# Automated reasoning in some local extensions of ordered structures

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## Abstract

*We give a uniform method for automated reasoning in several types of extensions of ordered algebraic structures (definitional extensions, extensions with boundedness axioms or with monotonicity axioms). We show that such extensions are local and, hence, efficient methods for hierarchical reasoning exist in all these cases.*

## 1 Introduction

We present a uniform method for automated reasoning in (i) extensions with functions which can be uniquely defined in terms of existing operations (definitional extensions), (ii) extensions with functions which satisfy general monotonicity laws and (iii) extensions with functions satisfying boundedness conditions (and possibly also monotonicity).

In previous work [19] we gave a resolution-based decision procedure for the universal fragment of the class of distributive lattices (or Boolean algebras) with join/meet hemimorphisms satisfying certain residuation conditions. For this, we used extensions of representation theorems for Boolean algebras [23] or distributive lattices [16] (cf. e.g. [8, 17]). This allowed us to use, instead of algebraic models, relational structures (usually much simpler). The main obstacle we faced when attempting to extend these methods to lattice-based structures or to monotone operators is that, although representation theorems exist, the dual spaces are quite complex: in the case of representation theorems for lattices [24, 13] they are doubly-ordered spaces (with a closure operator); in representation theorems for distributive lattices with monotone operators [4] they are endowed with maps having as values sets of elements.

In this paper we use a different approach: We show that extensions with monotone functions (and also several other theory extensions) satisfy an embeddability property of partial into total algebras, and thus are local (cf. [20]). In [20] we showed that in a local extension $\mathcal{T}_1$ of a theory $\mathcal{T}_0$ testing satisfiability of ground clauses w.r.t. $\mathcal{T}_1$ can be reduced to testing satisfiability of certain types of formulae w.r.t. $\mathcal{T}_0$. This also allows to express, parametrically, the decidability and complexity for the universal theory of $\mathcal{T}_1$ in terms of the decidability (complexity) of a certain fragment of $\mathcal{T}_0$.

For this, we use a generalization of ideas of Burris [2] and Ganzinger [5] which we proposed in [20]. Burris [2] proved that if a quasi-variety axiomatized by a set $\mathcal{K}$ of Horn clauses has the property that every finite partial algebra which is a partial model of the axioms in $\mathcal{K}$ can be extended to a total algebra model of $\mathcal{K}$, then the uniform word problem for $\mathcal{K}$ is decidable in polynomial time (this generalizes previous results by Skolem and Evans). A link between these embeddability properties and a proof theoretic concept (local theories) was established by Ganzinger [5]. In [20], we extend these results to special types of *theory extensions*, which we call *local* (cf. definitions in Sect.3).

In this paper we analyze the applications of the results on local theory extensions in [20] to automated reasoning in algebraic structures related to non-classical logics. The main contributions of the paper are summarized below:

- We give a new criterion for recognizing locality of a theory extension (Sect. 3.1, Thm. 3).
- We show that extensions with functions defined in terms of existing operators are local. As an illustration, we obtain a new decision procedure for the universal fragment of the class $\mathcal{MV}$ of MV-algebras, and show that the problem of deciding validity of sets of clauses w.r.t. $\mathcal{MV}$ is co-NP complete (Sect. 4).
- We prove that certain extensions with functions satisfying generalized monotonicity laws and/or boundedness conditions are also local (Sect. 5 and Sect. 6).

We illustrate the ideas (and the hierarchical reasoning method) for the case of MV-algebras with a monotone operator satisfying a boundedness condition (Sect. 6.1).

## 2 Preliminaries

**Posets and lattices.** We assume known standard notions, such as partially ordered set and lattice. For definitions and further information we refer to [3].

