Hierarchical and Modular Reasoning in Complex Theories: The Case of Local Theory Extensions

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Abstract. We present an overview of results on hierarchical and modular reasoning in complex theories. We show that for a special type of extensions of a base theory, which we call *local*, hierarchic reasoning is possible (i.e. proof tasks in the extension can be hierarchically reduced to proof tasks w.r.t. the base theory). Many theories important for computer science or mathematics fall into this class (typical examples are theories of data structures, theories of free or monotone functions, but also functions occurring in mathematical analysis). In fact, it is often necessary to consider complex extensions, in which various types of functions or data structures need to be taken into account at the same time. We show how such local theory extensions can be identified and under which conditions locality is preserved when combining theories, and we investigate possibilities of efficient modular reasoning in such theory combinations.

We present several examples of application domains where local theories and local theory extensions occur in a natural way. We show, in particular, that various phenomena analyzed in the verification literature can be explained in a unified way using the notion of locality.

1 Introduction

Many problems in mathematics and computer science can be reduced to proving the satisfiability of conjunctions of literals in a background theory (which can be the extension of a base theory with additional functions – e.g., free, monotone, or recursively defined – or a combination of theories). It is therefore very important to identify situations where reasoning in complex theories can be done efficiently and accurately. Efficiency can be achieved for instance by:

(1) reducing the search space (preferably without loosing completeness);
(2) modular reasoning, i.e., delegating some proof tasks which refer to a specific theory to provers specialized in handling formulae of that theory.

Identifying situations where the search space can be controlled without loss of completeness is of utmost importance, especially in applications where efficient algorithms (in space, but also in time) are essential. To address this problem, essentially very similar ideas occurred in various areas: proof theory and automated deduction, databases, algebra and verification.

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Local inference systems. Possibilities of restricting the search space in inference systems without loss of completeness were studied by McAllester and Givan in [17,21,18]. They introduced so-called "local inference systems", which can be modeled by sets of rules (or sets of Horn clauses N) with the property that for any ground Horn clause G, it is guaranteed that if G can be proved using N then G can already be proved by using only those instances $N[G]$ of N containing only ground terms occurring in G or in N. For local inference systems, validity of ground Horn clauses can be checked in polynomial time. In [5,4], Ganzinger and Basin define a more general notion, *order locality*, and establish links between order-locality and saturation w.r.t. ordered resolution with a special notion of redundancy. These results were used for automated complexity analysis.

The work on local inference systems and local theories can be seen as an extension of ideas which occurred in the study of *deductive databases*. The inference rules of a deductive database are usually of a special form (known as datalog program): typically a set of universal Horn clauses which do not contain function symbols. Any datalog program defines an inference relation for which entailment of ground clauses is decidable in polynomial time [33,34].

Locality and algebra. Similar ideas also occurred in algebra. To prove that the uniform word problem for lattices is decidable in polynomial time, Skolem [26] used the following idea: replace the lattice operations $\vee$ and $\wedge$ by ternary relations $r_\vee$ and $r_\wedge$, required to be functional, but not necessarily total. The lattice axioms were translated to a relational form, by flattening them and then replacing every atom of the form $x \vee y \approx z$ with $r_\vee(x, y, z)$ (similarly for $\wedge$-terms). Additional axioms were added, stating that equality is an equivalence and that the relations are compatible with equality and functional. This new presentation, consisting only of Horn, function-free clauses, can be used for deciding in polynomial time the uniform word problem for lattices. The correctness and completeness of the method relies on the fact that every partially-ordered set (where $\vee$ and $\wedge$ are partially defined) embeds into a lattice. A similar idea was used by Evans in the study of classes of algebras with a PTIME decidable word problem [12]. The idea was extended by Burris [8] to quasi-varieties of algebras. He proved that if a quasi-variety axiomatized by a set $\mathcal{K}$ of Horn clauses has the property that *every finite partial algebra which is a partial model of the axioms in* $\mathcal{K}$ *can be extended to a total algebra model of* $\mathcal{K}$ then the uniform word problem for $\mathcal{K}$ is decidable in polynomial time. In [13], Ganzinger established a link between the proof theoretic notion of locality and embeddability of partial into total algebras. In [14,27] the notion of locality for Horn clauses is extended to the notion of *local extension* of a base theory.

Locality and verification. Apparently independently, similar phenomena were studied in the verification literature, mainly motivated by the necessity of devising methods for efficient reasoning in theories of data structures. In [23], McPeak and Necula investigate local reasoning in pointer data structures, with the goal of efficiently proving invariants in programs dealing with pointers. They present a methodology of specifying shapes of data structures using a class of specifications which they call *local*. It is then shown that the class of local specifications has the property that in order to disprove a (ground) formula, only certain ground instances of the specification are needed, referring to a part of the data structure which is situated in the "neighborhood" of the counterexample. The essence of the method devised by McPeak and Necula is to perform case analysis based on memory writes, and generate facts that must be proved unsatisfiable by using a *finite number of instantiations* of the axioms defining the properties of the data structures. They show that for local specifications finite sets of instances can be found, without loss of completeness. Locality considerations also occur in the study of a theory of arrays by Bradley, Manna and Sipma [6]. They identified a fragment of the theory of arrays for which universal quantification over indices can be replaced by taking a (well-determined) set of ground instances for the index variables. We will show that all these phenomena are instances of a general concept, and present possibilities of recognizing various types of locality of theories and theory extensions.

It is equally important to be able to reason efficiently in complex theories.

Modular reasoning in combinations of theories. Modular methods for checking satisfiability of conjunctions of ground literals in combinations of theories which have *disjoint signatures*, or only share constants are well studied. The Nelson-Oppen combination procedure [24] for instance, can be applied for combining decision procedures of *stably infinite* theories over disjoint signatures. Resolution-based methods have also been used in this context [2,1]. Recently, attempts have been made to extend the Nelson-Oppen combination procedure to more general theories. Extensions have been achieved either by relaxing the requirement that the theories to be combined are stably-infinite [32][1]; or by relaxing the requirement that the theories to be combined have disjoint signatures [3,31,15]. Note that these extensions are still restrictive, as the conditions imposed on the base theory and on the component theories are very strong, and of a model theoretic nature. For instance, due to the limitations on the shared theory for decidability transfer in combinations of theories as studied in [15], only locally finite shared theories (hence no numerical domains) can be handled when applying these results to verification in [16]. In contrast, the notion of local extensions we studied [27] imposes no restrictions on using numerical domains as a base theory.

The notion of locality of a theory extension allows us to address at the same time the two aspects important for efficient reasoning mentioned above, namely restricting the search space and modular reasoning. Locality is used for restricting the search space, but as a side-effect it allows to reduce proof tasks in the extension, hierarchically, to proof tasks w.r.t. the base theory. We will present these results here. We will also present recent results on preservation of locality of theories (resp. theory extensions) under theory combination, and on possibilities of modular reasoning in such combinations. In particular, we are interested in characterizing the type of information which needs to be exchanged between

[1] It is interesting that, although the approach of [32] is orthogonal to the notion of locality of a theory extension [27], many of the examples considered there can also be explained using semantical characterizations of locality.

provers for the component theories in order to guarantee completeness of the procedure. This last problem is strongly related to the problem of studying the form of interpolants in theory combinations. This is why we will also mention some of our results on interpolation in local theory extensions.

Structure of the paper: The paper is structured as follows: Section 2 contains generalities on theories, local theories, partial algebras, weak validity and embeddability of partial algebras into total algebras. In Section 3 the notion of local theory extension is introduced, and a method for hierarchical reasoning in such extensions is presented. Section 4 presents two possibilities of identifying local theory extensions: one based on links between embeddability and locality (Section 4.1) and one which in addition allows to use resolution-based methods (Section 4.2). These methods are used in Section 5 to provide various examples of local theory extensions. We illustrate the method for hierarchical reasoning in Section 3.1 on several examples. In Section 6 we investigate conditions under which locality is preserved when combining theories, and in Section 7 we investigate possibilities of efficient modular reasoning in such theory combinations.

2 Preliminaries

Theories. Theories can be regarded as sets of formulae or as sets of models. Let $\mathcal{T}$ be a Π-theory and ϕ, ψ be Π-formulae. We say that $\mathcal{T} \wedge \phi \models \psi$ (written also $\phi \models_{\mathcal{T}} \psi$) if ψ is true in all models of $\mathcal{T}$ which satisfy ϕ.

In what follows we consider extensions of theories, in which the signature is extended by new *function symbols* (i.e. we assume that the set of predicate symbols remains unchanged in the extension). If a theory is regarded as a set of formulae, then its extension with a set of formulae is set union. If $\mathcal{T}$ is regarded as a collection of models then its extension with a set $\mathcal{K}$ of formulae consists of all structures (in the extended signature) which are models of $\mathcal{K}$ and whose reduct to the signature of $\mathcal{T}_0$ is in $\mathcal{T}_0$.

Let $\mathcal{T}_0$ be an arbitrary theory with signature $\Pi_0 = (S_0, \Sigma_0, \mathsf{Pred})$, where S_0 is a set of sorts, Σ_0 a set of function symbols, and Pred a set of predicate symbols. We consider extensions $\mathcal{T}_1$ of $\mathcal{T}_0$ with signature $\Pi = (S, \Sigma, \mathsf{Pred})$, where the set of sorts is $S = S_0 \cup S_1$ and the set of function symbols is $\Sigma = \Sigma_0 \cup \Sigma_1$ (i.e. the signature is extended by new sorts and function symbols). We assume that $\mathcal{T}_1$ is obtained from $\mathcal{T}_0$ by adding a set $\mathcal{K}$ of (universally quantified) clauses in the signature Π. Thus, $\mathsf{Mod}(\mathcal{T}_1)$ consists of all Π-structures which are models of $\mathcal{K}$ and whose reduct to Π_0 is a model of $\mathcal{T}_0$.

Local theories. This notion was introduced by Givan and McAllester in [17,21]. A *local theory* is a set of Horn clauses $\mathcal{K}$ such that, for any ground Horn clause C, $\mathcal{K} \models C$ only if already $\mathcal{K}[C] \models C$ (where $\mathcal{K}[C]$ is the set of instances of $\mathcal{K}$ in which all terms are subterms of ground terms in either $\mathcal{K}$ or $\mathcal{C}$).

The size of $\mathcal{K}[G]$ is polynomial in the size of G for a fixed $\mathcal{K}$. Since satisfiability of sets of ground Horn clauses can be checked in linear time [11], it follows that for local theories, validity of ground Horn clauses can be checked in polynomial time. Givan and McAllester proved that every problem which is decidable in PTIME can be encoded as an entailment problem of ground clauses w.r.t. a local theory [18]. An example of a local theory (cf. [18]) is the set of axioms of a monotone function w.r.t. a transitive relation $\leq$:

$$\mathcal{K} = \{x \leq y \ \wedge \ y \leq z \ \rightarrow \ x \leq z, \quad x \leq y \rightarrow f(x) \leq f(y)\}.$$

Another example provided in [18] is a local axiom set for reasoning about a lattice (similar to that proposed by Skolem in [26]). In [5,4], Ganzinger and Basin defined the more general notion of *order locality* and showed how to recognize (order-)local theories and how to use these results for automated complexity analysis. Given a term ordering $\succ$, we say that a set $\mathcal{K}$ of clauses entails a clause C bounded by $\succ$ (notation: $\mathcal{K} \models_{\preceq} C$), if and only if there is a proof of $\mathcal{K} \models C$ from those ground instances of clauses in $\mathcal{K}$ in which (under $\succeq$) each term is smaller than or equal to some term in C.

