposition 5.1. $\mathsf{RDLO}_{\Sigma,\mathsf{Res}} \models \phi$ *if and only if the set* $\mathcal{C}_{\mathsf{Res}} \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ *is unsatisfiable.*

Proof. Direct consequence of Theorem 3.1, Lemmas 5.1 and 5.2. □

We now analyse the form of the clauses in $\mathcal{C}_{\mathsf{Res}} \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ and their resolvents. If $t, t_1, \ldots, t_n$ are terms, we use the following notations:

$\mathcal{P}(t)$ abbreviates $(\neg)P_1(t) \vee \cdots \vee (\neg)P_m(t)$
$\mathcal{P}(t_1, \ldots, t_n)$ abbreviates $\mathcal{P}(t_1) \vee \cdots \vee \mathcal{P}(t_n)$.

The set $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ contains the following types of clauses:

$\mathcal{G}$	clauses of type $\mathcal{P}(c)$ for some constant c.
$\mathcal{V}$	clauses of type $\mathcal{P}(x)$ for some variable x.
$\mathcal{V}(h)$	clauses of type $\mathcal{P}(x, h(x))$, where $h \in Lh \cup La$.
$\mathcal{V}(f(e_1, \ldots, e_n))$	clauses of type $\mathcal{P}(x, c_1^{f(e_1,\ldots,e_n)}(x), \ldots, c_n^{f(e_1,\ldots,e_n)}(x))$, where $f(e_1, \ldots, e_n) \in ST(\phi)$, $f \in \Sigma \backslash (Lh \cup La)$.
$\mathcal{R}^+$	the clauses in $\Sigma(2) = \bigcup \Sigma_{\varepsilon_1 \ldots \varepsilon_n \to \varepsilon}(2)$
$\mathcal{R}^-$	factors of clauses in $\Sigma(3) = \bigcup \Sigma_{\varepsilon_1 \ldots \varepsilon_n \to \varepsilon}(3)$

with the additional property that the function and predicate symbols occurring in these classes are exactly those occurring in $\mathcal{C}_\Sigma(\phi)$. We denote by $\mathcal{D}(\phi)$ the set of all clauses of type $\mathcal{G}$, $\mathcal{V}$, $\mathcal{V}(h)$, $\mathcal{V}(f(e_1, \ldots, e_n))$, $\mathcal{R}^+$, or $\mathcal{R}^-$.

If $\mathsf{Res} \neq \emptyset$, the arguments of the R-literals might be permuted by inferences with clauses in $\mathcal{C}_{\mathsf{Res}}$. We take this into account and define a larger class of clauses to which all resolvents of clauses in $\mathcal{C}_{\mathsf{Res}} \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ belong.

Let $\mathcal{D}_{\mathsf{Res}}(\phi)$ be the class obtained from $\mathcal{D}(\phi)$ by adding $\mathcal{C}_{\mathsf{Res}}$, and by replacing $\mathcal{R}^+$ with the set of all clauses of type:

$\mathcal{V}(f(e_1, \ldots, e_n), R)$	clauses of the form $\mathcal{P}(x, c_1^{f(e_1,\ldots,e_n)}(x), \ldots, c_n^{f(e_1,\ldots,e_n)}(x)) \vee$ $\vee R(\pi(x, c_1^{f(e_1,\ldots,e_n)}(x), \ldots, c_n^{f(e_1,\ldots,e_n)}(x)))$, where π returns a permutation of its arguments and $f, R \in \Sigma \backslash (Lh \cup La)$

with the additional property that the function and predicate symbols occurring in these classes are exactly those occurring in $\mathcal{C}_\Sigma(\phi)$.

Lemma 5.3. (a) *The set* $\mathcal{F}(\phi)$ *of all clauses of type* $\mathcal{G}$, $\mathcal{V}$, $\mathcal{V}(h)$, $\mathcal{V}(f(e_1, \ldots, e_n))$ *is closed under* $R_S^{\succ}$.

(b) *The set* $\mathcal{D}(\phi)$ *of all clauses of type* $\mathcal{G}$, $\mathcal{V}$, $\mathcal{V}(h)$, $\mathcal{V}(f(e_1, \ldots, e_n))$, $\mathcal{R}^+$, $\mathcal{R}^-$ *is closed under* $R_S^{\succ}$.

(c) *The set* $\mathcal{D}_{\mathsf{Res}}(\phi)$ *of all clauses of the form* $\mathcal{G}$, $\mathcal{V}$, $\mathcal{V}(h)$, $\mathcal{V}(f(e_1, \ldots, e_n))$, $\mathcal{V}(f(e_1, \ldots, e_n), R)$, $\mathcal{R}^-$, *and* $\mathcal{C}_{\mathsf{Res}}$ *(for a given set* Res *of residuation rules) is closed under* $R_S^{\succ}$.

Proof. (a) We show that the set of all clauses in $\mathcal{F}(\phi)$ is closed under $R_S^{\succ}$.

(1) The resolvent of two clauses in $\mathcal{G}$, and the resolvent of a clause in $\mathcal{G}$ and one in $\mathcal{V}$, are again in $\mathcal{G}$. No inferences are possible between clauses in $\mathcal{G}$ and clauses in $\mathcal{V}(h)$, or $\mathcal{V}(f(e_1, \ldots, e_n))$ since no constant c_i is unifiable with a term $f(x)$.

(2) The resolvent of two clauses in $\mathcal{V}$ is again in $\mathcal{V}$.

The resolvent of a clause in $\mathcal{V}$ and one in $\mathcal{V}(h)$ is a clause in $\mathcal{V}$ or in $\mathcal{V}(h)$. Indeed, let $C \vee L$ be a clause in $\mathcal{V}$ and $D \vee L'$ a clause in $\mathcal{V}(h)$, where L, L' are the literals that are resolved upon. By the definition of $\mathcal{V}$, $L = P_e(y)$ or its negation, y is the only variable occurring in $C \vee L$, and no constants, function symbols, or R-literals occur in $C \vee L$. By the ordering restrictions and the definition of $\mathcal{V}(h)$, $L' = P_e(h(x))$ or its negation, x is the only variable occurring in $D \vee L'$, and no constants, no functions different from h, and no R-literals occur in $D \vee L'$. The resolvent is obtained by using the substitution that replaces y with $h(x)$. If the resolvent contains the term $h(x)$, then it is in $\mathcal{V}(h)$, otherwise it is in $\mathcal{V}$.