We denote the dual of a partially ordered set $P = (X, \leq)$ by $P^\partial = (X, \leq^\partial) = (X, \geq)$. For $x \in P$, $x^\downarrow := \{y \in P \mid$

$y \leq x\}$ and $x^{\uparrow} := \{y \in P \mid y \geq x\}$. For $A \subseteq P$, $A^u = \{x \mid \forall a \in A(x \geq a)\}$ and $A^l = \{x \mid \forall a \in A(x \leq a)\}$ denote the set of upper and resp. lower bounds of A.

The *Dedekind-MacNeille* completion $\mathrm{DM}(P)$ of a poset $P$ consists of all subsets of $P$ satisfying $A^{ul} = A$, ordered by inclusion. The map $x \mapsto x^{\downarrow}$ embeds $P$ into $\mathrm{DM}(P)$ and preserves infima and suprema if they exist. The map $^l : \mathrm{DM}(P^{\partial}) \rightarrow \mathrm{DM}(P)^{\partial}$, sending a set to the set of its lower bounds, is an isomorphism; its inverse is the map $^u$.

**Theories and models.** Let $\Pi = (S, \Sigma, \mathsf{Pred})$ be a signature where $S$ is a set of sorts, $\Sigma$ is a set of function symbols and $\mathsf{Pred}$ a set of predicate symbols. A $\Pi$-structure is a tuple

$$\mathcal{M} = (\{M_s\}_{s\in S}, \{f_{\mathcal{M}}\}_{f\in\Sigma}, \{P_{\mathcal{M}}\}_{P\in\mathsf{Pred}}),$$

where for every $s \in S$, $M_s \neq \emptyset$, for all $f \in \Sigma$ with arity $a(f)=s_1\times\ldots\times s_n\rightarrow s$, $f_{\mathcal{M}} : \prod_{i=1}^n M_{s_i}\rightarrow M_s$ and for all $P \in \mathsf{Pred}$ with arity $a(P) = s_1\times\ldots\times s_n$, $P_{\mathcal{M}} \subseteq \prod_{i=1}^n M_{s_i}$. We consider formulae over variables in a (many-sorted) family $X = \{X_s \mid s \in S\}$, where for every $s \in S$, $X_s$ is a set of variables of sort $s$. Theories can be regarded as sets of formulae or as sets of models. A model of a set $\mathcal{T}$ of $\Pi$-formulae is a $\Pi$-structure satisfying all formulae of $\mathcal{T}$. In this paper, whenever we speak about a theory $\mathcal{T}$ we implicitly refer to the set of all models of $\mathcal{T}$.

A *partial* $\Pi$-*structure* $(\{M_s\}_{s\in S}, \{f_{\mathcal{M}}\}_{f\in\Sigma}, \{P_{\mathcal{M}}\}_{P\in\mathsf{Pred}})$ is a structure where for every $s \in S$, $M_s \neq \emptyset$ and for every $f \in \Sigma$ with arity $s_1\times\ldots\times s_n\rightarrow s$, $f_{\mathcal{M}}$ is a partial function from $\prod_{i=1}^n M_{s_i}$ to $M_s$. The notion of evaluating a term $t$ with variables $X = \{X_s \mid s \in S\}$ w.r.t. an assignment $\{\beta_s{:}X_s \rightarrow M_s \mid s \in S\}$ (notation: $\beta : X \rightarrow \mathcal{M}$) in a partial structure $\mathcal{M}$ is the same as for total structures, except that the evaluation is undefined if $t = f(t_1, \ldots, t_n)$ with $a(f)=s_1\times\ldots\times s_n\rightarrow s$, and at least one of $\beta_{s_i}(t_i)$ is undefined, or else $(\beta_{s_1}(t_1), \ldots, \beta_{s_n}(t_n))$ is not in the domain of $f_{\mathcal{M}}$. Let $\mathcal{M}$ be a partial $\Pi$-structure, $C$ a clause and $\beta : X \rightarrow \mathcal{M}$. We say that $(\mathcal{M}, \beta) \models_w C$ iff either

(i) for some term $t$ in $C$, $\beta(t)$ is undefined, or else

(ii) $\beta(t)$ is defined for all terms $t$ of $C$, and there exists a literal $L$ in $C$ s.t. $\beta(L)$ is true in $\mathcal{M}$.