Definition 1 ([5,4]). *A set of clauses $\mathcal{K}$ is local with respect to $\succ$ if whenever $\mathcal{K} \models C$ for a ground clause C, then $\mathcal{K} \models_{\preceq} C$.*

Theorem 1 ([5,4]). *Let $\succ$ be a (possibly partial) term ordering and $\mathcal{K}$ be a set of clauses. Assume that $\mathcal{K}$ is saturated with respect to $\succ$-ordered resolution, and let C be a ground clause. Then $\mathcal{K} \models C$ for a ground clause C if and only if $\mathcal{K} \models_{\preceq} C$, i.e. $\mathcal{K}$ is local with respect to $\succ$.*

The converse of this theorem is not true in general. Ganzinger and Basin established conditions under which the converse holds – they use a hyperresolution calculus and identify conditions when for Horn clauses order locality is equivalent to so-called *peak saturation* (Theorems 4.4–4.7 in [4]). These results are obtained for first-order logic without equality. In [13], Ganzinger established a link between proof theoretic and semantic concepts for polynomial time decidability of uniform word problems which had already been studied in algebra [26,12,8]. He defined two notions of locality for equational Horn theories, and established relationships between these notions of locality and corresponding semantic conditions, referring to embeddability of partial algebras into total algebras. Theorem 1 also can be used for recognizing equational Horn theories:

Theorem 2 ([13]). *Let $\mathcal{K}$ be a set of Horn clauses. Then $\mathcal{K}$ is a local theory in logic with equality if and only if $\mathcal{K} \cup EQ$ is a local theory in logic without equality, where EQ denotes the set of equality axioms consisting of reflexivity, symmetry, transitivity, and of congruence axioms for each function symbol in the signature.*

Theorems 2 and 1 were used in [13] for proving the locality of the following presentation Int of the set of integers with successor and predecessor by saturation:

(1)	$p(x) \approx y \rightarrow s(y) \approx x$	(3)	$p(x) \approx p(y) \rightarrow y \approx x$
(2)	$s(x) \approx y \rightarrow p(y) \approx x$	(4)	$s(x) \approx s(y) \rightarrow y \approx x$

The presentation Int' of integers with successor and predecessor consisting of the axioms (1) and (2) alone (without the injectivity conditions (3) and (4)) is not local but it is *stably local*: in order to disprove a ground set G of clauses only those ground instances $\mathsf{Int}'^{[G]}$ of Int' are needed where variables are mapped to subterms occurring in G. (Note that $\mathsf{Int}' \cup EQ$ is not saturated under ordered resolution; when saturating it the injectivity axioms are generated.)

In [14,27] the notion of locality for Horn clauses is extended to the notion of *local extension* of a base theory (cf. Section 3). In the study of local theory extensions we will refer to *total* models of a theory and to *partial models* of a theory. The necessary notions on partial structures are defined below.

Partial structures. Let $\Pi = (S, \Sigma, \mathsf{Pred})$ be a S-sorted signature where Σ is a set of function symbols and Pred a set of predicate symbols. A *partial Π-structure* is a structure $A = (\{A_s\}_{s \in S}, \{f_A\}_{f \in \Sigma}, \{P_A\}_{P \in \mathsf{Pred}})$, where for every $s \in S$, A_s is a non-empty set and for every $f \in \Sigma$ with arity $s_1 \dots s_n \to s$, f_A is a partial function from $\prod_{i=1}^n A_{s_i}$ to A_s. A is called a *total structure* if all functions f_A are total. (In the one-sorted case we will denote both an algebra and its support with the same symbol.) Details on partial algebras can be found in [7]. The notion of evaluating a term t with variables $X = \{X_s \mid s \in S\}$ w.r.t. an assignment $\{\beta_s{:}X_s \to A_s \mid s \in S\}$ for its variables in a partial structure A is the same as for total many-sorted algebras, except that the evaluation is undefined if $t = f(t_1, \dots, t_n)$ with $a(f) = (s_1 \dots s_n \to s)$, and at least one of $\beta_{s_i}(t_i)$ is undefined, or else $(\beta_{s_1}(t_1), \dots, \beta_{s_n}(t_n))$ is not in the domain of f_A.

A *weak Π-embedding* between the partial structures $A = (\{A_s\}_{s \in S}, \{f_A\}_{f \in \Sigma}, \{P_A\}_{P \in \mathsf{Pred}})$ and $B = (\{B_s\}_{s \in S}, \{f_B\}_{f \in \Sigma}, \{P_B\}_{P \in \mathsf{Pred}})$ is a (many-sorted) family $i = (i_s)_{s \in S}$ of total maps $i_s : A_s \to B_s$ such that

- if $f_A(a_1, \dots, a_n)$ is defined then also $f_B(i_{s_1}(a_1), \dots, i_{s_n}(a_n))$ is defined and $i_s(f_A(a_1, \dots, a_n)) = f_B(i_{s_1}(a_1), \dots, i_{s_n}(a_n))$, provided $a(f) = s_1 \dots s_n \to s$;
- for each s, i_s is injective and an embedding w.r.t. Pred i.e. for every $P \in \mathsf{Pred}$ with arity $s_1 \dots s_n$ and every $a_1, \dots, a_n$ where $a_i \in A_{s_i}$, $P_A(a_1, \dots, a_n)$ if and only if $P_B(i_{s_1}(a_1), \dots, i_{s_n}(a_n))$.

In this case we say that A *weakly embeds* into B.

In what follows we will denote a many-sorted variable assignment $\{\beta_s{:}X_s \to A_s \mid s \in S\}$ as $\beta : X \to \mathcal{A}$. For the sake of simplicity all definitions below are given for the one-sorted case. They extend in a natural way to many-sorted structures.

Definition 2 (Weak validity). *Let $(A, \{f_A\}_{f \in \Sigma}, \{P_A\}_{P \in \mathsf{Pred}})$ be a partial structure and $\beta : X \to A$.*

(1) $(A, \beta) \models_w t \approx s$ if and only if (a) $\beta(t)$ and $\beta(s)$ are both defined and equal; or (b) at least one of $\beta(s)$ and $\beta(t)$ is undefined.
(2) $(A, \beta) \models_w t \not\approx s$ if and only if (a) $\beta(t)$ and $\beta(s)$ are both defined and different; or (b) at least one of $\beta(s)$ and $\beta(t)$ is undefined.
(3) $(A, \beta) \models_w P(t_1, \dots, t_n)$ if and only if (a) $\beta(t_1), \dots, \beta(t_n)$ are all defined and $(\beta(t_1), \dots, \beta(t_n)) \in P_A$; or (b) at least one of $\beta(t_1), \dots, \beta(t_n)$ is undefined.
(4) $(A, \beta) \models_w \neg P(t_1, \dots, t_n)$ if and only if (a) $\beta(t_1), \dots, \beta(t_n)$ are all defined and $(\beta(t_1), \dots, \beta(t_n)) \notin P_A$; or (b) at least one of $\beta(t_1), \dots, \beta(t_n)$ is undefined.

(A, β) weakly satisfies a clause C *(notation:* $(A, \beta) \models_w C$*) if* $(A, \beta) \models_w L$ *for at least one literal* L *in* C. A weakly satisfies C *(notation:* $A \models_w C$*) if* $(A, \beta) \models_w C$ *for all assignments* β. A weakly satisfies a set of clauses $\mathcal{K}$ *(notation:* $A \models_w \mathcal{K}$*) if* $A \models_w C$ *for all* $C \in \mathcal{K}$.

Definition 3 (Evans validity). *Evans validity is defined similarly, with the difference that (1) is replaced with:*

(1') $(A, \beta) \models t \approx s$ *if and only if (a)* $\beta(t)$ *and* $\beta(s)$ *are both defined and equal; or (b)* $\beta(s)$ *is defined,* $t = f(t_1, \dots, t_n)$ *and* $\beta(t_i)$ *is undefined for at least one of the direct subterms of* t*; or (c) both* $\beta(s)$ *and* $\beta(t)$ *are undefined.*

Evans validity extends to (sets of) clauses in the usual way. We use the notation: $(A, \beta) \models L$ *for a literal* L*;* $(A, \beta) \models C$ *and* $A \models C$ *for a clause* C*, etc.*

Example 1. *Let* A *be a partial* Σ*-algebra, where* $\Sigma = \{\mathsf{car}/1, \mathsf{nil}/0\}$*. Assume that* nil_A *is defined and* $\mathsf{car}_A(\mathsf{nil}_A)$ *is not defined. Then:*

- $A \not\models \mathsf{car}(\mathsf{nil}) \approx \mathsf{nil}$ *(since* $\mathsf{car}_A(\mathsf{nil})$ *is undefined in* A*, but* nil *is defined in* A*);*
- $A \models \mathsf{car}(\mathsf{nil}) \not\approx \mathsf{nil}$*;*
- $A \models_w \mathsf{car}(\mathsf{nil}) \approx \mathsf{nil}$*,* $A \models_w \mathsf{car}(\mathsf{nil}) \not\approx \mathsf{nil}$ *(since* $\mathsf{car}(\mathsf{nil})$ *is not defined in* A*).*

Definition 4. *A partial* Π*-algebra* A *is a* weak partial model *(resp. partial model) of* $\mathcal{T}_1$ *with totally defined* Σ_0*-function symbols if (i)* $A_{|\Pi_0}$ *is a model of* $\mathcal{T}_0$ *and (ii)* $A \models_w \mathcal{K}$ *(resp.* $A \models \mathcal{K}$*).*

If the base theory $\mathcal{T}_0$ and its signature are clear from the context, we will refer to *(weak) partial models* of $\mathcal{T}_1$. We will use the following notation:

- $\mathsf{PMod}(\Sigma_1, \mathcal{T}_1)$ is the class of all partial models of $\mathcal{T}_1$ in which the functions in Σ_1 are partial, and all other function symbols are total;
- $\mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ is the class of all weak partial models of $\mathcal{T}_1$ in which the Σ_1-functions are partial and all the other function symbols are total;
- $\mathsf{Mod}(\mathcal{T}_1)$ denotes the class of all total models of $\mathcal{T}_1$.

We will also consider small variations of the notion of weak partial model:

- $\mathsf{PMod}^{\mathsf{f}}_{\mathsf{w}}(\Sigma_1, \mathcal{T}_1)$ is the class of all finite weak partial models of $\mathcal{T}_1$ in which the Σ_1-functions are partial and all the other function symbols are total;
- $\mathsf{PMod}^{\mathsf{fd}}_{\mathsf{w}}(\Sigma_1, \mathcal{T}_1)$ is the class of all weak partial models of $\mathcal{T}_1$ in which the Σ_1-functions are partial and their definition domain is a finite set, and all the other function symbols are total.

and similar variations $\mathsf{PMod}^{\mathsf{f}}(\Sigma_1, \mathcal{T}_1)$, $\mathsf{PMod}^{\mathsf{fd}}(\Sigma_1, \mathcal{T}_1)$ of the notion of partial model.