Similar considerations show that the resolvent of a clause in $\mathcal{V}$ and one in $\mathcal{V}(f(e_1, \ldots, e_n))$ is in $\mathcal{V}$ or $\mathcal{V}(f(e_1, \ldots, e_n))$.

(3) The resolvent of two clauses in $\mathcal{V}(h)$ is in $\mathcal{V}$ or $\mathcal{V}(h)$. Indeed, let $C \vee L$, $D \vee L'$ be two clauses in $\mathcal{V}(h)$. We can assume without loss of generality that $L = P_{e_1}(h(x))$ and $L' = \neg P_{e_2}(h(y))$, x is the only variable occurring in $C \vee L$, y the only variable occurring in $D \vee L'$, and no constants, no functions different from h, and no R-literals occur in $C \vee L$ and $D \vee L'$. Then $P_{e_1} = P_{e_2}$ and the resolvent is obtained by using the substitution that replaces y with x. If the resolvent contains the term $h(x)$, then it is in $\mathcal{V}(h)$, otherwise it is in $\mathcal{V}$.

No inferences are possible between a clause in $\mathcal{V}(h)$ and one in $\mathcal{V}(k)$ if h and k are different, since the maximal literals in the two clauses are $P_{e_1}(h(x))$ and $\neg P_{e_2}(k(y))$ or vice-versa; and $h(x)$ and $k(y)$ are not unifiable. No inferences are possible between a clause in $\mathcal{V}(h)$ and one in $\mathcal{V}(f(e_1, \ldots, e_n))$ for similar reasons.

(4) The resolvent of two clauses in $\mathcal{V}(f(e_1, \ldots, e_n))$ is in $\mathcal{V}(f(e_1, \ldots, e_n))$ or in $\mathcal{V}$. No inferences are possible between a clause in $\mathcal{V}(f(e_1, \ldots, e_n))$ and one in $\mathcal{V}(g(e'_1, \ldots, e'_m))$ if $f(e_1, \ldots, e_n) \neq g(e'_1, \ldots, e'_m)$.

(b) We now show that $\mathcal{D}(\phi)$ is closed under $\mathsf{R}^{\succ}_S$.

We proved that $\mathcal{F}(\phi)$ is closed under $\mathsf{R}^{\succ}_S$. No inferences are possible between two clauses in $\mathcal{R}^+$, since for every clause in $\mathcal{R}^+$ the maximal literal is a positive R-literal and no other literal is selected. No inferences are possible between two clauses in $\mathcal{R}^-$ since all such clauses contain a selected negative R-literal. The resolvent of a clause in $\mathcal{R}^+$ and one in $\mathcal{R}^-$ is a clause in $\bigcup_{f \in \Sigma \setminus (Lh \cup La)} \mathcal{V}(f(e_1, \ldots, e_n))$. Indeed, let $C \vee \neg R(y_1, \ldots, y_n, y)$ be a clause in $\mathcal{R}^-$, and let $D \vee R(c_1^{f(e_1,\ldots,e_n)}(x), \ldots, c_n^{f(e_1,\ldots,e_n)}(x), x)$ one in $\mathcal{R}^+$. Then (i) all terms in C are variables, and $R(y_1, \ldots, y_n, y)$ contains all variables in C; and (ii) the terms in D are among $\{x, c_1^{f(e_1,\ldots,e_n)}(x), \ldots, c_n^{f(e_1,\ldots,e_n)}(x)\}$; and (iii) no R-literal occurs in C or D. The resolvent is obtained by using the substitution that replaces y with x and y_i with $c_i^{f(e_1,\ldots,e_n)}(x)$, for all $1 \leq i \leq n$. The resolvent is therefore of type $\mathcal{V}(f(e_1, \ldots, e_n))$.

As $\mathcal{D}(\phi) = \mathcal{F}(\phi) \cup \mathcal{R}^+ \cup \mathcal{R}^-$ the closure of $\mathcal{D}(\phi)$ under $\mathsf{R}^{\succ}_S$ follows immediately from the fact that no inferences are possible between clauses in $\mathcal{R}^+ \cup \mathcal{R}^-$ and clauses in $\mathcal{F}(\phi)$, since only clauses in $\mathcal{R}^+ \cup \mathcal{R}^-$ contain R-literals and they are either (strictly) maximal or selected.

(c) We show that $\mathcal{D}_{\mathsf{Res}}(\phi)$ is closed under $\mathsf{R}^{\succ}_S$.

No inferences are possible between clauses in $\mathcal{F}\phi$ and clauses in $\mathcal{D}_{\mathsf{Res}}(\phi) \setminus \mathcal{F}\phi$, nor between two clauses in $\bigcup_{f,R} \mathcal{V}(f(e_1, \ldots, e_n), R)$. The resolvent of a clause in $\mathcal{R}^-$ and one

in $\mathcal{V}(f(e_1,\ldots,e_n), R)$ is in $\mathcal{V}(f(e_1,\ldots,e_n))$. The resolvent of a clause in $\mathcal{C}_{\mathsf{Res}}$ and one in $\mathcal{V}(f(e_1,\ldots,e_n), R)$ is in $\bigcup_Q \mathcal{V}(f(e_1,\ldots,e_n), Q)$. No inferences are possible between two clauses in $\mathcal{C}_{\mathsf{Res}} \cup \mathcal{R}^-$ since all such clauses contain a negative (hence, selected) R-literal. □

Lemma 5.4. *If the arity of operators in Σ is bounded then $|\mathcal{F}(\phi)|$, $|\mathcal{D}(\phi)|$, and $|\mathcal{D}_{\mathsf{Res}}(\phi)|$ are exponential in the size of ϕ. Otherwise, $|\mathcal{F}(\phi)|$, $|\mathcal{D}(\phi)|$, and $|\mathcal{D}_{\mathsf{Res}}(\phi)|$ are exponential in the square of the size of ϕ.*

Proof. We first give an upper bound for the number of clauses in $\mathcal{F}(\phi)$. All literals of clauses of the form $\mathcal{P}(c_i)$ have c_i as argument, so $|ST(\phi)|$ possible atoms can occur in such a clause. Hence, there are at most $3^{|ST(\phi)|}$ clauses of type $\mathcal{P}(c_i)$ in $\mathcal{G}$. Since there exist m constants, $c_1,\ldots,c_m$, $\mathcal{G}$ contains at most $m \cdot 3^{|ST(\phi)|}$ clauses. Similar considerations show that (up to renaming of variables) $\mathcal{V}$ contains at most $3^{|ST(\phi)|}$ clauses.