$\mathcal{M}$ *weakly satisfies* $C$ (notation $\mathcal{M}\models_w C$) if $(\mathcal{M}, \beta)\models_w C$ for all $\beta : X \rightarrow \mathcal{M}$. $\mathcal{M}$ *is a weak partial model of a set of clauses* $\mathcal{K}$ (notation $\mathcal{M}\models_w\mathcal{K}$) if $\mathcal{M}\models_w C$ for all $C \in \mathcal{K}$.

## 3 Local theory extensions

Let $\mathcal{T}_0$ be a theory with signature $\Pi_0 = (S_0, \Sigma_0, \mathsf{Pred})$. We consider extensions $\mathcal{T}_1$ of $\mathcal{T}_0$ with signature $\Pi = (S, \Sigma, \mathsf{Pred})$, where $S = S_0 \cup S_1, \Sigma = \Sigma_0 \cup \Sigma_1$ (i.e. the signature is extended by new sorts and function symbols) obtained by adding a set $\mathcal{K}$ of (universally quantified) clauses. Let $\mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ be the class of all weak partial models $P$ of $\mathcal{K}$, in which the $\Sigma_1$-functions are partial and such that $P_{|\Pi_0}$ is a total model of $\mathcal{T}_0$. In what follows, when referring to sets $G$ of ground clauses we assume they are in the signature $\Pi^c = (S, \Sigma \cup \Sigma_c, \mathsf{Pred})$ where $\Sigma_c$ is a set of new constants.

An extension $\mathcal{T}_0 \subseteq \mathcal{T}_1{=}\mathcal{T}_0{\cup}\mathcal{K}$ is *local* if satisfiability of any set $G$ of ground clauses w.r.t. $\mathcal{T}_0{\cup}\mathcal{K}$ only depends on $\mathcal{T}_0$ and those instances $\mathcal{K}[G]$ of $\mathcal{K}$ in which the terms starting with extension functions are in the set $\mathsf{st}(\mathcal{K}, G)$ of ground terms which already occur in $G$ or $\mathcal{K}$. Formally, $\mathcal{T}_0 \subseteq \mathcal{T}_1{=}\mathcal{T}_0 \cup \mathcal{K}$ is a local extension if it satisfies condition $(\mathsf{Loc})$:

$(\mathsf{Loc})$ For every set $G$ of ground clauses $G \models_{\mathcal{T}_1} \perp$ iff there is no partial $\Pi^c$-structure $P$ such that $P_{|\Pi_0}$ is a total model of $\mathcal{T}_0$, all terms in $\mathsf{st}(\mathcal{K}, G)$ are defined in $P$, and $P$ weakly satisfies $\mathcal{K}[G] \wedge G$.

**Embeddability and locality.** In [20] we show that embeddability of weak partial models into total models $(\mathsf{Emb_w})$ implies locality of an extension.

$(\mathsf{Emb_w})$ Every $A \in \mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ weakly embeds into a total model of $\mathcal{T}_1$.

A non-ground clause is $\Sigma_1$-*flat* if function symbols (including constants) do not occur as arguments of functions in $\Sigma_1$. A $\Sigma_1$-flat non-ground clause is called $\Sigma_1$-*linear* if whenever a variable occurs in two terms in the clause which start with a function symbol in $\Sigma_1$, the two terms are identical, and if no term which starts with a function in $\Sigma_1$ contains two occurrences of the same variable.

**Theorem 1 ([20])** *Let $\mathcal{K}$ be a set of clauses which are $\Sigma_1$-flat and $\Sigma_1$-linear, and let $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$. If $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies* $(\mathsf{Emb_w})$ *then it satisfies* $(\mathsf{Loc})$.

Examples of local theory extensions were given in [20, 21]. In this paper we give other examples, including extensions with general monotonicity conditions.