Embeddability. For theory extensions $\mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$, where $\mathcal{K}$ is a set of clauses, we consider the following conditions:

(Emb) Every $A \in \mathsf{PMod}(\Sigma_1, \mathcal{T}_1)$ weakly embeds into a total model of $\mathcal{T}_1$.
(Emb$_\mathsf{w}$) Every $A \in \mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ weakly embeds into a total model of $\mathcal{T}_1$.

We also define a stronger notion of embeddability, which we call *completability*:

($\mathsf{Comp_w}$) Every $A \in \mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ weakly embeds into a total model B of $\mathcal{T}_1$ such that $A_{|\Pi_0}$ and $B_{|\Pi_0}$ are isomorphic.

(Comp) is defined analogously (w.r.t. $\mathsf{PMod}(\Sigma_1, \mathcal{T}_1)$).

Conditions which only refer to embeddability of *finite* partial models are denoted by ($\mathsf{Emb^f_w}$), ($\mathsf{Comp^f_w}$), resp. ($\mathsf{Emb^f}$), ($\mathsf{Comp^f}$). Conditions referring to embeddability of partial models in which the extension functions have a finite definition domain (i.e. in $\mathsf{PMod^{fd}_w}(\Sigma_1, \mathcal{T}_1)$) are denoted by ($\mathsf{Emb^{fd}_w}$), resp. ($\mathsf{Comp^{fd}_w}$).

3 Local Theory Extensions

The notion of local theories introduced and studied by Givan and McAllester [17,21,18] can be extended in a natural way to extensions of a base theory with a set of additional function symbols constrained by a set $\mathcal{K}$ if clauses.

Let $\mathcal{K}$ be a set of clauses in the signature $\Pi = (S, \Sigma, \mathsf{Pred})$, where $S = S_0 \cup S_1$ and $\Sigma = \Sigma_0 \cup \Sigma_1$. In what follows, when we refer to sets G of ground clauses we assume that they are in the signature $\Pi^c = (S, \Sigma \cup \Sigma_c, \mathsf{Pred})$, where Σ_c is a set of new constants. If Ψ is a set of ground $\Sigma_0 \cup \Sigma_1 \cup \Sigma_c$-terms, we denote by $\mathcal{K}_\Psi$ the set of all instances of $\mathcal{K}$ in which all terms starting with a Σ_1-function symbol are ground terms in the set Ψ. We denote by $\mathcal{K}^\Psi$ the set of all instances of $\mathcal{K}$ in which all variables occurring below a Σ_1-function symbol are instantiated with ground terms in the set $T_{\Sigma_0}(\Psi)$ of Σ_0-terms generated by Ψ.

If G is a set of ground clauses and $\Psi = \mathsf{st}(\mathcal{K}, G)$ is the set of ground subterms occurring in either $\mathcal{K}$ or G then we write $\mathcal{K}[G] := \mathcal{K}_\Psi$, and $\mathcal{K}^{[G]} := \mathcal{K}^\Psi$.

We will focus on the following type of locality of a theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$, where $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ with $\mathcal{K}$ a set of (universally quantified) clauses:

(Loc) For every set G of ground clauses $\mathcal{T}_1 \cup G \models \perp$ iff $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G$ has no weak partial model in which all terms in $\mathsf{st}(\mathcal{K}, G)$ are defined.

(SLoc) For every set G of ground clauses $\mathcal{T}_1 \cup G \models \perp$ iff $\mathcal{T}_0 \cup \mathcal{K}^{[G]} \cup G$ has no partial model in which all terms in $\mathsf{st}(\mathcal{K}, G)$ are defined.

Weaker notions ($\mathsf{Loc^f}$), resp. ($\mathsf{SLoc^f}$) can be defined if we require that the respective conditions only hold for *finite* sets G of ground clauses. An intermediate notion of locality ($\mathsf{Loc^{fd}}$) can be defined if we require that the respective conditions only hold for sets G of ground clauses containing only a finite set of terms starting with a function symbol in Σ_1. A more general notion of locality (ELoc) is presented at the end of Section 4.1.

An extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ is *local* (*stably local*) if it satisfies condition ($\mathsf{Loc^f}$) (resp. ($\mathsf{SLoc^f}$)). A local (stably local) theory [13] is a local extension of the empty theory. In (stably) local theory extensions hierarchical reasoning is possible.

3.1 Hierarchical Reasoning in Local Theory Extensions

Consider a local theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$. The locality conditions defined above require that, for every set G of ground clauses, $\mathcal{T}_1 \cup G$ is satisfiable if and only if $\mathcal{T}_0 \cup \mathcal{K}{*}[G] \cup G$ has a (Evans, weak, finite) partial model with additional properties, where, depending on the notion of locality, $\mathcal{K} * [G]$ is $\mathcal{K}[G]$ or $\mathcal{K}^{[G]}$. All clauses in $\mathcal{K} * [G] \cup G$ have the property that the function symbols in Σ_1 have as arguments only ground terms. Therefore, $\mathcal{K} * [G] \cup G$ can be flattened and purified (i.e. the function symbols in Σ_1 are separated from the other symbols) by introducing, in a bottom-up manner, new constants c_t for subterms $t = f(g_1, \dots, g_n)$ with $f \in \Sigma_1$, g_i ground $\Sigma_0 \cup \Sigma_c$-terms (where Σ_c is a set of constants which contains the constants introduced by flattening, resp. purification), together with corresponding definitions $c_t \approx t$. The set of clauses thus obtained has the form $\mathcal{K}_0 \cup G_0 \cup D$, where D is a set of ground unit clauses of the form $f(g_1, \dots, g_n) \approx c$, where $f \in \Sigma_1$, c is a constant, $g_1, \dots, g_n$ are ground terms without function symbols in Σ_1, and $\mathcal{K}_0$ and G_0 are clauses without function symbols in Σ_1. Flattening and purification preserve both satisfiability and unsatisfiability with respect to total algebras, and also with respect to partial algebras in which all ground subterms which are flattened are defined [27].

For the sake of simplicity in what follows we will always flatten and then purify $\mathcal{K} * [G] \cup G$. Thus we ensure that D consists of ground unit clauses of the form $f(c_1, \dots, c_n) \approx c$, where $f \in \Sigma_1$, and $c_1, \dots, c_n, c$ are constants.

Lemma 3 ([27]). *Let $\mathcal{K}$ be a set of clauses and G a set of ground clauses, and let $\mathcal{K}_0 \cup G_0 \cup D$ be obtained from $\mathcal{K} * [G] \cup G$ by flattening and purification, as explained above. Assume that $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is a local theory extension. Then the following are equivalent:*

(1) $\mathcal{T}_0 \cup \mathcal{K}{*}[G] \cup G$ has a partial model in which all terms in $\mathsf{st}(\mathcal{K}, G)$ are defined.
(2) $\mathcal{T}_0 \cup \mathcal{K}_0 \cup G_0 \cup D$ has a partial model with all terms in $\mathsf{st}(\mathcal{K}_0, G_0, D)$ defined.
(3) $\mathcal{T}_0 \cup \mathcal{K}_0 \cup G_0 \cup N_0$ has a (total) model, where

$$N_0 = \{ \bigwedge_{i=1}^{n} c_i \approx d_i \rightarrow c = d \mid f(c_1, \dots, c_n) \approx c, f(d_1, \dots, d_n) \approx d \in D \}.$$

Theorem 4 ([27]). *Assume that the theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ either (1) satisfies condition $(\mathsf{Loc}^\mathsf{f})$, or (2) satisfies condition $(\mathsf{SLoc}^\mathsf{f})$ and $\mathcal{T}_0$ is locally finite. Then:*

(a) If all variables in the clauses in $\mathcal{K}$ occur below some function symbol from Σ_1 and if the universal theory of $\mathcal{T}_0$ is decidable, then the universal theory of $\mathcal{T}_1$ is decidable.
(b) Assume some variables in $\mathcal{K}$ do not occur below any function symbol in Σ_1. If the $\forall\exists$ theory of $\mathcal{T}_0$ is decidable then the universal theory of $\mathcal{T}_1$ is decidable.

In case (a) above locality allows to reduce reasoning in $\mathcal{T}_1$ to reasoning in an extension of $\mathcal{T}_0$ with free function symbols (for this an SMT procedure can be used). In case (b) this is not possible, as $\mathcal{K} * [G]$ is not a set of ground clauses.

We will illustrate the applicability of Lemma 3 and Theorem 4 for specific examples of local theory extensions in Section 5.

4 Identifying Local Theory Extensions

We discuss two different ways of recognizing the locality of a theory extension. The first is semantical, based on possibilities of embedding partial models of a theory extension into total models. The second is proof theoretical, and at the moment part of work in progress: we present some results based on possibilities of saturating the extension axioms with respect to ordered resolution.

4.1 Locality and Embeddability

Links between *locality of a theory* and *embeddability* were established by Ganzinger in [13]. Similar results can also be obtained for *local theory extensions.*

In what follows we say that a non-ground clause is Σ_1-*flat* if function symbols (including constants) do not occur as arguments of function symbols in Σ_1. A Σ_1-flat non-ground clause is called Σ_1-*linear* if whenever a variable occurs in two terms in the clause which start with function symbols in Σ_1, the two terms are identical, and if no term which starts with a function symbol in Σ_1 contains two occurrences of the same variable.

For sets of Σ_1-flat clauses locality implies embeddability. This generalizes results presented in the case of local theories in [13].

Theorem 5. *Assume that $\mathcal{K}$ is a family of Σ_1-flat clauses in the signature Π.*

(1) If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1 := \mathcal{T}_0 \cup \mathcal{K}$ satisfies (Loc) *then it satisfies* $(\mathsf{Emb_w})$.
(2) If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1 := \mathcal{T}_0 \cup \mathcal{K}$ satisfies $(\mathsf{Loc^f})$ *then it satisfies* $(\mathsf{Emb_w^f})$.
(3) If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1 := \mathcal{T}_0 \cup \mathcal{K}$ satisfies $(\mathsf{Loc^{fd}})$ *then it satisfies* $(\mathsf{Emb_w^{fd}})$.
(4) If $\mathcal{T}_0$ is compact and the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies $(\mathsf{Loc^f})$*, then $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies* $(\mathsf{Emb_w})$.

Conversely, embeddability implies locality. The following results appear in [27], [30] and allow us to give several examples of local theory extensions (cf. Sect. 5).

Theorem 6 ([27,30]). *Let $\mathcal{K}$ be a set of Σ_1-flat and Σ_1-linear clauses.*

(1) If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies $(\mathsf{Emb_w})$ *then it satisfies* (Loc).
(2) Assume that $\mathcal{T}_0$ is a locally finite universal theory, and that $\mathcal{K}$ contains only finitely many ground subterms. If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies $(\mathsf{Emb_w^f})$*, then $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies* $(\mathsf{Loc^f})$.
(3) $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies $(\mathsf{Emb_w^{fd}})$*. Then $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies* $(\mathsf{Loc^{fd}})$.