Let $h \in Lh \cup La$. All literals of clauses of type $\mathcal{P}(x, h(x))$ contain as arguments either x or $h(x)$. Hence, $2|ST(\phi)|$ atoms can occur in a clause of this type. Hence, there are at most $3^{2|ST(\phi)|}$ clauses in $\mathcal{V}(h)$ (up to renaming of variables).

Every clause of the form $\mathcal{P}(x, c_1^{f(e_1,\ldots,e_n)}(x),\ldots,c_n^{f(e_1,\ldots,e_n)}(x))$ has only P-literals that have one of $(n+1)$ possible arguments. Hence, $(n+1)\cdot|ST(\phi)| \leq (mp(\phi)+1)\times|ST(\phi)|$ atoms can occur in such a clause, where $mp(\phi)$ is the maximum arity of an operator occurring in ϕ. This shows that, up to renaming of variables, there are at most $3^{(mp(\phi)+1)\cdot|ST(\phi)|}$ clauses in $\mathcal{V}(f(e_1,\ldots,e_n))$. In conclusion,

$$\begin{aligned}|\mathcal{F}(\phi)| &= |\mathcal{G}| + |\mathcal{V}| + \sum_{\substack{h\in Lh\cup La\\ h(e)\in ST(\phi)}} |\mathcal{V}(h)| + \sum_{\substack{f(e_1\ldots e_n)\in ST(\phi)\\ f\in\Sigma\setminus(Lh\cup La)}} |\mathcal{V}(f(e_1\ldots e_n))|\\ &\leq 2|ST(\phi)|\cdot 3^{2|ST(\phi)|} + |ST(\phi)|\cdot 3^{(mp(\phi)+1)\cdot|ST(\phi)|}\\ &\leq 4|ST(\phi)|\cdot 3^{(mp(\phi)+2)\cdot|ST(\phi)|}.\end{aligned}$$

We now give an upper bound for the number of clauses in $\mathcal{D}(\phi)$. The number of clauses in $\mathcal{R}^+$ is equal to the number of clauses in $\Sigma(2)$, i.e. at most $|ST(\phi)|$.The number of all factors of a clause in $\Sigma(3)$, in variables say $\{y_1,\ldots,y_n,x\}$, is bounded by the number of all equivalence relations on $\{y_1,\ldots,y_n\}$, hence it is at most $2^{mp(\phi)^2}$. This shows that, up to renaming of variables,

$$\begin{aligned}|\mathcal{D}(\phi)| = |\mathcal{F}(\phi)| + |\mathcal{R}^+\cup\mathcal{R}^-| &\leq 4|ST(\phi)|\cdot 3^{(mp(\phi)+2)\cdot|ST(\phi)|}\\ &\quad + |ST(\phi)|(1+2^{mp(\phi)^2}).\end{aligned}$$

We now estimate the number of clauses in $\mathcal{D}_{\mathsf{Res}}(\phi)$. The number of clauses in $\mathcal{C}_{\mathsf{Res}}$ is $2\cdot|\mathsf{Res}|$, where $|\mathsf{Res}|$ is the number of residuation rules in Res. We give an upper bound for the number of clauses in $\mathcal{V}(f(e_1,\ldots,e_n), R)$. Note first that all terms occurring in such a clause are of the form $c_1^{f(e_1,\ldots,e_n)}(x),\ldots,c_n^{f(e_1,\ldots,e_n)}(x)$ or x. Hence, at most $(n+1)\cdot ST(\phi)$ P-atoms can be formed with these terms. There are at most $(n+1)! \leq (n+1)^n \leq 2^{mp(\phi)^2}$ possibilities of choosing the arguments of the (unique) positive R-literal

in such a clause. Hence,

$$\left|\bigcup_{f,R\in\Sigma\setminus(Lh\cup La)} \mathcal{V}(f(e_1,\ldots,e_n),R)\right| \leq |ST(\phi)|^2\cdot 3^{(mp(\phi)+1)\cdot|ST(\phi)|}\cdot 2^{mp(\phi)^2}$$
$$\leq |ST(\phi)|^2\cdot 3^{(2mp(\phi)+1)\cdot|ST(\phi)|}.$$

Therefore, up to renaming of variables,

$$|\mathcal{D}_{\mathsf{Res}}(\phi)| = |\mathcal{F}(\phi)| + |\mathcal{R}^-| + \left|\bigcup_{f,R}\mathcal{V}(f(e_1\ldots e_n),R)\right| + |\mathcal{C}_{\mathsf{Res}}|$$
$$\leq 5|ST(\phi)|\cdot 3^{(mp(\phi)+2)\cdot|ST(\phi)|} + |ST(\phi)|^2\cdot 3^{(2mp(\phi)+1)\cdot|ST(\phi)|}$$
$$+2|\mathsf{Res}| \leq 5\cdot 3^{(2mp(\phi)+2)\cdot|ST(\phi)|} + 2\cdot|\mathsf{Res}|.$$

This shows that, if the arity of operators in Σ is bounded then $|\mathcal{F}(\phi)|$, $|\mathcal{D}(\phi)|$, and $|\mathcal{D}_{\mathsf{Res}}(\phi)|$ are exponential in the size of ϕ. If the arity of operators in Σ is not bounded then, for every ϕ, $mp(\phi)$ and $|ST(\phi)|$ are smaller than the size of ϕ, so $|\mathcal{F}(\phi)|$, $|\mathcal{D}(\phi)|$, and $|\mathcal{D}_{\mathsf{Res}}(\phi)|$ are exponential in the square of the size of ϕ. □

Lemma 5.5. *The length of the clauses in $\mathcal{D}(\phi)$ and $\mathcal{D}_{\mathsf{Res}}(\phi)$ is polynomial in the size of ϕ.*

Proof. The length of clauses in $\mathcal{C}_{\mathsf{Res}}$ is 2. The proof of Lemma 5.4 shows that the size of the clauses in $\mathcal{G}$, $\mathcal{V}$, $\mathcal{V}(h)$, $\mathcal{R}^+$ and $\mathcal{R}^-$ is linear in $|ST(\phi)|$, and the size of the clauses in $\mathcal{V}(f(e_1,\ldots,e_n))$ and $\mathcal{V}(f(e_1,\ldots,e_n),R)$ is bounded by $mp(\phi)\cdot|ST(\phi)|$. □

Theorem 5.1. $\mathsf{R}_S^{\succ}$ *decides the universal clause theory of* $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ *in exponential time.*