**Hierarchic reasoning in local theory extensions.** Let $\mathcal{T}_0 \subseteq \mathcal{T}_1{=}\mathcal{T}_0 \cup \mathcal{K}$ be a local theory extension. To check the satisfiability of a set $G$ of ground clauses w.r.t. $\mathcal{T}_1$ we can proceed as follows (for details cf. [20]):

*Step 1: Use locality.* By the locality condition, $G$ is unsatisfiable w.r.t. $\mathcal{T}_1$ iff $\mathcal{K}[G] \wedge G$ has no weak partial model in which all the subterms of $\mathcal{K}[G] \wedge G$ are defined, and whose restriction to $\Pi_0$ is a total model of $\mathcal{T}_0$.

*Step 2: Flattening and purification.* We purify and flatten $\mathcal{K}[G] \wedge G$ by introducing new constants for the arguments of the extension functions as well as for the (sub)terms $t = f(g_1, \ldots, g_n)$ starting with extension functions $f \in \Sigma_1$, together with new corresponding definitions $c_t \approx t$. The set of clauses thus obtained has the form $\mathcal{K}_0 \wedge G_0 \wedge D$, where $D$ is a set of ground unit clauses of the form $f(c_1, \ldots, c_n) \approx c$, where $f \in \Sigma_1$ and $c_1, \dots, c_n, c$ are constants, and $\mathcal{K}_0, G_0$ are clause sets without function symbols in $\Sigma_1$.

*Step 3: Reduction to testing satisfiability in $\mathcal{T}_0$.* We reduce the problem to testing satisfiability in $\mathcal{T}_0$ by replacing $D$ with the following set of clauses:

$$\mathsf{Con}[D] = \bigwedge \{ \bigwedge_{i=1}^{n} c_i = d_i \rightarrow c = d \,|\, f(c_1, \dots, c_n) = c \in D, \; f(d_1, \dots, d_n) = d \in D \}.$$

**Theorem 2 ([20])** *If $\mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ satisfies $(\mathsf{Loc})$ then (with the notations above) the following are equivalent:*

*(1) $\mathcal{T}_0 \wedge \mathcal{K} \wedge G$ has a model.*

*(2) $\mathcal{T}_0 \wedge \mathcal{K}[G] \wedge G$ has a partial model in $\mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ (where all terms in $\mathsf{st}(\mathcal{K}, G)$ are defined).*

*(3) $\mathcal{T}_0 \wedge \mathcal{K}_0 \wedge G_0 \wedge D$ has a partial model in $\mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ (where all terms in $\mathsf{st}(\mathcal{K}, G)$ are defined).*

*(4) $\mathcal{T}_0 \wedge \mathcal{K}_0 \wedge G_0 \wedge \mathsf{Con}[D]$ has a (total) $\Pi_0$-model.*

### 3.1 A finite locality criterion

Note that $(\mathsf{Loc})$ refers to satisfiability of *arbitrary* sets $G$ of ground clauses. We are however interested in *finite locality* $(\mathsf{Loc_f})$ (the same as $(\mathsf{Loc})$, except it has to hold for *finite* sets $G$ of clauses) [20]. We say that *the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ is finitely local* (*$\mathcal{T}_1$ is a finitely local extension of $\mathcal{T}_0$*) if $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies condition $(\mathsf{Loc_f})$. In [20] we showed that finite locality is guaranteed if (i) $\mathcal{T}_0$ is universal and locally finite, (ii) $\mathcal{K}$ contains finitely many ground terms, and (iii) all *finite* models in $\mathsf{PMod}(\Sigma_1, \mathcal{T}_1)$ embed into total models.

An easy change in the proof of Theorem 1 yields an alternative criterion for recognizing finite locality.

Let $\mathsf{PMod_w^{fd}}(\Sigma_1, \mathcal{T}_1)$ be the subclass of $\mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ in which the $\Sigma_1$-functions are defined on a finite set.