Theorem 7 ([27]). *Let $\mathcal{T}_0$ be a universal theory and $\mathcal{K}$ be a set of clauses. Then:*

(1) If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies (Emb) *then it satisfies* (SLoc).
(2) Assume that $\mathcal{T}_0$ is a locally finite universal theory, and that $\mathcal{K}$ contains only finitely many ground subterms. If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies $(\mathsf{Emb^f})$*, then $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies* $(\mathsf{SLoc^f})$.

Analyzing the proofs of Theorems 6 and 7 we notice that the embeddability conditions (Comp) and ($\mathsf{Comp_w}$) imply, in fact, stronger locality conditions. Consider a theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ with a set $\mathcal{K}$ of formulae of the form $\forall x_1 \dots x_n(\Phi(x_1, \dots, x_n) \vee C(x_1, \dots, x_n))$, where $\Phi(x_1, \dots, x_n)$ is an *arbitrary first-order formula* in the base signature Π_0 with free variables $x_1, \dots, x_n$, and $C(x_1, \dots, x_n)$ is a *clause* in the signature Π.

We can extend the notion of locality of an extension accordingly:

(ELoc) For every formula $\Gamma = \Gamma_0 \cup G$, where Γ_0 is a Π_0-sentence and G is a set of ground clauses, $\mathcal{T}_1 \cup \Gamma \models\perp$ iff $\mathcal{T}_0 \cup \mathcal{K}[\Gamma] \cup \Gamma$ has no weak partial model in which all terms in $\mathsf{st}(\mathcal{K}, G)$ are defined.

A stable locality condition (ESLoc) can be defined similarly. The proofs of Theorems 6 and 7 can be adapted with minimal changes to prove a stronger result:

Theorem 8 ([27]). *(1) Assume all terms of $\mathcal{K}$ starting with a Σ_1-function are flat and linear. If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies* ($\mathsf{Comp_w}$) *then it satisfies* (ELoc).

(2) Assume that $\mathcal{T}_0$ is a universal theory. If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies (Comp) *then it satisfies* (ESLoc).

4.2 Locality and Saturation

In Section 2, the results of Basin and Ganzinger [5,4] were mentioned, in which links between saturation w.r.t. ordered resolution of a set of clauses $\mathcal{K}$ and the (order)-locality of $\mathcal{K}$ were established. It is natural to ask if similar results can be obtained for local theory extensions. Local theory extensions are extensions $\mathcal{T}_1$ of a base theory $\mathcal{T}_0$ by means of a set of new sorts S_1 and a set of new function symbols Σ_1 constrained by a set of clauses $\mathcal{K}$. We investigate the link between the locality of the set $\mathcal{K}$ of clauses and the locality of the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$. This is work in progress, from which we present a first result:

Theorem 9. *Let $\mathcal{T}_0$ be a first-order theory with signature $\Pi_0 = (S_0, \Sigma_0, \mathsf{Pred})$. Let $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ with signature $\Pi = (S_0 \cup S_1, \Sigma_0 \cup \Sigma_1, \mathsf{Pred})$. Assume that:*

- *all functions in Σ_1 occurring in $\mathcal{K}$ have their output sort in S_1;*
- *$\mathcal{K}$ is a set of clauses which only contain function symbols in Σ_1;*
- *the set $\mathcal{K}$ of clauses is local (resp. stably local).*

Then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is also local (resp. stably local).

Proof: (Sketch) Let $P = (\{P_s\}_{s \in S_0 \cup S_1}, \{f_P\}_{f \in \Sigma_0 \cup \Sigma_1}, \{R_P\}_{R \in \mathsf{Pred}})$ be a weak partial model of $\mathcal{T}_0 \cup \mathcal{K}$ in which all Σ_0-functions are totally defined. We will denote by $P_{|\Sigma_1}$ the partial structure obtained from P by forgetting all operation symbols in Σ_0. P (hence also $P_{|\Sigma_1}$) is a weak partial model of $\mathcal{K}$. By the locality of $\mathcal{K}$, $P_{|\Sigma_1} = (\{P_s\}_{s \in S_0 \cup S_1}, \{f_P\}_{f \in \Sigma_1}, \{R_A\}_{R \in \mathsf{Pred}})$ weakly embeds (via an embedding i) into a total model $A = (\{A_s\}_{s \in S_0 \cup S_1}, \{f_A\}_{f \in \Sigma_1}, \{R_A\}_{R \in \mathsf{Pred}})$ of $\mathcal{K}$. Let A^* be the substructure of A having the same supports as A for the sorts in S_1 and support $i_s(P_s)$ for each sort $s \in S_0$. (Since we assumed that all function symbols in Σ_1 have output sort in S_1, A^* is closed under all Σ_1-operations.)

Let $B = (\{B_s\}_{s \in S_0 \cup S_1}, \{f_B\}_{f \in \Sigma_0 \cup \Sigma_1}, \{R_B\}_{R \in \mathsf{Pred}})$, where for $s \in S_0$, $B_s = i_s(P_s)$, for $s \in S_1$, $B_s = A_s$, for $f \in \Sigma_0$, f_B coincides with f_P, for $f \in \Sigma_1$, f_B coincides with f_{A^*}, and all predicate symbols coincide with those in A^*. Then $B_{|\Pi_0}$ is isomorphic to $P_{|\Pi_0}$, hence is a model of $\mathcal{T}_0$ and $B_{|\Sigma_1} = A^*$, hence $B \models \mathcal{K}$. □

The locality of $\mathcal{K}$ can be checked e.g. by testing whether $\mathcal{K} \cup EQ$ is saturated under ordered resolution (w.r.t. the (strict) subterm ordering) using Theorems 1 and 2 (cf. also [5,4,13]) but now extended to a many-sorted framework. The advantage is that even if $\mathcal{K}$ is not saturated, if $\mathcal{K}^*$ is a finite saturation of $\mathcal{K} \cup EQ$ under ordered resolution, then $\mathcal{T}_0 \cup \mathcal{K}^*$ can be used to extract a presentation which defines a (local) theory extension which has the same total models as $\mathcal{T}_0 \cup \mathcal{K}$.

Note: The idea in the proof of Theorem 9 can also be used to show that (under the assumptions in Theorem 9) if $\mathcal{K}$ satisfies Comp (resp. (Comp_w)) then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ also satisfies Comp (resp. (Comp_w)).

5 Examples of Local Theory Extensions

We present several examples of theory extensions for which embedding conditions among those mentioned above hold and are thus local, and illustrate the possibilities of local reasoning in such extensions.

5.1 Extensions with Free Functions

Any extension $\mathcal{T}_0 \cup \mathsf{Free}(\Sigma)$ of a theory $\mathcal{T}_0$ with a set Σ of free function symbols satisfies condition $(\mathsf{Comp_w})$.

Example 2. Let $\mathcal{T}_0$ be the theory $\mathsf{LI}(\mathbb{Q})$ of linear rational arithmetic, and let $\mathcal{T}_1 = \mathsf{LI}(\mathbb{Q}) \cup \mathsf{Free}(\{f, g, h\})$ be the extension of $\mathcal{T}_0$ with the free functions f, g, h, and let $G = g(a) = c+5 \wedge f(g(a)) \geq c+1 \wedge h(b) = d+4 \wedge d = c+1 \wedge f(h(b)) < c+1$. We show that G is unsatisfiable in $\mathsf{LI}(\mathbb{Q}) \cup \mathsf{Free}(\{f, g, h\})$ as follows:

Step 1: Flattening; purification. G is purified and flattened by replacing the terms starting with f, g, h with new variables. We obtain the following purified form:

$$\begin{array}{ll} G_0: & a_1 = c+5 \quad \wedge \quad a_2 \geq c+1 \quad \wedge \quad b_1 = d+4 \quad \wedge \quad d = c+1 \quad \wedge \quad b_2 < c+1, \\ \mathsf{Def}: & a_1 = g(a) \quad \wedge \quad a_2 = f(a_1) \quad \wedge \quad b_1 = h(b) \quad \wedge \quad b_2 = f(b_1). \end{array}$$

Step 2: Hierarchical reasoning. By Lemma 3, G is unsatisfiable in $\mathsf{LI}(\mathbb{Q}) \cup \mathsf{Free}(\{f, g, h\})$ iff $G_0 \wedge N_0$ is unsatisfiable in $\mathsf{LI}(\mathbb{Q})$, where N_0 corresponds to the consequences of the congruence axioms for those ground terms which occur in the definitions Def for the newly introduced variables.

Def	G_0	N_0
$a_1{=}g(a) \wedge a_2{=}f(a_1)$	$a_1 = c+5 \wedge a_2 \geq c+1$	$N_0:\ b_1{=}a_1 \rightarrow b_2{=}a_2$
$b_1{=}h(b) \wedge b_2{=}f(b_1)$	$b_1 = d+4 \wedge d = c+1 \wedge b_2 < c+1$	

To prove that $G_0 \wedge N_0$ is unsatisfiable in $\mathsf{LI}(\mathbb{Q})$, note that $G_0 \models_{\mathsf{LI}(\mathbb{Q})} a_1 = b_1$. Hence, $G_0 \wedge N_0$ entails $a_2 = b_2 \wedge a_2 \geq c+1 \wedge b_2 < c+1$, which is inconsistent.

5.2 Shallow Theory Extensions

We now consider the case of shallow extensions of a base theory $\mathcal{T}_0$ first considered in [14]. Let Π be the signature of the theory extension. We assume that all extension functions have a codomain in the set S_0 of (base) sorts. A Π-clause is called *shallow* if extension function symbols in Σ_1 occur in C only positively and only at the root of terms. The theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is *shallow* if $\mathcal{K}$ consists only of shallow clauses. Typical examples of shallow clauses are tail-recursive definitions of an extension function.

Theorem 10 ([14]). *Assume that $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is a theory extension by a set $\mathcal{K}$ of shallow clauses w.r.t. the family of all extension functions with a codomain in S_0. Then the extension satisfies condition* Comp *and hence is stably local.*

5.3 Extensions with Monotone Functions

In [27] and [30] we analyzed extensions with monotonicity conditions for an n-ary function f w.r.t. a subset $I \subseteq \{1, \dots, n\}$ of its arguments:

$$(\mathsf{Mon}_f^I) \qquad \bigwedge_{i \in I} x_i \leq_i y_i \wedge \bigwedge_{i \notin I} x_i = y_i \rightarrow f(x_1, .., x_n) \leq f(y_1, .., y_n).$$

If $I = \{1, \dots, n\}$ we speak of monotonicity in all arguments; we denote $\mathsf{Mon}_f^{\{1,\dots,n\}}$ by Mon_f. If $I = \emptyset$, $\mathsf{Mon}_f^\emptyset$ is equivalent to the congruence axiom for f. Monotonicity in some arguments and antitonicity in other arguments is modeled by considering functions $f : \prod_{i \in I} P_i^{\sigma_i} \times \prod_{j \notin I} P_j \rightarrow P$ with $\sigma_i \in \{-, +\}$, where $P_i^+ = P_i$ and $P_i^- = P_i^\partial$, the order dual of the poset P_i. The corresponding axioms are denoted by Mon_f^σ, where for $i \in I$, $\sigma(i) = \sigma_i \in \{-, +\}$, and for $i \notin I$, $\sigma(i) = 0$.