Proof. By Corollary 3.2, $\mathsf{RDLO}_{\Sigma,\mathsf{Res}} \models \phi$ if and only if $\mathcal{C}_{\mathsf{RDLO}(\Sigma,\mathsf{Res})}(\phi)$ is unsatisfiable. It is easy to see that $|\mathcal{C}_{\mathsf{RDLO}(\Sigma,\mathsf{Res})}(\phi)|$ is linear in the length of ϕ. By Proposition 5.1, $\mathcal{C}_{\mathsf{RDLO}(\Sigma,\mathsf{Res})}(\phi)$ is unsatisfiable if and only if $(\mathsf{Ren})\cup(\mathsf{P})\cup(\mathsf{N})\cup\mathcal{C}_{\mathsf{Res}}$ is unsatisfiable. In Lemma 5.3 we showed that all possible clauses which can be obtained from $(\mathsf{Ren})\cup(\mathsf{P})\cup(\mathsf{N})\cup\mathcal{C}_{\mathsf{Res}}$ by inferences in $\mathsf{R}_S^{\succ}$ are in the set $\mathcal{D}_{\mathsf{Res}}(\phi)$. By Lemma 5.4, the number of clauses in $\mathcal{D}_{\mathsf{Res}}(\phi)$ is exponential in the size of ϕ if there is a bound on the arity of the symbols in Σ, and exponential in the square of the size of ϕ otherwise.

The fact that $\mathsf{R}_S^{\succ}$ decides the universal clause theory of $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$ in exponential time now follows from the fact that ordering and selection constraints can be checked in polynomial time, and every subsumption check is polynomial in the number of clauses. □

Corollary 5.1. $\mathsf{R}_S^{\succ}$ *decides in exponential time the universal clause theory of the class* RD_{01} *of bounded distributive lattices with two binary operators satisfying a residuation condition.*

5.2. Example 2: the class DLO_Σ

Let $\phi = \forall x_1,\ldots,x_k(\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^m t_{j1} = t_{j2})$. By Corollary 3.1, $\mathsf{DLO}_\Sigma \models \phi$ if and only if $\mathcal{C}_\Sigma(\phi)$ is unsatisfiable. This corresponds to the situation when the set Res of residuation conditions is empty. In this case, all clauses in $(\mathsf{Ren})\cup(\mathsf{P})\cup(\mathsf{N})$ and their $\mathsf{R}_S^{\succ}$-resolvents are in the class $\mathcal{D}(\phi)$. If Res is empty, the method presented in

Section 5.1 specializes to a decision procedure for the universal clause theory of DLO_Σ which is exponential in the size of the input if the arity of operators in Σ has an upper bound, and exponential in the square of the size of the input in general. The selection strategy we adopted for this purpose shows, as a by-product (cf. Proposition 5.1), that also in this case inferences with the clauses containing the $\leq$ symbol are not needed for refutational completeness. The following result is a direct corollary of Theorem 5.1.

Theorem 5.2 (Sofronie-Stokkermans, 1999a). $\mathsf{R}_S^{\succ}$ *decides the universal clause theory of* DLO_Σ *in exponential time.*

Similar arguments apply for the class of all Boolean algebras with operators.

Theorem 5.3. $\mathsf{R}_S^{\succ}$ *decides the universal clause theory of* BAO_Σ *in exponential time.*

Corollary 5.2. $\mathsf{R}_S^{\succ}$ *decides in exponential time the universal clause theory of* (i) *the class* D_{01} *of all bounded distributive lattices, and* (ii) *the class* $\mathsf{DLO}_{Lh\cup La}$ *of all bounded distributive lattices with lattice homomorphisms in Lh and antimorphisms in La.*

Corollary 5.3. *For every universal clause formula* ϕ *in the language of* DLO_Σ, $\mathsf{DLO}_\Sigma \models \phi$ *if and only if* $\mathsf{BAO}_\Sigma \models \phi$.

Proof. [7]Let ϕ be a formula in the language of DLO_Σ. By Proposition 5.1, $\mathsf{DLO}_\Sigma \models \phi$ if and only if $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ is unsatisfiable. The class BAO_Σ satisfies condition (K), where $\mathcal{K}$ is the set of all structures in RT_Σ for which the order is discrete. So $\mathsf{BAO}_\Sigma \models \phi$ if and only if $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ is unsatisfiable. □

Proposition 5.2. *Assume that if* e_1 *contains* e_2 *as a subterm then* $P_{e_1} \succ P_{e_2}$. *Let* t *be a term in the language of* D_{01}. *Then* $\mathsf{R}_S^{\succ}$ *decides whether* $\mathsf{D}_{01} \models t = 1$ *in polynomial time.*

Proof. Due to the choice of $\succ$, (Ren) is saturated under $\mathsf{R}_S^{\succ}$. Hence, in a first resolution step the only possible inferences are between $(\mathsf{N}_1)\neg P_t(c)$ and one of the clauses

$$P_{e_1}(x) \wedge P_{e_2}(x) \rightarrow P_{e_1 \wedge e_2}(x) \tag{9}$$

$$P_{e_i'}(x) \rightarrow P_{e_1' \vee e_2'}(x), \qquad i = 1, 2. \tag{10}$$

All the resolvents are ground and negative. The same argument applies for the resolvents, and so on. This shows that $(\mathsf{Ren}) \cup (\mathsf{N}_1)$ is unsatisfiable if and only if $(\mathsf{Ren}_g^H) \cup (\mathsf{N}_1)$, where (Ren_g^H) consists of all instances of clauses in (Ren) of the form (9) or (10) in which the variable x is replaced with c. The conclusion follows from the fact that $(\mathsf{Ren}_g^H) \cup (\mathsf{N}_1)$ is a set of ground Horn clauses with size linear in the length of t. □

5.3. Example 3: Heyting algebras

If $\phi = \forall x_1, \ldots, x_k(\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^m t_{j1} = t_{j2})$ then, by Corollary 3.3, we know that $\mathsf{H} \models \phi$ if and only if the set of clauses $\mathcal{C}_\mathsf{H}(\phi)$ is unsatisfiable, where $\mathcal{C}_\mathsf{H}(\phi)$ is obtained by adjoining to $(\mathsf{RT}) \cup (\mathsf{Her}) \cup (\mathsf{Ren})(0, 1) \cup (\mathsf{Ren})(\wedge) \cup (\mathsf{Ren})(\vee) \cup (\mathsf{P}) \cup (\mathsf{N})$ the clauses corresponding to $(\mathsf{Ren})(\neg)$ and $(\mathsf{Ren})(\Rightarrow)$ (cf. also Table 2).