$(\mathsf{Emb_w^{fd}})$ Every $A \in \mathsf{PMod_w^{fd}}(\Sigma_1, \mathcal{T}_1)$ weakly embeds into a total model of $\mathcal{T}_1$.

**Theorem 3** *Let $\mathcal{K}$ be a set of clauses which are $\Sigma_1$-flat and $\Sigma_1$-linear, and let $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$. Assume that $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies $(\mathsf{Emb_w^{fd}})$. Then $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies $(\mathsf{Loc_f})$.*

**Theorem 4 (Decidability)** *Assume that the theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies condition $(\mathsf{Loc_f})$. Then:*

*(a) If all variables in the clauses in $\mathcal{K}$ occur below some $\Sigma_1$-function symbol and if the universal theory of $\mathcal{T}_0$ is decidable, then the universal theory of $\mathcal{T}_1$ is decidable.*

*(b) If the $\forall\exists$ theory of $\mathcal{T}_0$ is decidable then the universal theory of $\mathcal{T}_1$ is decidable.*

**Theorem 5 (Complexity)** *Let $\mathcal{T}_0$ be a theory for which the satisfiability of a set of ground clauses of size $n$ can be checked in time at most $g(n)$, and let $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ be a local theory extension where in every clause in $\mathcal{K}$ each variable occurs below some extension function. The validity of a set of clauses in the extension can be checked in time $g(c \cdot n^k)$, where $c$ is a constant and $k$ is the maximum number of extension terms in a clause in $\mathcal{K}$ or in a congruence axiom.*

## 4 Definitional extensions

We start by studying extensions of a $\Sigma_0$-theory $\mathcal{T}_0$ (i.e. a class of $\Sigma_0$-algebras) with operators in a set $\Sigma$ which are defined in terms of the operations in $\Sigma_0$.

**Theorem 6** *Let $\mathcal{T}_0$ be a $\Sigma_0$-theory and $\Sigma_1$ be a set of operation symbols. Assume that for every $f \in \Sigma_1$ we have a definition of $f$, i.e. a conjunction $\mathsf{Def}(f)$ of formulae:*

$$\bigwedge_{i=1}^{k} \forall \overline{x} (\phi_i(x_1, \dots, x_n) \rightarrow f(x_1, \dots, x_n) = t_i(x_1, \dots, t_n))$$

*where $t_i$ are $\Sigma_0$-terms, and $\phi_i$ are $\Pi_0$-clauses such that for $i \neq j$, $\phi_i \wedge \phi_j$ is unsatisfiable w.r.t. $\mathcal{T}_0$. Then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \{\mathsf{Def}_f \mid f \in \Sigma_1\}$ is local.*

*Proof*: Any $(A, \{f_A\}_{f \in \Sigma_0 \cup \Sigma_1}) \in \mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ can be completed to a total model of $\mathcal{T}_1$ by fixing an element $a_0 \in A$ and defining (if $f(a_1, \dots, a_n)$ is undefined) $f(a_1, \dots, a_n) = t_i(a_1, \dots, a_n)$ whenever $\phi_i(a_1, \dots, a_n)$ is true, and $f(a_1, \dots, a_n) = a_0$ if $\phi_i(a_1, \dots, a_n)$ are all false.

**Example: MV-algebras (universal theory).** Methods for automatically testing validity of formulae in (in)finite-valued propositional Lukasiewicz and G¨odel logics, and in propositional product logic are known (cf. e.g. [11]). It has been proved that in all cases above the problem of testing validity is co-NP complete [15, 9, 10, 12].

Theorems 2 and 6 yield a simple and easy to implement alternative decision procedure for the *universal theory* of the class $\mathcal{MV}$ of MV-algebras. As a consequence, we show that the problem of testing the validity of (sets of) clauses w.r.t. the class $\mathcal{MV}$ of all $\mathcal{MV}$-algebras is co-NP complete.