Theorem 11 ([27,30]). *The following hold:*

1. *Let $\mathcal{T}_0$ be a class of (many-sorted) bounded semilattice-ordered Σ_0-structures. Let Σ_1 be disjoint from Σ_0 and $\mathcal{T}_1 = \mathcal{T}_0 \cup \{\mathsf{Mon}_f^\sigma | f \in \Sigma_1\}$. Then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies* $(\mathsf{Comp}_\mathsf{w}^\mathsf{fd})$, *hence is local.*
2. *Any extension of the theory of posets with functions in a set Σ_1 satisfying $\{\mathsf{Mon}_f^\sigma \mid f \in \Sigma_1\}$ satisfies condition* $(\mathsf{Emb}_\mathsf{w})$, *hence is local.*

This provides us with a large number of concrete examples.

Corollary 12 ([27,30]). *The extensions with functions satisfying monotonicity axioms Mon_f^σ of the following classes of algebras are local:*

(1) any class of algebras with a bounded (semi)lattice reduct, a bounded distributive lattice reduct, or a Boolean algebra reduct ($(\mathsf{Comp}_\mathsf{w}^\mathsf{fd})$ *holds);*
(2) $\mathcal{T}$, the class of totally-ordered sets; $\mathcal{DO}$, the theory of dense totally-ordered sets ($(\mathsf{Comp}_\mathsf{w}^\mathsf{fd})$ *holds);*
(3) the class $\mathcal{P}$ of partially-ordered sets ($(\mathsf{Emb}_\mathsf{w})$ *holds).*

Corollary 13 ([27,30]). *Any (possibly many-sorted) extension of a class of algebras with a semilattice reduct, a (distributive) lattice reduct, or a Boolean algebra reduct, as well as any extension of the theory of reals (integers) with functions satisfying* Mon_f^σ *into an infinite numeric domain is local* ($(\mathsf{Comp}_\mathsf{w}^\mathsf{fd})$ *holds).*

Example 3. Let $\mathcal{T}_0$ be a theory (with a binary predicate $\leq$), and $\mathcal{T}_1$ a local extension of $\mathcal{T}_0$ with two monotone functions f, g. Consider the following problem:

$$\mathcal{T}_0 \cup \mathsf{Mon}_f \cup \mathsf{Mon}_g \models \forall x, y, z, u, v(x \leq y \wedge f(y \vee z) \leq g(u \wedge v) \rightarrow f(x) \leq g(v))$$

The problem reduces to the problem of checking whether $\mathcal{T}_0 \cup \mathsf{Mon}_f \cup \mathsf{Mon}_g \cup G \models \perp$, where $G = c_0 \leq c_1 \wedge f(c_1 \vee c_2) \leq g(c_3 \wedge c_4) \wedge f(c_0) \not\leq g(c_4)$.

The locality of the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ means that, in order to test if $\mathcal{T}_0 \cup \mathsf{Mon}_f \cup \mathsf{Mon}_g \cup G \models \perp$, it is sufficient to test whether $\mathcal{T}_0 \cup \mathsf{Mon}_f[G] \cup \mathsf{Mon}_g[G] \cup G \models_w \perp$, where $\mathsf{Mon}_f[G], \mathsf{Mon}_g[G]$ consist of those instances of the monotonicity axioms for f and g in which the terms starting with f and g already occur in G:

$$\mathsf{Mon}_f[G] = \begin{array}{l} c_0 \leq c_1 \vee c_2 \rightarrow f(c_0) \leq f(c_1 \vee c_2) \\ c_1 \vee c_2 \leq c_0 \rightarrow f(c_1 \vee c_2) \leq f(c_0) \end{array} \qquad \mathsf{Mon}_g[G] = \begin{array}{l} c_4 \leq c_3 \wedge c_4 \rightarrow g(c_4) \leq g(c_3 \wedge c_4) \\ c_3 \wedge c_4 \leq c_4 \rightarrow g(c_3 \wedge c_4) \leq g(c_4) \end{array}$$

In order to check the satisfiability of the latter formula, we purify it, introducing definitions for the terms below the extension functions $d_1 = c_1 \vee c_2, d_2 = c_3 \wedge c_4$ as well as for the terms starting with the extension functions themselves: $f(d_1) = e_1, f(c_0) = e_3, g(c_4, e_4), g(d_2, e_2)$, and add the following (purified) instances of the congruence axioms: $d_1 = c_0 \rightarrow e_1 = e_3$ and $c_4 = d_2 \rightarrow e_4 = e_2$. We obtain the following set of clauses:

Def	G_0	N_0	$\mathcal{K}_0$
$f(d_1) = e_1$	$c_0 \leq c_1$	$d_1 = c_0 \rightarrow e_1 = e_3$	$d_1 \leq c_0 \rightarrow e_1 \leq e_3$
$f(c_0) = e_3$	$d_1 = c_1 \vee c_2$	$d_2 = c_4 \rightarrow e_2 = e_4$	$c_0 \leq d1 \rightarrow e_3 \leq e_1$
$g(c_4) = e_4$	$d_2 = c_3 \wedge c_4$		$d_2 \leq c_4 \rightarrow e_2 \leq e_4$
$g(d_2) = e_2$	$e_1 \leq e_2 \quad \wedge \quad e_3 \not\leq e_4$		$d_4 \leq d_2 \rightarrow e_4 \leq e_2$

We illustrate the hierarchical reduction to testing satisfiability in the base theory for the following examples of local extensions:

(1) Let $\mathcal{T}_0 = \mathcal{DL}$, the theory of distributive lattices or $\mathcal{T}_0 = \mathcal{B}$, the theory of Boolean algebras. The universal clause theory of $\mathcal{DL}$ (resp. $\mathcal{B}$) is the theory of the two element lattice (resp. two element Boolean algebra), so testing Boolean satisfiability is sufficient (this is in NP); any SAT solver can be used for this.

(2) If $\mathcal{T}_0 = \mathcal{L}$ we can reduce the problem above to the problem of checking the satisfiability of a set of ground Horn clauses. This can be checked in PTIME.

(3) If $\mathcal{T}_0 = \mathbb{R}$ we first need to explain what $\vee$ and $\wedge$ are. For this, we replace $d_1 = c_1 \vee c_2$ with $(c_1 \leq c_2 \rightarrow d_1 = c_2) \wedge (c_2 < c_1 \rightarrow d_1 = c_2)$ and similarly for $d_2 = c_3 \wedge c_4$. We proved unsatisfiability using the REDLOG demo [10].

We can therefore conclude that in all cases above:

$\mathcal{T}_1 \models \forall x, y, z, u, v(x \leq y \wedge f(y \vee z) \leq g(u \wedge v) \rightarrow f(x) \leq g(v))$. □

Blockwise and piecewise monotonicity also define local theory extensions if the base theory (for indices) $\mathcal{T}_0$ is a theory endowed with a total order relation $\leq$. The extensions below are similar to some examples considered in [6] but slightly more general since we do not restrict $\mathcal{T}_0$ to be Presburger arithmetic.

Theorem 14. *Let $\mathcal{T}_0$ be a theory endowed with a total order relation $\leq$.*

Piecewise monotonicity. *Assume that $l_1, \dots, l_m, u_1, \dots, u_m$ are constants such that $l_1 \leq u_1 < l_2 \leq u_2 < \dots < l_m \leq u_m$. Let f be a unary function symbol. Any* piecewise-monotone *extension $\mathcal{T}_0 \wedge (\mathsf{GMon}_f)$ of $\mathcal{T}_0$ is local. Here $(\mathsf{GMon}_f) = (\mathsf{GMon}_f^{[l_1,u_1]}) \wedge \dots \wedge (\mathsf{GMon}_f^{[l_m,u_m]})$, where:*

$$(\mathsf{GMon}_f^{[l_i,u_i]}) \qquad \forall x, y (l_i \leq x \leq y \leq u_i \quad \rightarrow \quad f(x) \leq f(y)).$$

Blockwise monotonicity. *Assume that $l_1, \dots, l_m, u_1, \dots, u_m$ are given constants such that $l_1 \leq u_1 < l_2 \leq u_2 < \dots < l_m \leq u_m$. Let f be a unary function. Any* blockwise-monotone *extension $\mathcal{T}_0 \wedge (\mathsf{BMon}_f)$ of $\mathcal{T}_0$ is local. Here $(\mathsf{BMon}_f) = \bigwedge_{i=1}^{m-1} (\mathsf{BMon}_f^{[l_i,u_i],[l_{i+1},u_{i+1}]})$, where:*

$$(\mathsf{BMon}_f^{[l_i,u_i],[l_{i+1},u_{i+1}]}) \quad \forall x, y (l_i \leq x \leq u_i < l_{i+1} \leq y \leq u_{i+1} \rightarrow f(x) \leq f(y)).$$

Similar conditions can be defined for n-ary functions and/or many-sorted functions bridging several theories endowed with total orders.

Strict monotonicity. Strict monotonicity can be handled too, under the assumption of density of the codomain of the functions [20].

5.4 Boundedness Conditions

Any extension of a theory for which $\leq$ is reflexive with functions satisfying (Mon_f^σ) and boundedness (Bound_f^t) conditions is local [28,30].

$$(\mathsf{Bound}_f^t) \qquad \forall x_1, \dots, x_n (f(x_1, \dots, x_n) \leq t(x_1, \dots, x_n))$$

where $t(x_1, \dots, x_n)$ is a term in the base signature Π_0 with variables among $x_1, \dots, x_n$ (such that in any model the associated function has the same monotonicity as f). Similar results can be established for *guarded monotonicity conditions* with mutually disjoint guards [28].

Theorem 15. *Any extension of $\mathcal{T}_0$ with a function $f \not\in \Sigma_0$ satisfying boundedness (Bound_f^t) or guarded boundedness (GBound_f^t) conditions is local.*

$$(\mathsf{Bound}_f^t) \qquad \forall x_1, \dots, x_n (f(x_1, \dots, x_n) \leq t(x_1, \dots, x_n))$$
$$(\mathsf{GBound}_f^t) \quad \forall x_1, \dots, x_n (\phi(x_1, \dots, x_n) \rightarrow f(x_1, \dots, x_n) \leq t(x_1, \dots, x_n))$$

where $t(x_1, \dots, x_n)$ is a term in the base signature Π_0 with variables among $x_1, \dots, x_n$ and $\phi(x_1, \dots, x_n)$ a conjunction of literals in signature Π_0 with variables among $x_1, \dots, x_n$.