[7] An algebraic proof of Corollary 5.3 uses the fact that every lattice in DLO_Σ embeds into a Boolean algebra with operators in Σ. (Of course, there exist subclasses of DLO_Σ for which this is not true.)

Table 2
The set of clauses in $\mathcal{C}_{\mathsf{H}}(\phi)$

(R)	$\{x \leq x\}$,	
(T)	$\{\neg x \leq y, \neg y \leq z, x \leq z\}$	
(Her)	$\{\neg x \leq y, \neg P_e(x), P_e(y)\}$,	
(Ren)		
(1)	$\{P_1(x)\}$,	
(0)	$\{\neg P_0(x)\}$,	
($\wedge$)	$\{\neg P_{e_1 \wedge e_2}(x), P_{e_i}(x)\}, i = 1, 2$	$\{\neg P_{e_1}(x), \neg P_{e_2}(x), P_{e_1 \wedge e_2}(x)\}$,
($\vee$)	$\{\neg P_{e_1 \vee e_2}(x), P_{e_1}(x), P_{e_2}(x)\}$,	$\{\neg P_{e_i}(x), P_{e_1 \vee e_2}(x)\}, i = 1, 2$
($\neg$)	$\{\neg P_{\neg e}(x), \neg x \leq y, \neg P_e(y))\}$,	$\{x \leq C_{\neg e}(x), P_{\neg e}(x)\}$,
		$\{P_e(C_{\neg e}(x)), P_{\neg e}(x)\}$,
($\Rightarrow$)	$\{x \leq C_{e_1 \Rightarrow e_2}(x), P_{e_1 \Rightarrow e_2}(x)\}$,	$\{\neg P_{e_1 \Rightarrow e_2}(x), \neg x \leq y, \neg P_{e_1}(y), P_{e_2}(y)\}$,
	$\{P_{e_1}(C_{e_1 \Rightarrow e_2}(x)), P_{e_1 \Rightarrow e_2}(x)\}$,	
	$\{\neg P_{e_2}(C_{e_1 \Rightarrow e_2}(x)), P_{e_1 \Rightarrow e_2}(x)\}$,	
(P)	$\{\neg P_{s_{i1}}(x), P_{s_{i2}}(x)\}$,	$\{P_{s_{i1}}(x), \neg P_{s_{i2}}(x)\}, i = 1, \ldots, n$,
(N)	$\{P_{t_{j1}}(c_j), P_{t_{j2}}(c_j)\}$,	$\{\neg P_{t_{j1}}(c_j), \neg P_{t_{j2}}(c_j)\}, j = 1, \ldots, m$.

where the predicate symbols are indexed by subterms in $ST(\phi)$.

The arguments used in the previous sections cannot be used in this case to show that inferences with clauses in $\mathcal{C}_{\mathsf{H}}(\phi)$ that contain $\leq$-literals are not necessary. Thus, in particular, the reflexivity and transitivity axioms, which are extremely prolific in the context of resolution-based theorem proving, cannot be neglected. In order to compute efficiently with non-symmetric reflexive and transitive relations, Bachmair and Ganzinger (1998) devised the *ordered chaining* calculi. Ordered chaining is a family of calculi, parametrized by total orderings on ground expressions. Some additional properties of the orderings are required to guarantee completeness. Additionally, selection can be used for controlling inferences. Ordered chaining with selection was used to obtain decision procedures for the relational translation of propositional modal logics with modal operators satisfying the axioms D, T or $\mathsf{4}$ in Ganzinger et al. (2001). We show that the results in Ganzinger et al. (2001) can be adapted to yield a doubly exponential decision procedure for the universal Horn theory of the class of Heyting algebras.

Lemma 5.6. *Let* $\phi = \forall x_1 \ldots \forall x_k (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^{m} t_{j1} = t_{j2})$*, and for every* $j \in \{1, \ldots, m\}$ *let* $\phi_j = \forall x_1 \ldots \forall x_k (\bigwedge_{i=1}^{n} s_{ii1} = s_{i2} \rightarrow t_{j1} = t_{j2})$*. Then* $\mathsf{H} \models \phi$ *if and only if there exists* $j \in \{1, \ldots, m\}$ *such that* $\mathsf{H} \models \phi_j$*.*

Proof. McKinsey (1943) showed that for every class $\mathcal{V}$ of Σ-algebras which is closed under direct products, $\mathcal{V} \models \forall x_1 \ldots \forall x_k (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^{m} t_{j1} = t_{j2})$ if and only if there exists $j \in \{1, \ldots, m\}$ such that $\mathcal{V} \models \forall x_1 \ldots \forall x_k (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \rightarrow t_{j1} = t_{j2})$. As H is closed under direct products, the lemma follows immediately. □

We now analyse the form of the clauses in $\mathcal{C}_{\mathsf{H}}(\phi_j)$, where $j \in \{1, \ldots, m\}$ is arbitrary, but fixed. Together with the abbreviations used in Section 5.1 we will also use the following notation. If $\overline{x}$ is the sequence $x_1, \ldots, x_n$, and t a term, then:

$\neg(\overline{x} \leq t)$ abbreviates $\bigvee_{i=1}^{n} \neg(x_i \leq t)$,
$\neg(t \leq \overline{x})$ abbreviates $\bigvee_{i=1}^{n} \neg(t \leq x_i)$.

Let $\mathcal{D}_{\mathsf{H}}(\phi_j)$ be the class of clauses consisting of the reflexivity axiom and all clauses of the form $\mathcal{P}(c_j)$, $\mathcal{P}(x) \vee (x \leq f(x))$ and

$$\mathcal{P}(\overline{x}) \vee \neg(\overline{x} \leq y) \vee \mathcal{P}(\overline{z}) \vee \neg(\overline{z} \leq f(y)) \vee \mathcal{P}(y) \vee \mathcal{P}(f(y)) \tag{11}$$

(containing only function and predicate symbols occurring in $\mathcal{C}_{\mathsf{H}}(\phi_j)$), such that, additionally, for every clause C in the last class of clauses, if x is a variable occurring in a monadic atom $P(x)$ in C and if C contains a (negative) $\leq$-literal, then x occurs in at least one such literal. This is exactly the class of clauses analysed in Ganzinger et al. (2001). (We restrict to clauses which contain only one constant, c_j, because of some technical details in the proofs of Ganzinger et al., 2001.)