As $\mathcal{MV}$ is closed under products, it is sufficient to give a decision procedure for the universal Horn theory of $\mathcal{MV}$ (then its clause theory, and hence the universal theory is decidable). For this, note first that $\mathcal{MV}$ is the quasi-variety generated by the real unit interval $[0, 1]$ with the Lukasiewicz connectives $\{\vee, \wedge, \circ, \Rightarrow\}$, i.e. the algebra $[0, 1]_L = ([0, 1], \vee, \wedge, \circ, \Rightarrow)$ (cf. [7], Corollary 7.2). Therefore, the following are equivalent:

(1) $\mathcal{MV} \models \forall \overline{x} \bigwedge_{i=1}^{n} s_i(\overline{x}) = t_i(\overline{x}) \rightarrow s(\overline{x}) = t(\overline{x})$

(2) $[0, 1]_L \models \forall \overline{x} \bigwedge_{i=1}^{n} s_i(\overline{x}) = t_i(\overline{x}) \rightarrow s(\overline{x}) = t(\overline{x})$

(3) $\mathcal{T}_0 \cup \mathsf{Def}_L \wedge \bigwedge_{i=1}^{n} s_i(\overline{c}) = t_i(\overline{c}) \wedge s(\overline{c}) \neq t(\overline{c}) \models \perp$,

where $\mathcal{T}_0$ consists of the real unit interval $[0, 1]$ with the operations $+, -$ and predicate symbol $\leq$, and $\mathsf{Def}_L$ is the following set of definitions for the Lukasiewicz connectives:

| | | |
|---|---|---|
| $(\mathsf{Def}_\vee)$ | $x \leq y \rightarrow x \vee y = y$ | $x > y \rightarrow x \vee y = x$ |
| $(\mathsf{Def}_\wedge)$ | $x \leq y \rightarrow x \wedge y = x$ | $x > y \rightarrow x \wedge y = y$ |
| $(\mathsf{Def}_{\circ_L})$ | $x + y < 1 \rightarrow x \circ y = 0$ | $x + y \geq 1 \rightarrow x \circ y = x + y - 1$ |
| $(\mathsf{Def}_{\Rightarrow_L})$ | $x \leq y \rightarrow x \Rightarrow y = 1$ | $x > y \rightarrow x \Rightarrow y = 1 - x + y$ |

To check (3), we proceed as follows. Let $G$ be the set of clauses $\bigwedge_{i=1}^n s_i(\overline{c}) = t_i(\overline{c}) \wedge s(\overline{c}) \neq t(\overline{c})$.

*Step 1:* By the locality of the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathsf{Def}_L$, we only need to consider those instances $\mathsf{Def}_L[G]$ of $\mathsf{Def}_L$ which correspond to the ground instances occurring in $G$.

*Step 2:* We flatten $\mathsf{Def}_L[G] \wedge G$ by introducing new constants for the arguments of Lukasiewicz connectives as well as for the subterms starting with such connectives, together with corresponding definitions $c_t = t$ (stored in a set $D$). We thus obtain a set $\mathsf{Def}_L^0 \wedge G_0 \wedge D$ of ground clauses.

*Step 3:* $D$ is replaced by the set $\mathsf{Con}[D]$ of functionality axioms corresponding to the instances $f(c_1, \dots, c_n) = c$ in $D$. By Theorem 2 it is sufficient to check that $\mathsf{Def}_L^0 \wedge G_0 \wedge \mathsf{Con}[D]$ (a conjunction of ground Horn clauses in linear arithmetic over $[0,1]$) has a $\mathcal{T}_0$-model. For this, one can use, for instance, a $DPLL(T)$ method for SAT-solving modulo the theory of reals or rationals [6].

As the problem of testing satisfiability of arbitrary disjunctions of linear constraints over the reals (or rationals) is NP-complete [22], testing the validity of (sets of) clauses w.r.t. the class $\mathcal{MV}$ of all $\mathcal{MV}$-algebras is co-NP complete.