Theorem 16 (Piecewise boundedness for free functions). *Let $m \in \mathbb{N}$. For $i \in \{1, \dots, m\}$, let $t_i(x_1, \dots, x_n)$ and $s_i(x_1, \dots, x_n)$ be terms in the signature Π_0 with variables among $x_1, \dots, x_n$, and let $\phi_i(x_1, \dots, x_n)$, $i \in \{1, \dots, m\}$ be conjunctions of literals in the base signature Π_0, with variables among $x_1, \dots, x_n$, i.e. such that for every $i \neq j$, $\phi_i \wedge \phi_j \models_{\mathcal{T}_0} \perp$. Any "piecewise-bounded" extension $\mathcal{T}_0 \wedge (\mathsf{GBound}_f)$, where $f \notin \Sigma_0$, is local. Here $(\mathsf{GBound}_f) = \bigwedge_{i=1}^m (\mathsf{GBound}_f^{[s_i,t_i],\phi_i})$;*

$$(\mathsf{GBound}_f^{[s_i,t_i],\phi_i}) \quad \forall \overline{x} (\phi_i(\overline{x}) \rightarrow s_i(\overline{x}) \leq f(\overline{x}) \leq t_i(\overline{x})).$$

Combinations of (strict) monotonicity with (guarder) boundedness often occur in applications. We present a simple example in the verification of a train controller (for details and more realistic rules we refer to [20]).

Example 4 ([20]). We consider a controller which communicates with all the trains on a given linear track. Trains report their position in given time intervals (Δt) and the controller then communicates them how they can move. The trains adjust their speed accordingly (between given minimum and maximum speeds). These update rules can be described by the following set of clauses where the positions of trains are stored in arrays a (for the current moment of time) and a' for their positions at the next evaluation point (after Δt seconds).

$$
\begin{array}{lll}
(\mathsf{F1}) & \forall i & (i = 0 \quad \rightarrow \quad a(i) + \Delta t {*} \mathsf{min} \leq_{\mathbb{R}} a'(i) \leq_{\mathbb{R}} a(i) + \Delta t {*} \mathsf{max}) \\
(\mathsf{F2}) & \forall i & (0 < i < n \quad \wedge \quad a(p(i)) >_{\mathbb{R}} 0 \quad \wedge \quad a(p(i)) - a(i) \geq_{\mathbb{R}} l_{\mathsf{alarm}} \\
 & & \qquad \rightarrow a(i) + \Delta t * \mathsf{min} \leq_{\mathbb{R}} a'(i) \leq_{\mathbb{R}} a(i) + \Delta t {*} \mathsf{max}) \\
(\mathsf{F3}) & \forall i & (0 < i < n \quad \wedge \quad a(p(i)) >_{\mathbb{R}} 0 \quad \wedge \quad a(p(i)) - a(i) <_{\mathbb{R}} l_{\mathsf{alarm}} \\
 & & \qquad \rightarrow a'(i) = a(i) + \Delta t {*} \mathsf{min}) \\
(\mathsf{F4}) & \forall i & (0 < i < n \quad \wedge \quad a(p(i)) \leq_{\mathbb{R}} 0 \quad \rightarrow \quad a'(i) = a(i)).
\end{array}
$$

The following constants are considered either given or parameters: $\Delta t > 0$ (time between evaluations of the system); minimum/maximum speed of trains $0 \leq \mathsf{min} \leq \mathsf{max}$; l_{alarm} (the distance between trains which is deemed secure); n (the number of trains). An example of an invariant to be checked is collision freeness. At a very abstract level, this can be expressed as a monotonicity axiom,

$$\mathsf{CF}(a) \quad \forall i, j \;\; (0 \leq i < j \leq n \rightarrow a(i) >_{\mathbb{R}} a(j)),$$

where $<$ is an ordering which expresses train precedence and $>_{\mathbb{R}}$ is the usual ordering on the real numbers (i.e. for all trains i, j on the track, if i precedes j then i should be positioned strictly ahead of j). For a more realistic encoding of collision freeness which takes into account the length of trains cf. [20]. To check that collision freeness is an invariant, we check that the initial state is collision free and that collision freeness is preserved by the updating rules $\mathcal{K} = \{\mathsf{F}_1, ..., \mathsf{F}_4\}$.

Let $\mathcal{T}_0$ be a many-sorted combination of real arithmetic – for reasoning about positions, sort num – with an index theory – for describing precedence between trains, sort i. Let $\mathcal{T}$ be the extension of $\mathcal{T}_0$ with the two functions a and a'. We need to check that $\mathcal{T} \models \mathcal{K} \wedge \mathsf{CF}(a) \rightarrow \mathsf{CF}(a')$, i.e.

$$\mathcal{T} \wedge \mathcal{K} \wedge \mathsf{CF}(a) \wedge \neg \mathsf{CF}(a') \models \perp .$$

For this, in [20] we considered two successive extensions of the base theory $\mathcal{T}_0$:

- the extension $\mathcal{T}_1$ of $\mathcal{T}_0$ with a strictly monotone function a, of sort $\mathsf{i} \rightarrow \mathsf{num}$,
- the extension $\mathcal{T}_2$ of $\mathcal{T}_1$ with a function a' satisfying the update axioms $\mathcal{K}$.

By using the previous results we can prove that both these extensions are local. We can theorefore reduce successively, in a hierarchical way (using Lemma 3) the test of satisfiability of a set G of ground clauses w.r.t. $\mathcal{T}_2$ first to a satisfiability test for a set G' of ground clauses w.r.t. $\mathcal{T}_1$, and then to a satisfiability test for a set G'' of ground clauses w.r.t. $\mathcal{T}_0$. This hierarchical approach can also be used for determining constraints between the parameters of the system $(\Delta t, \mathsf{min}, \mathsf{max}, n, l_{\mathsf{alarm}})$ which guarantee collision freeness.

5.5 Data Structures: Theories of Constructors/Selectors

Many data structures important in verification have local or stably local axiomatizations, or can be defined by using chains of local theory extensions. Some examples are given below.

Extensions with selector functions [27]. Let $\mathcal{T}_0$ be a theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, let $c \in \Sigma_0$ with arity n, and let $\Sigma_1 = \{s_1, \dots, s_n\}$ consist of n unary function symbols. Let $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathsf{Sel}_c$ (a theory with signature $\Pi = (\Sigma_0 \cup \Sigma_1, \mathsf{Pred})$) be the extension of $\mathcal{T}_0$ with the set Sel_c of clauses below. Assume that $\mathcal{T}_0$ satisfies the (universally quantified) formula $\mathsf{Inj_c}$ (i.e. c is injective in $\mathcal{T}_0$) then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies condition $(\mathsf{Comp_w})$ [27].

$$
\begin{array}{lll}
(\mathsf{Sel}_c) & s_i(c(x_1, \dots, x_n)) \approx x_i & i \in \{1, \dots, n\} \\
& x \approx c(x_1, \dots, x_n) \rightarrow c(s_1(x), \dots, s_n(x)) \approx x & \\
(\mathsf{Inj_c}) & c(x_1, \dots, x_n) \approx c(y_1, \dots, y_n) \rightarrow (\bigwedge_{i=1}^{n} x_i \approx y_i) &
\end{array}
$$

A general study of the locality of various presentations of theories of constructors and selectors, as well as of theories of arrays is subject of ongoing work jointly with Swen Jacobs and Carsten Ihlemann [19]; it will not be mentioned here. Below, we present a simple example concerning an axiomatization of doubly-linked lists with additional information fields. Then we analyze a class of alternative axiomatizations of pointer structures (studied by Necula and McPeak [23]) and show that these also define stably local theory extensions.

Example 5. Let $\mathcal{T}_1$ be the theory of doubly-linked lists with information on elements, with sorts cell (list cell) and s (scalar, referring to the information stored in the cells). The signature contains the functions s and p (arity $\mathsf{cell} \rightarrow \mathsf{cell}$) and a family of functions $\{\mathsf{info}_i\}_{i \in I}$ (arity $\mathsf{cell} \rightarrow \mathsf{s}$). We assume that s and p satisfy the axioms (1)–(4) in Section 2 (listed directly after Theorem 2 as an axiomatization for Int) and $\{\mathsf{info}_i \mid i \in I\}$ are not constrained by any other axioms. We can view the theory $\mathcal{T}_1$ as the extension of the theory Int in Section 2 with an additional sort s and free function symbols $\{\mathsf{info}_i \mid i \in I\}$. Thus, satisfiability tests for ground clauses w.r.t. $\mathcal{T}_1$ can be reduced in a hierarchical way, in one step to satisfiability tests for ground clauses w.r.t. Int (a local theory). A direct locality proof can also be given. By imposing additional axioms on the info_i functions we still define local extensions: this would for instance be the case if adding guarded boundedness constraints on some of the info_i's, or local sets of axioms $\mathcal{K}$ on $\{\mathsf{info}_i \mid i \in I\}$ subject to the conditions in Theorem 9.

5.6 Verification of Pointer Programs: Local Data Structures

In [23], McPeak and Necula investigate reasoning in pointer data structures. The language used has two sorts (a pointer sort p and a scalar sort s). Sets Σ_p and Σ_s of pointer resp. scalar fields are given. They can be modeled by functions of sort $\mathsf{p} \rightarrow \mathsf{p}$ and $\mathsf{p} \rightarrow \mathsf{s}$, respectively. A constant null of sort p exists. The only predicate of sort p is equality between pointers; predicates of scalar sort can have any arity. In this language one can define pointer (dis)equalities and arbitrary scalar constraints. The local axioms considered in [23] are of the form

$$\forall p \;\; \mathcal{E} \vee \mathcal{C} \tag{1}$$

where $\mathcal{E}$ contains disjunctions of pointer equalities and $\mathcal{C}$ contains scalar constraints (sets of both positive and negative literals). It is assumed that for all terms $f_1(f_2(\dots f_n(p)))$ occurring in the body of an axiom, the axiom also contains the disjunction $p = \mathsf{null} \vee f_n(p) = \mathsf{null} \vee \dots \vee f_2(\dots f_n(p))) = \mathsf{null}$. This has the rôle of excluding null pointer errors. Examples of axioms (for doubly linked data structures with state and priorities) which are considered there are:

$$\begin{array}{ll}
\forall p \;\; p \neq \mathsf{null} \wedge \mathsf{next}(p) \neq \mathsf{null} & \rightarrow \mathsf{prev}(\mathsf{next}(p)) = p \\
\forall p \;\; p \neq \mathsf{null} \wedge \mathsf{next}(p) \neq \mathsf{null} & \rightarrow \mathsf{state}(p) = \mathsf{state}(\mathsf{next}(p)) \\
\forall p \;\; p \neq \mathsf{null} \wedge \mathsf{next}(p) \neq \mathsf{null} \wedge \mathsf{state}(p) = \mathsf{RUN} & \rightarrow \mathsf{priority}(p) \geq \mathsf{priority}(\mathsf{next}(p))
\end{array}$$

(the first axiom states that prev is a left inverse for next, the second axiom tells how a state can be updated; the third axiom is a monotonicity condition on the function $\mathsf{priority}$ with values in a partially ordered domain).

For the sake of simplicity, in what follows we assume that in (1) the disjunctions contain also definedness guards on the scalar fields. The special form of the axioms ensures that all partial models can be embedded into total models.