Let $C_S^{\succ}$ be the ordered chaining calculus (with ordering $\succ$ and of the selection function S) defined in Ganzinger et al. (2001).

Theorem 5.4 (Ganzinger et al., 2001). (1) *The set $\mathcal{D}_{\mathsf{H}}(\phi_j)$ contains only finitely many condensed clauses (modulo variable renaming). $\mathcal{D}_{\mathsf{H}}(\phi_j)$ is closed under the ordered chaining calculus $C_S^{\succ}$.*

(2) *Any derivation from $\mathcal{C}_{\mathsf{H}}(\phi_j)$ in the ordered chaining calculus $C_S^{\succ}$ with eager condensation terminates, and $\mathcal{C}_{\mathsf{H}}(\phi_j)$ is unsatisfiable if and only if its saturation under $C_S^{\succ}$ (with eager condensation) contains the empty clause.*

The decision procedure in Ganzinger et al. (2001) is doubly exponential[8]. The main reason for the doubly-exponential bound is that clauses of type (11) may be exponentially long in the size of the signature.

Corollary 5.4. *For every formula $\phi = \forall x_1, \ldots, x_k(\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^{m} t_{j1} = t_{j2})$, ordered chaining with eager condensation and selection decides (in at most doubly exponential time and exponential space with respect to the length of ϕ) whether $\mathsf{H} \models \phi$.*

Proof. By Lemma 5.6, $\mathsf{H} \models \phi$ if and only if there exists $j \in \{1, \ldots, m\}$ such that $\mathsf{H} \models \phi_j$. By Theorem 5.4, for every $j \in \{1, \ldots, m\}$, the unsatisfiability of $\mathcal{C}_{\mathsf{H}}(\phi_j)$ can be decided (in at most doubly exponential time and exponential space with respect to the length of ϕ_j) by ordered chaining with selection and eager condensation. □

6. Comparisons

6.1. Comparison with methods based on rewriting

We now present some examples of universal Horn formulae that hold in DLO_{Σ} or $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$. These examples illustrate the type of problems that can be solved with the method described in this paper, and the way this method compares to a more traditional approach, based on rewriting. The unsatisfiability of the resulting sets of clauses was checked by SPASS (Weidenbach et al., 1996).

[8] In Ganzinger et al. (2001) it is actually showed that a more space-economic (single-exponential) representation can be obtained by splitting the clauses into their variable-disjoint regions and connecting them with the help of auxiliary monadic predicates.

1. *Every distributive lattice is modular.*
   Modularity can be expressed by

$$\phi_1 = \forall a\, \forall b\, \forall c (a \leq b \rightarrow a \vee (c \wedge b) = (a \vee c) \wedge b)$$

2. or $\phi_2 = \forall a\, \forall b\, \forall c (a \leq b \rightarrow a \vee (c \wedge b) = (a \vee c) \wedge (a \vee b))$.
3. *In a distributive lattice, complements are unique if they exist.*
   Uniqueness of complements can be expressed by the formula[9]:

$$\phi_3 = \forall a\, \forall b\, \forall c \left(\left\{ \begin{array}{lll} a \vee b = 1 & \& & a \wedge b = 0 \\ a \vee c = 1 & \& & a \wedge c = 0 \end{array} \right\} \rightarrow b = c \right).$$

4. *In every distributive lattice the simplification rule holds.*
   The simplification rule can be expressed by the following formula:

$$\phi_4 = \forall a\, \forall b\, \forall c \left(\left\{ \begin{array}{l} a \wedge b = c \wedge b \\ a \vee b = c \vee b \end{array} \right\} \rightarrow a = c \right).$$

5. *In every bounded distributive lattice endowed with a lattice antimorphism k the following holds (cf. Blyth and Varlet, 1994, p. 77, Example 5.1):*
   If

$$k^2(a) \leq k(a) \vee k(b) \vee k(c), \qquad k^2(a) \leq a \vee k(a),$$
$$k^3(b) \leq k(a) \vee k(b) \vee k(c), \qquad k^3(b) \leq a \vee k(a),$$

   then $k^2(a \vee k(b)) \leq (a \wedge k(b \wedge c)) \vee k(a)$.
   The corresponding formula is:

$$\phi_5 = \forall a\, \forall b\, \forall c \left(\left\{ \begin{array}{l} k^2(a) \leq a \vee k(a) \\ k^3(b) \leq a \vee k(a) \\ k^2(a) \leq k(a) \vee k(b) \vee k(c) \\ k^3(b) \leq k(a) \vee k(b) \vee k(c) \end{array} \right\} \rightarrow \begin{array}{l} k^2(a \vee k(b)) \leq \\ (a \wedge k(b \wedge c)) \vee k(a) \end{array} \right).$$

   (Note: In equational form, ϕ_5 turns out to be quite difficult to prove. As can be seen in Table 3, it is quite easy to prove it with the method we describe in this paper.)
6. *The composition of a unary join antihemimorphism and a lattice antimorphism is a join hemimorphism.*
   The formula expressing that if k is a lattice antimorphism and f is a join antihemimorphism, then $f \circ k$ is a join hemimorphism is

$$\phi_6 = \forall a\, \forall b\, f(k(a \vee b)) = f(k(a)) \vee f(k(b)),$$

   where $f \in Jh$ and $k \in La$.

[9] As a notational convention, the curly brackets and the symbol & indicate that the formulae on the left of the implication symbol $\rightarrow$ are combined conjunctively.

7. *Extended simplification rule for join hemimorphisms.*
   The extended simplification rule for a binary join hemimorphism f can be expressed by the formula:

$$\phi_7 = \forall a\, \forall b\, \forall c\, \forall d \left(\begin{array}{c} \left. \begin{array}{rcl} f(a \vee b, d) & = & f(c \vee b, d) \\ f(a, d) \wedge f(b, d) & = & f(c, d) \wedge f(b, d) \end{array} \right\} \\ \rightarrow\ f(a, d) = f(c, d) \end{array} \right).$$

8. *If there exists a right residuation associated with a binary join hemimorphism, then it is unique.*
   Let $\Sigma = \{\circ, \Rightarrow_1, \Rightarrow_2\}$, and Res consisting of two rules, stating that $\Rightarrow_1, \Rightarrow_2$ are right residuations associated with $\circ$. The fact that $\Rightarrow_1$ and $\Rightarrow_2$ are equal is expressed by the following formula:

$$\phi_8 = \forall a\, \forall b (a \Rightarrow_1 b = a \Rightarrow_2 b).$$

9. *The following formulae hold in every right residuated bounded distributive lattice*

$$\phi_9 = \forall a\, \forall b (b \leq a \Rightarrow (a \circ b)) \qquad \phi_{10} = \forall a\, \forall b (a \circ (a \Rightarrow b) \leq b).$$

In Table 3 we present a comparison between **RT**, the method described in this paper which uses relational structures, and **DLat**, a method that uses equational reasoning (in first-order logic with equality).