**Note:** Reasoning in $[0,1]_G = ([0,1], \wedge, \vee, \circ_G, \Rightarrow_G)$ where $\circ_G, \Rightarrow_G$ are the G¨odel connectives is similar. In this case, the signature of $\mathcal{T}_0$ only needs to contain $\leq$; we obtain a reduction to testing the satisfiability of a set of ground Horn clauses in a restricted fragment of linear arithmetic over $[0,1]$, where the atoms have the form $c \leq d$ or $c = d$. For $[0,1]_\Pi = ([0,1], \wedge, \vee, \circ_\Pi, \Rightarrow_\Pi)$ where $\circ_\Pi, \Rightarrow_\Pi$ are the product logic operations one can proceed similarly. In this case, the signature of $\mathcal{T}_0$ needs to contain $\{\leq, *, /\}$. Similar methods can be used for extensions with projection operators $\Delta, \nabla$, defined by $(\Delta(1) = 1) \wedge \forall x (x < 1 \rightarrow \Delta(x) = 0)$; resp. $(\nabla(0) = 0) \wedge \forall x (x > 0 \rightarrow \nabla(x) = 1)$, and can, in principle, be used for other subclasses of BL-logic [14] whose connectives are definable by terms in real arithmetic.

## 5 Extensions with monotone functions

We are interested in monotonicity axioms for $n$-ary functions w.r.t. a subset $I \subseteq \{1, \dots, n\}$ of their arguments:

$$(\mathsf{Mon}_f^I) \bigwedge_{i \in I} x_i {\leq_i} y_i \wedge \bigwedge_{i \notin I} x_i {=} y_i \rightarrow f(x_1, .., x_n) \leq f(y_1, .., y_n).$$

**Notation.** $\mathsf{Mon}_f^\emptyset$ is the congruence axiom for $f$. If $I = \{1, \dots, n\}$ we speak of monotonicity in all arguments; we denote $\mathsf{Mon}_f^{\{1,\dots,n\}}$ by $\mathsf{Mon}_f$. Monotonicity in some arguments and antitonicity in other arguments is modeled by considering functions $f : \prod_{i \in I} P_i^{\sigma_i} \times \prod_{j \notin I} P_j \rightarrow P$ with $\sigma_i \in \{-, +\}$, where $P_i^+ = P_i$ and $P_i^- = P_i^\partial$, the dual of $P_i$. The corresponding axioms are denoted by $\mathsf{Mon}_f^\sigma$, where for $i \in I$, $\sigma(i) = \sigma_i \in \{-, +\}$, and for $i \notin I$, $\sigma(i) = 0$.

We identify extensions $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \bigwedge_{f \in \Sigma_1} \mathsf{Mon}_f^{\sigma(f)}$ with property $\mathsf{Emb_w}$ or $\mathsf{Emb_w^{fd}}$ and which are, therefore, local.

**Theorem 7** *Let $(P_1, P_2, \dots, P_n, P, f)$ be a weak partial model of $\mathsf{Mon}_f^\sigma$, i.e. such that $f\colon \prod_{i \in I} P_i^{\sigma_i} \times \prod_{j \notin I} P_i \rightarrow P$ is a partial function weakly satisfying $\mathsf{Mon}_f^I$.*

*(1) If $P$ is a $\vee$-semilattice with 0 (or dually) and the definition domain of $f$ is finite, then $f$ has a total extension, $\overline{f} : \prod_{i \in I} P_i^{\sigma_i} \times \prod_{j \notin I} P_i \rightarrow P$ satisfying $\mathsf{Mon}_f^\sigma$.*

*(2) If $P$ is a poset (not necessarily a semilattice) then there exists a total function $\overline{f} : \prod_{i=1}^n \mathrm{DM}(P_i)^{\sigma_i} \rightarrow \mathrm{DM}(P)$ satisfying $\mathsf{Mon}_f^I$ and a (many-sorted) weak embedding $\iota\colon (P_1, ..P_n, P, f) \hookrightarrow (\mathrm{DM}(P_1), ..\mathrm{DM}(P_n), \mathrm{DM}(P), \overline{f})$.*