Theorem 17. *Let $\mathcal{T}_1$ be the two-sorted extension $\mathcal{T}_0 \cup \mathcal{K}$ of a Π_0-theory $\mathcal{T}_0$ (or sort s, the theory of scalars), with signature $\Pi = (S, \Sigma, \mathsf{Pred})$, where $S = \{\mathsf{p}, \mathsf{s}\}$, $\Sigma = \Sigma_p \cup \Sigma_s \cup \Sigma_0$ axiomatized by a set $\mathcal{K}$ of axioms $\forall p(\mathcal{E} \vee \mathcal{C})$ of type (1). Then $\mathcal{T}$ is a stably local extension of $\mathcal{T}_0$.*

Proof: Let $P = (P_p, P_s, \{f_P\}_{f \in \Sigma_p \cup \Sigma_s} \cup \{g_P\}_{g \in \Sigma_0}, \{R_P\}_{R \in \mathsf{Pred}}) \in \mathsf{PMod}(\Sigma_p \cup \Sigma_s, \mathcal{T}_1)$. We construct a total model A starting from P as follows. The universes of A are the same as those of P. For every $f \in \Sigma_p$ and every $p \in P_p$, we define $f_A(p) := f_P(p)$ if $f_P(p)$ is defined, and $f_A(p) := \mathsf{null}$ otherwise. For every $f \in \Sigma_s$ and every $q \in P_p$ we define $f_A(q) := f_P(q)$ if $f_P(q)$ is defined, and $f_A(p) := \mathsf{null}_s$ otherwise. We show that B is a model of $\mathcal{T}_1$: Clearly, $B_{|\Pi_0} = P_{|\Pi_0} = (P_s, \{g_P\}_{g \in \Sigma_0}, \{R_P\}_{R \in \mathsf{Pred}})$ is a total model of $\mathcal{T}_0$. We show that $B \models \mathcal{K}$. Let $C = \forall p(\mathcal{E} \vee \mathcal{C}) \in \mathcal{K}$ and let $\beta : X_p \rightarrow B_p$. If there exists any $t = \mathsf{null}$ in $\mathcal{E}$ with any $\beta(t) = \mathsf{null}$, $\beta \models C$. Assume now that $\beta(t) \neq \mathsf{null}$ for all terms occurring below a function symbol in Σ_p or Σ_s in C. This means that $\beta(t)$ is defined also in P for all such terms. As $(P, \beta) \models C$, there exists a literal L in C such that $(P, \beta) \models L$. We distinguish the following cases: (a) $L = t \approx s$ and both $\beta(t), \beta(s)$ are defined in P. Then they are defined and equal also in B, so $(B, \beta) \models C$. (b)

$L = t \approx s$, $\beta(s)$ is defined, $t = f(t_1, \dots, t_n)$, where $f \in \Sigma_p \cup \Sigma_s$, and $\beta(t_i)$ is undefined for at least one of $t_1, \dots, t_n$. This case cannot occur since we assumed that $\beta(t_i) \neq \mathsf{null}$ for all i; if they were undefined in P the value assigned to them in B would have been null. (c) $L = t \approx s$, $\beta(t)$ and $\beta(s)$ are both undefined. Then the value assigned to them in B is either null if they are of pointer sort, or null_s if they are of scalar sort. Therefore $(B, \beta) \models C$ also in this case. (d) $L = (\neg)R(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ are terms of scalar sort. An argument similar to that used in (b) shows that (if clauses are guarded by definedness conditions for the scalar terms) $\beta(t_i)$ is defined in P for all i so $(P, \beta) \models L$, i.e. $(B, \beta) \models C$. Thus, $(B, \beta) \models C$ for all $\beta : X_p \rightarrow B_p$ and all $C \in \mathcal{K}$. This shows that $B \in \mathsf{Mod}(\mathcal{T}_1)$.[2] □

6 Combinations of Local Extensions

In this section we study the locality of combinations of local theory extensions. In the light of the results in Section 4.1 we concentrate on studying which embeddability properties are preserved under combinations of theories. For the sake of simplicity, in what follows we only consider conditions $(\mathsf{Emb_w})$ and $(\mathsf{Comp_w})$. Analogous results can be given for conditions $(\mathsf{Emb_w^f})$, $(\mathsf{Comp_w^f})$, resp. $(\mathsf{Emb_w^{fd}})$, $(\mathsf{Comp_w^{fd}})$ and combinations thereof. Full proofs are contained in [29].

We first consider the situation when both components satisfy the embeddability condition $(\mathsf{Comp_w})$.

Theorem 18. *Let $\mathcal{T}_0$ be a first order theory with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$ and (for $i \in \{1, 2\}$) $\mathcal{T}_i = \mathcal{T}_0 \cup \mathcal{K}_i$ be an extension of $\mathcal{T}_0$ with signature $\Pi_i = (\Sigma_0 \cup \Sigma_i, \mathsf{Pred})$. Assume that both extensions $\mathcal{T}_0 \subseteq \mathcal{T}_1$ and $\mathcal{T}_0 \subseteq \mathcal{T}_2$ satisfy condition $(\mathsf{Comp_w})$, and that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Then the extension $\mathcal{T}_0 \subseteq \mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ satisfies condition $(\mathsf{Comp_w})$. If, additionally, in $\mathcal{K}_i$ all terms starting with a function symbol in Σ_i are flat and linear, for $i = 1, 2$, then the extension is local.*

Example 6. *The following combinations of theories (seen as extensions of a first-order theory $\mathcal{T}_0$) satisfy condition $(\mathsf{Comp_w})$ (in case (4) condition $(\mathsf{Comp_w^{fd}})$):*

(1) $\mathcal{T}_0 \cup \mathsf{Free}(\Sigma_1)$ and $\mathcal{T}_0 \cup \mathsf{Sel}_c$ if $\mathcal{T}_0$ is a theory and $c \in \Sigma_0$ is injective in $\mathcal{T}_0$.

(2) $\mathbb{R} \cup \mathsf{Free}(\Sigma_1)$ and $\mathbb{R} \cup \mathsf{Lip}_c^\lambda(f)$, where $f \notin \Sigma_1$. Here $\mathsf{Lip}_c^\lambda(f)$ is the λ-Lipschitz condition[3] for f at point $c \in \mathbb{R}$ (for $\lambda > 0$):

$$(\mathsf{Lip}_c^\lambda(f)) \qquad \forall x \; |f(x) - f(c)| \leq \lambda \cdot |x - c|.$$

(3) $\mathbb{R} \cup \mathsf{Lip}_{c_1}^{\lambda_1}(f)$ and $\mathbb{R} \cup \mathsf{Lip}_{c_2}^{\lambda_2}(g)$, where $f \neq g$.

[2] Analyzing the proof of Theorem 7 (embeddability implies stable locality), one can see that for any valuation β and set Ψ of ground terms of sort p closed under subterms the proof only uses embeddability into total models of a special type of partial models P, namely those for which for $f \in \Sigma_p \cup \Sigma_s$, $f_P(p_1, \dots, p_n)$ is defined iff there exist terms $t_1, \dots t_n$ in Ψ which evaluate to $p_1, \dots, p_n$ w.r.t. β. With such restrictions the definedness guards on scalar terms are not necessary for proving stable locality.

[3] We proved in [27] that for every function f and constants c and λ with $\lambda > 0$ the extension $\mathbb{R} \subseteq \mathbb{R} \cup (\mathsf{Lip_c^\lambda(f)})$ satisfies $(\mathsf{Comp_w})$, hence it is local.

(4) $\mathcal{T}_0 \cup \mathsf{Free}(\Sigma_1)$ *and* $\mathcal{T}_0 \cup \mathsf{Mon}_f^\sigma$, *where* $f \notin \Sigma_1$ *has arity* n, $\sigma : \{1, \dots, n\} \rightarrow \{-1, 1, 0\}$, *if* $\mathcal{T}_0$ *is, e.g., a theory of algebras with a bounded semilattice reduct.*

This result can be extended to the more general situation in which one extension satisfies condition $(\mathsf{Emb_w})$ and the other satisfies $(\mathsf{Comp_w})$ or $(\mathsf{Emb_w})$.

Theorem 19. *Let* $\mathcal{T}_0$ *be a first order theory with signature* $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, *and let* $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}_1$ *and* $\mathcal{T}_2 = \mathcal{T}_0 \cup \mathcal{K}_2$ *be two extensions of* $\mathcal{T}_0$ *with signatures* $\Pi_1 = (\Sigma_0 \cup \Sigma_1, \mathsf{Pred})$ *and* $\Pi_2 = (\Sigma_0 \cup \Sigma_2, \mathsf{Pred})$, *respectively. Assume that:*

(1) $\mathcal{T}_0 \subseteq \mathcal{T}_1$ *satisfies condition* $(\mathsf{Comp_w})$,
(2) $\mathcal{T}_0 \subseteq \mathcal{T}_2$ *satisfies condition* $(\mathsf{Emb_w})$,
(3) $\mathcal{K}_1$ *is a set of* Σ_1*-flat clauses in which all variables occur below a* Σ_1*-function.*

Then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ *satisfies* $(\mathsf{Emb_w})$. *If in* $\mathcal{K}_i$ *all terms starting with a function symbol in* Σ_i *are flat and linear (for* i=1, 2*) the extension is local.*

Theorem 20. *Let* $\mathcal{T}_0$ *be an arbitrary theory in signature* $\Pi_0 = (\Sigma_0, \mathsf{Pred})$. *Let* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *be two sets of clauses over signatures* $\Pi_i = (\Sigma_0 \cup \Sigma_i, \mathsf{Pred})$, *where* Σ_1 *and* Σ_2 *are disjoint. We make the following assumptions:*

(A1) *The class of models of* $\mathcal{T}_0$ *is closed under direct limits of diagrams in which all maps are embeddings (or, equivalently,* $\mathcal{T}_0$ *is a* $\forall\exists$ *theory).*
(A2) $\mathcal{K}_i$ *is* Σ_i*-flat and* Σ_i*-linear for* $i = 1, 2$, *and* $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_i$, $i = 1, 2$ *are both local extensions of* $\mathcal{T}_0$.
(A3) *For all clauses in* $\mathcal{K}_1$ *and* $\mathcal{K}_2$, *every variable occurs below some extension function.*

Then $\mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ *is a local extension of* $\mathcal{T}_0$.

Example 7. *The following combinations of theories (seen as extensions of the theory* $\mathcal{T}_0$*) satisfy condition* $(\mathsf{Emb_w})$, *hence are local:*

(1) $\mathcal{E}q \subseteq \mathsf{Free}(\Sigma_1) \cup \mathcal{L}$, *where* $\mathcal{E}q$ *is the pure theory of equality, without function symbols, and* $\mathcal{L}$ *the theory of lattices.*
(2) $\mathcal{T}_0 \subseteq (\mathcal{T}_0 \cup \mathsf{Free}(\Sigma_1)) \cup (\mathcal{T}_0 \cup \mathsf{Mon}(\Sigma_2))$, *where* $\Sigma_1 \cap \Sigma_2 = \emptyset$, $\mathsf{Mon}(\Sigma_2) = \bigwedge_{f \in \Sigma_2} \mathsf{Mon}_f^{\sigma(f)}$ *and* $\mathcal{T}_0$ *is, e.g., the theory of posets.*
(3) *The combination of the theory of lattices and the theory of integers with injective successor and predecessor is local (local extension of the theory of pure equality).*

7 Modular Reasoning

In what follows we discuss some issues related to modular reasoning in combinations of local theory extensions. We analyze, in particular, the form of information which needs to be exchanged between provers for the component theories when reasoning in combinations of local theory extensions.