The translation to clause form in **RT** used the results in Theorem 3.1 and those in Proposition 5.1, where we showed that inferences with clauses containing $\leq$ are not necessary for DLO_Σ and $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$. In addition, to reduce the number of clauses generated, an inequality $a \leq b$ is not transformed into $a \wedge b = a$ or $a \vee b = b$, but is directly replaced by $\forall x (P_a(x) \rightarrow P_b(x))$.

In **DLat** we simply attempted to prove that the conjunction of the negation of the formulae above and the axioms for bounded distributive lattices with operators is unsatisfiable. Since the distributivity of the lattice can be expressed by either of the axioms $(D_\vee)$ or $(D_\wedge)$, in **DLat** we experimented with three possible axiom systems for bounded distributive lattices, namely;

j:	distributivity was expressed by $D_\vee$;
m:	distributivity was expressed by $D_\wedge$;
b:	distributivity was expressed by both $D_\vee$ and $D_\wedge$.

In both cases we indicate the number of input clauses, number of clauses derived, memory and time needed by SPASS V1.0.

It can be seen that for Examples 1 (j), 2 (j), and 6 (j, m, b), due to the special form of the formulae, a proof was found very quickly with the second approach. Examples 1 and 2 show that, due to some regularities in the form of the formula one of the possibilities **j** or **m** may be more convenient than the other. Which is best seems difficult to decide beforehand. Considering both distributivity axioms does not result in an improved performance. Example 6 suggests that rewriting-based approaches may be more efficient for the equational theory of DLO_Σ. The results above show that, except for very regular and simple formulae, or for equational formulae, the first method, based on the result presented in this paper, behaves better than the second, based on equational reasoning.

Table 3
Some run examples

		RT					DLat			
ϕ	$\mathcal{V}$	cl in	cl der	mem (KB)	Time (ms)		cl in	cl der	mem (KB)	Time (ms)
ϕ_1	D_{01}	15	20	444	30	j	13	1	390	20
						m	13	337	570	80
						b	14	464	618	120
ϕ_2	D_{01}	18	26	450	50	j	13	2	391	30
						m	13	338	572	80
						b	14	464	618	110
ϕ_3	D_{01}	18	9	421	30	j	16	539	643	120
						m	16	539	643	120
						b	17	7054	3719	9550
ϕ_4	D_{01}	18	14	444	30	j	14	3062	2120	2850
						m	14	3397	2325	3530
						b	15	5533	3565	10 250
ϕ_5	DLO_{La}	43	58	471	40	j	20	5506	4624	16 930
						m	20			∞
						b	21			∞
ϕ_6	$\mathsf{DLO}_{La,Ja}$	23	51	461	40	j	18	1	394	20
						m	18	1	393	20
						b	19	0	393	20
ϕ_7	DLO_{Jh}	38	57	498	60	j	22	5725	4411	11 970
						m	22	6445	5370	21 570
						b	23			∞
ϕ_8	$\mathsf{DLO}_{\Sigma,\mathsf{Res}}$	14	16	429	30	j	28	4908	3258	6560
						m	28			∞
						b	29			∞
ϕ_9	RD_{01}	12	7	426	20	j	22	1699	1425	990
						m	22	983	1071	430
						b	23	2898	2054	3090
ϕ_{10}	RD_{01}	12	10	427	20	j	22	3392	2442	3510
						m	22	874	999	350
						b	23	2317	1711	2060

Where "cl in" respectively "cl der" represents the number of input, respectively derived clauses, and ∞ indicates the fact that execution did not terminate after more than 2 min.

6.2. *Comparison with other methods of proving decidability*

The examples in the previous section show that the embedding into first-order logic without equality described in this paper reduces problems which are quite difficult for existing equational theorem provers (such as SPASS or Waldmeister) to very easy problems. We now sketch other possible ways of obtaining similar decidability results, as well as some plans for future work.

6.2.1. Other refinements of resolution

We used resolution with A-orderings (Fermüller et al., 1993) for the type of clauses generated from uniform word problems for the class of all bounded distributive lattices (without any operators) and showed that it provides an exponential time decision procedure. For the class DLO_Σ, inferences with clauses containing $\leq$ are difficult to control in the absence of a selection function. A decision procedure by resolution with A-orderings and selection should be very similar to the one presented in this paper.

6.2.2. Embedding into decidable fragments of first-order logic

Embedding into the class $\mathcal{S}^+$. The set $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ of clauses generated from a uniform word problem in DLO_Σ is in the class $\mathcal{S}^+$ (Fermüller et al., 1993)[10]. In Fermüller et al. (1993, p. 114ff), a resolution-based decision procedure for the class $\mathcal{S}^+$ is given. The decision method is based on a refinement R_m of resolution with A-orderings that uses monadization and splitting. The set $(\mathsf{Ren})\cup(\mathsf{P})\cup(\mathsf{N})$ can therefore be decided by R_m-resolution with splitting. Unfortunately the complexity of the procedure is still rather high.

Embedding into the guarded fragment (GF) (with equality in the guards). Alternatively, for the classes DLO_Σ and $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$, Theorem 3.1 provides an embedding into a fragment of first-order logic that can easily be seen to be a subset of the GF (Andréka et al., 1998). It is easy to see that the formulae obtained from the direct implications in the renaming rules for operators in Σ, and in the inverse implication in the rules for operators of type $\varepsilon_1 \dots \varepsilon_n \to -1$ are in the GF. The inverse implications in the renaming rules for Lh and La are not in the GF, but can be made guarded by adding (always true) guards of the form $x = x$. The same also holds for the renaming rules for operators of type $\varepsilon_1 \dots \varepsilon_n \to 1$. A doubly-exponential decision procedure for the GF with equality, that uses superposition, was given in Ganzinger and de Nivelle (1999). Due to the fact that the clauses we consider are less general, we obtain a better complexity bound than that following from the embedding into the GF.