7.1 Reasoning in Local Combinations of Theory Extensions

Let $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}_1$ and $\mathcal{T}_2 = \mathcal{T}_0 \cup \mathcal{K}_2$ be theories with signatures $\Pi_1 = (\Sigma_0 \cup \Sigma_1, \mathsf{Pred})$ and $\Pi_2 = (\Sigma_0 \cup \Sigma_2, \mathsf{Pred})$, and G a set of ground clauses in the joint signature with additional constants $\Pi^c = (\Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_c, \mathsf{Pred})$. We want to decide whether $\mathcal{T}_1 \cup \mathcal{T}_2 \cup G \models \perp$.

The set G of ground clauses can be flattened and purified as explained above. For the sake of simplicity, everywhere in what follows we will assume w.l.o.g. that $G = G_1 \wedge G_2$, where G_1, G_2 are flat and linear sets of clauses in the signatures Π_1, Π_2 respectively, i.e. for $i = 1, 2$, $G_i = G_i^0 \wedge G_0 \wedge D_i$, where G_i^0 and G_0 are clauses in the base theory and D_i a conjunction of unit clauses of the form $f(c_1, \dots, c_n) = c, f \in \Sigma_i$.

Corollary 21. *Assume that $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}_1$ and $\mathcal{T}_2 = \mathcal{T}_0 \cup \mathcal{K}_2$ are local extensions of a theory $\mathcal{T}_0$ with signature $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, and that the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is local. Let $G = G_1 \wedge G_2$ be a set of flat, linear are purified ground clauses, such that $G_i = G_i^0 \wedge G_0 \wedge D_i$ are as explained above. Then the following are equivalent:*

(1) $\mathcal{T}_1 \cup \mathcal{T}_2 \cup (G_1 \wedge G_2) \models \perp$,
(2) $\mathcal{T}_0 \cup (\mathcal{K}_1 \cup \mathcal{K}_2)[G_1 \wedge G_2] \cup (G_1^0 \wedge G_0 \wedge D_1) \wedge (G_2^0 \wedge G_0 \wedge D_2) \models \perp$,
(3) $\mathcal{T}_0 \cup \mathcal{K}_1^0 \cup \mathcal{K}_2^0 \cup (G_1^0 \cup G_0) \cup (G_2^0 \cup G_0) \cup N_1 \cup N_2 \models \perp$, *where*

$$N_i = \{\bigwedge_{i=1}^{n} c_i \approx d_i \rightarrow c = d \mid f(c_1, \dots, c_n) \approx c, f(d_1, \dots, d_n) \approx d \in D_i\}, i = 1, 2,$$

and $\mathcal{K}_i^0$ is the formula obtained from $\mathcal{K}_i[G_i]$ after purification and flattening, taking into account the definitions from D_i.

A more precise characterization of the formulae that need to be exchanged between provers for the components is provided by results on interpolation.

7.2 Interpolation in Local Theory Extensions

A theory $\mathcal{T}$ has *interpolation* if, for all formulae ϕ and ψ in the signature of $\mathcal{T}$, if $\phi \models_{\mathcal{T}} \psi$ then there exists a formula I containing only symbols which occur in both ϕ and ψ such that $\phi \models_{\mathcal{T}} I$ and $I \models_{\mathcal{T}} \psi$. First order logic has interpolation [9], but for an arbitrary first order theory $\mathcal{T}$, the interpolants may contain (alternations of) quantifiers even for very simple formulae ϕ and ψ. It is important to identify situations in which ground clauses have ground interpolants.

> A theory $\mathcal{T}$ has the *ground interpolation property* if for all ground clauses $A(\overline{c}, \overline{d})$ and $B(\overline{c}, \overline{e})$, if $A(\overline{c}, \overline{d}) \wedge B(\overline{c}, \overline{e}) \models_{\mathcal{T}} \perp$ then there exists a ground formula $I(\overline{c})$, containing only the constants $\overline{c}$ occurring both in A and B, such that $A(\overline{c}, \overline{d}) \models_{\mathcal{T}} I(\overline{c})$ and $B(\overline{c}, \overline{e}) \wedge I(\overline{c}) \models_{\mathcal{T}} \perp$.

In [28] we identify a class of theory extensions $\mathcal{T}_0 \subseteq \mathcal{T}_1$ for which interpolants can be computed hierarchically using a procedure for generating interpolants in the base theory $\mathcal{T}_0$. This allows to exploit specific properties of $\mathcal{T}_0$ for obtaining simple interpolants in $\mathcal{T}_1$. We make the following assumptions[4] about $\mathcal{T}_0$ and $\mathcal{T}_1$:

Assumption 1: $\mathcal{T}_0$ is *convex* w.r.t. the set Pred of all predicates (including equality $\approx$), i.e., for all conjunctions Γ of ground atoms, relations $R_1, \dots, R_m \in \mathsf{Pred}$ and ground tuples of corresponding arity $\overline{t}_1, \dots, \overline{t}_n$, if $\Gamma \models_{\mathcal{T}_0} \bigvee_{i=1}^m R_i(\overline{t}_i)$ then there exists a $j \in \{1, \dots, m\}$ such that $\Gamma \models_{\mathcal{T}_0} R_j(\overline{t}_j)$.

Assumption 2: $\mathcal{T}_0$ is *P-interpolating*, i.e. for all conjunctions A and B of ground literals, all binary predicates $R \in P$ and all constants a and b such that a occurs in A and b occurs in B (or vice versa), if $A \wedge B \models_{\mathcal{T}_0} aRb$ then there exists a term t containing only constants common to A and B with $A \wedge B \models_{\mathcal{T}_0} aRt \wedge tRb$.

Assumption 3: $\mathcal{T}_0$ has ground interpolation.

Assumption 4: $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$, where $\mathcal{K}$ consists of the combinations of clauses:

$$\begin{cases} x_1\, R_1\, s_1 \wedge \dots \wedge x_n\, R_n\, s_n \rightarrow f(x_1, \dots, x_n)\, R\, g(y_1, \dots, y_n) \\ x_1\, R_1\, y_1 \wedge \dots \wedge x_n\, R_n\, y_n \rightarrow f(x_1, \dots, x_n)\, R\, f(y_1, \dots, y_n) \end{cases} \quad (2)$$

where $n \geq 1$, $x_1, \dots, x_n$ are variables, $R_1, \dots, R_n, R$ are binary relations with $R_1, \dots, R_n \in P$ and R transitive, and each s_i is either a variable among the arguments of g, or a term of the form $f_i(z_1, \dots, z_k)$, where $f_i \in \Sigma_1$ and all the arguments of f_i are variables occurring among the arguments of g.

Theorem 22 ([28]). *If the theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies the assumptions above then ground interpolants for $\mathcal{T}_1$ exist and can be computed hierarchically.*

In [25] we adapt and apply the idea of Theorem 22 for efficiently computing interpolants with a simple form for extensions of linear arithmetic with free function symbols, as an alternative to the method proposed by McMillan [22].

As a consequence of the results in [28], the following theory extensions have ground interpolation, and interpolants can be computed hierarchically.

(a) Extensions with free function symbols of any of the base theories: $\mathcal{E}q$ (pure equality), $\mathcal{P}$ (posets), $\mathsf{LI}(\mathbb{Q}), \mathsf{LI}(\mathbb{R})$ (linear rational, resp. real arithmetic), $\mathcal{S}$ (semilattices), $\mathcal{DL}$ (lattices), $\mathcal{B}$ Boolean algebras.
(b) Extensions with monotone functions of any of the base theories: $\mathcal{P}$ (posets), $\mathcal{S}$ (semilattices), $\mathcal{DL}$ (lattices), $\mathcal{B}$ Boolean algebras.
(c) Extensions of any of the base theories in (b) with $\mathsf{Leq}(f, g) \wedge \mathsf{Mon}_f$.
(d) Extensions of any of the base theories in (b) with $\mathsf{SGc}(\mathsf{f}, \mathsf{g_1}) \wedge \mathsf{Mon}(f, g_1)$.
(e) Extensions of any of the base theories in (a) with Bound^t_f or GBound^t_f (where t is a term and ϕ a set of literals in the base theory).
(f) Extensions of any base theory in (b) with $\mathsf{Mon}_f \wedge \mathsf{Bound}^t_f$, if t is monotone.

7.3 Application: Information Exchange in Combinations of Theories

The method for hierarchic reasoning described in Corollary 21 is modular, in the sense that once the information about $\Sigma_1 \cup \Sigma_2$-functions was separated into a

[4] Examples of theories which have these properties are provided in [28].

Σ_1-part and a Σ_2-part, it does not need to be recombined again. For reasoning in the combined theory one can proceed as follows:

- Purify (and flatten) the goal G, and thus transform it into an equisatisfiable conjunction $G_1 \wedge G_2$, where G_i consists of clauses in the signature Π_i, for $i = 1, 2$, and $G_i = G_i^0 \wedge G_0 \wedge D_i$, as in Corollary 21.
- The problem of testing the validity of the formulae containing extension functions in the signature Σ_i, $\mathcal{K}_i[G_i] \wedge G_i$ are reduced (using Lemma 3) to testing the validity of the formula $\mathcal{K}_i^0 \wedge G_i^0 \wedge G_0 \wedge N_i$ in the base theory.
- The conjunction of all the formulae obtained this way, for all component theories, is used as input for a decision procedure for the base theory.

We show that, in fact, only information exchange over the shared signature (i.e. shared functions and constants) is necessary.

Theorem 23 ([28]). *Let $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_i$ be local extensions, $i = 1, 2$ where $\mathcal{K}_i$ are Σ_i-flat and Σ_i-linear and all variables in clauses in $\mathcal{K}_i$ occur below a Σ_i-symbol. Assume the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$ is local. Let $G = G_1 \wedge G_2$ be as constructed before with $\mathcal{T}_0 \cup (\mathcal{K}_1 \wedge G_1) \wedge (\mathcal{K}_2 \wedge G_2) \models \perp$. Then we can construct a ground formula I which contains only function symbols in $\Sigma_0 = \Sigma_1 \cap \Sigma_2$ and constants shared by G_1, G_2 such that $(\mathcal{T}_0 \cup \mathcal{K}_1) \wedge G_1 \models I$ and $(\mathcal{T}_0 \cup \mathcal{K}_2) \wedge G_2 \wedge I \models \perp$.*

8 Conclusions

We presented an overview of results on hierarchical and modular reasoning in complex theories. We show that for *local* and *stably local* theory extensions hierarchic reasoning is possible (i.e. proof tasks in the extension can be hierarchically reduced to proof tasks w.r.t. the base theory). We showed how local theory extensions can be identified and provided various examples from mathematics and verification. In particular, we identified phenomena analyzed in the verification literature which can be explained using the notion of locality.

We then presented criteria for recognizing situations in which combinations of theory extensions of a base theory are again local extensions of the base theory. These results allow to recognize even wider classes of local theory extensions, and open the way for studying possibilities of modular reasoning in such extensions. For this, it is interesting to analyze the exact amount of information which needs to be exchanged between provers for the component theories. We characterized the form of this information in the case of local combinations of local extensions. We plan to investigate whether there are any links between the results described in this paper and other methods for reasoning in combinations of theories over non-disjoint signatures e.g. by Ghilardi [15].

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