If the clauses containing the symbol $\leq$ cannot be neglected, as in the case of the class of Heyting algebras (with operators), then the translation in Theorem 3.1 leads to formulae outside the GF. It has actually been proved that in the presence of the transitivity axiom the decidability of the GF is lost: GF with three variables and transitive relations (Grädel, 1999), and even GF (without equality) with two variables and transitive relations (Ganzinger et al., 1999) are undecidable extensions of GF. It has been shown that the GF with transitive guards (an extension of GF in which some relations are transitive, transitive relations appear only in guards and equality may appear everywhere) is decidable and complete for deterministic double exponential time (Szwast and Tendera, 2001). The monotonicity axioms in C_Σ lead also outside this fragment. We hope that, at least in some particular situations, Theorem 3.1 can be used for obtaining an embedding into some decidable extension (yet to be defined) of the GF with transitive guards.

Another possibility would be to try to extend the ideas of Demri and Goré (2000) and de Nivelle (2001), in which a translation of $S4$ into the GF and the two variable fragment

[10] The class $\mathcal{S}^+$ is the class of all clauses C such that for all literals L in C (i) if t is a functional term occurring in C then $\mathsf{Var}(t) = \mathsf{Var}(C)$, and (ii) $|\mathsf{Var}(L)| \leq 1$ or $\mathsf{Var}(L) = \mathsf{Var}(C)$.

is given, so that the use of the transitivity of the relation on the Kripke structures for $S4$ is not needed. Demri used a similar translation for intuitionistic logic (Demri, 2001); we hope that similar ideas can be used for some classes of Heyting algebras with operators.

Translation to first-order monadic logic with equality. An alternative proof of the fact that the set $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ is decidable can be given by using a translation to first-order monadic logic with equality similar to the one used in Bachmair et al. (1993) (for more details we refer e.g. to Sofronie-Stokkermans, 1999a). From the complexity of first-order monadic logic with equality we obtain a NEXPTIME upper bound for the complexity of the universal Horn theory of such varieties, hence higher than the one we obtained by using ordered resolution with selection.

The reduction to the monadic class with equality can also be used if the relations satisfy additional conditions of the form

$$\forall x_1 \ldots \forall x_n (R_f(x_1, \ldots, x_n) \Longrightarrow R_g(x_1, \ldots, x_n))$$
$$\forall x_1 \ldots \forall x_n (R(x_1, \ldots, x_n) \text{ if and only if } R_1(x_1, \ldots, x_n) \vee R_2(x_1, \ldots, x_n))$$
$$\forall x_1 \ldots \forall x_n (R(x_1, \ldots, x_n) \text{ if and only if } R_1(x_1, \ldots, x_n) \wedge R_2(x_1, \ldots, x_n)).$$

Essential for the correctness of the translation is that in all these additional conditions the order in which the variables occur is the same. Hence, the translation to first-order monadic logic cannot be used for handling residuation conditions, since the rules induced on the associated relations involve changes in the order in which the variables occur within R-literals.

6.3. Set constraints

We point out the relationships of our method with methods for proving satisfiability of set constraints. In Section 3.2.1 we showed that the problem of checking whether a universal Horn formula holds in a class $\mathcal{V}$ of distributive lattices with operators can be reduced to the problem of checking a family of set constraints. (Set constraints with functions do not occur in a natural way in the context of this paper. Therefore, also in what follows, all our considerations refer to set constraints without functions.)

6.3.1. Relationships with Herbrand set constraints

Although it occurs in a different context, the structure-preserving translation to clause form established in Proposition 3.3 is, in some sense, similar to that used in Bachmair et al. (1993) for Herbrand set constraints. The difference between the notions of set constraints considered in Bachmair et al. (1993) and this paper is semantic: whereas satisfiability of Herbrand set constraints has to be checked only over the Herbrand universe, the satisfiability of the set constraints we consider in Proposition 3.2 needs to be checked over all possible first-order models. This would make it impossible to give a translation similar to the one in Bachmair et al. (1993) for set constraints with functions.

6.3.2. Relationship with Tarskian set constraints

The notion of Tarskian set constraints (McAllester et al., 1996) provides a generalization and uniform framework that encompasses sets calculi such as those occurring in the context of knowledge representation (concept languages, or terminological languages) and in the context of modal or temporal logics. Tarskian set expressions and

constraints occur in, and are motivated by, the work of Jónsson and Tarski (1951, 1952). The complexity of various types of Tarskian set constraints (with or without functions or recursion) has been studied by McAllester et al. (1996) and Mielniczuk and Pacholski (1998). In particular, they showed that Tarskian set constraints without recursion and without function symbols are EXPTIME complete. Arguments similar to those used in Proposition 3.3 show that checking satisfiability of a family of Tarskian set constraints without recursion and without function symbols can be reduced to checking the satisfiability of a family of first-order sentences. Arguments similar to the ones in Theorem 5.2 can then be used to show that satisfiability of Tarskian set constraints without recursion and without functions can be checked by resolution in EXPTIME.

7. Conclusions and plans for future work

In this paper we presented a resolution-based method for automated theorem proving in the universal theory of certain varieties of distributive lattices with operators. The method is based on extensions of the Priestley representation theorem to distributive lattices with operators. Based on it, we obtained decidability and complexity results for the universal word problem of D_{01}, DLO_{Σ}, $\mathsf{RDLO}_{\Sigma,\mathsf{Res}}$, as well as for the variety of Heyting algebras. The complexity results agree with those established for Tarskian set constraints in McAllester et al. (1996) and Mielniczuk and Pacholski (1998), but the methods we use are different. The fact that the same type of structures are used as relational models for distributive lattices and Heyting algebras (the only difference is the signature) shows that the restriction of the universal theory of Heyting algebras to the signature $\{0, 1, \vee, \wedge\}$ coincides with the universal theory of distributive lattices. This remark is consistent with the remarks in Struth (1998) on the similarity of the cut rules necessary for the calculus for distributive lattices developed there and the cut rules in intuitionistic logic.

These results open a promising field of research that we would like to explore in future work. We expect to be able to use similar ideas for other varieties of distributive lattices or Heyting algebras with operators, and even for considering many-sorted operators $f : L_1 \times \cdots \times L_n \to L$, where $L_1, \ldots, L_n, L$ are distributive lattices. One problem to be solved is to find conditions for such varieties that would give decidability results. It would also be important to find conditions which, given a variety $\mathcal{V}$ of distributive lattices with operators, ensure that a class $\mathcal{K}$ of (first-order definable) relational structures can be found, such that condition (K) is satisfied.

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