*Proof*: (1) Assume that $P$ is a $\vee$-semilattice with 0. Then the extension $\overline{f} : P_1 \times \cdots \times P_n \rightarrow P$ of $f$ defined by $\overline{f}(x_1, \dots, x_n) {=} \bigvee \{f(y_1, \dots, y_n) | y_i {\leq^{\sigma_i}} x_i, f(y_1, \dots, y_n)$ defined$\}$ (where $y_i \leq^+ x_i$ means $y_i \leq x_i$, $y_i \leq^- x_i$ means $y_i \geq x_i$, and $y_i \leq^0 x_i$ means $y_i = x_i$) has the desired properties. The case when $P$ is a $\wedge$-semilattice is similar.

(2) For each $j {\notin} I$ let $a_j$ be an arbitrary but fixed element of $P_j$, and let $\mathsf{max}_j : \mathrm{DM}(P_j) {\rightarrow} P_j$ be such that $\mathsf{max}_i(A)$ is a maximal element of $A$ if $A$ has one, and $a_i$ otherwise. Let $\overline{f} : \prod_{i=1}^n \mathrm{DM}(P_i)^{\sigma_i} {\rightarrow} \mathrm{DM}(P)$ be defined by $\overline{f}(A_1, \dots, A_n) = [f(u_1(A_1), \dots, u_n(A_n))]^{ul}$, where for $i {\in} I$, $u_i(A) = A$ if $\sigma_i = +$ and $u_i(A) = A^u$ otherwise; and if $i {\notin} I$ then $u_i(A) = \mathsf{max}_i(A)$. $\overline{f}$ has the desired properties.

**Theorem 8** *(1) Let $\mathcal{T}_0$ be a class of (many-sorted) bounded semilattice-ordered $\Sigma_0$-structures. Let $\Sigma_1$ be disjoint from $\Sigma_0$, and $\mathcal{T}_1 = \mathcal{T}_0 \cup \{\mathsf{Mon}^\sigma(f) | f \in \Sigma_1\}$. The extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ is finitely local.*

*(2) Any extension of the theory of posets with functions in a set $\Sigma_1$ satisfying $\{\mathsf{Mon}^\sigma(f) \mid f \in \Sigma_1\}$ is local.*

*Proof*: Direct consequence of Theorems 3, 1, and 7. □

**Example 1** *As a consequence of Theorem 7(1), and of the fact that in this case the support of the algebra does not change when extending $f$ to a total function, we can prove finite locality of the extensions with functions satisfying monotonicity axioms of the following (possibly many-sorted) classes of algebras:*

*(1) Any class of bounded (semi)lattices, distributive lattices, or Boolean algebras with operators.*

*(2) Any extension of a class of semilattices, (distributive) lattices, or Boolean algebras with operators, with monotone functions into a bounded numeric domain[1].*

*(3) $\mathcal{T}$, the class of totally-ordered sets; $\mathcal{DO}$, the theory of dense totally-ordered sets.*

*(4) Any extension of the theory of reals (integers) with monotone functions into a fixed numerical domain[2].*

[1] Of interest in non-classical logics (e.g. description logics) [18].

[2] Such extensions may be useful for reasoning about fuzzy notions.

**Theorem 9** *Assume that in $\mathcal{T}_0$ the satisfiability of a set of ground clauses of size $n$ can be checked in time at most $g(n)$. Let $\mathcal{T}_1 = \mathcal{T}_0 \cup \{\mathsf{Mon}_f^{\sigma(f)} \mid f \in \Sigma_1\}$ be an extension of $\mathcal{T}_0$ with monotone functions. The satisfiability of a set of ground clauses of size $n$ w.r.t. $\mathcal{T}_1$ can be checked in time $g(c \cdot n^2)$, where $c$ is a constant.*

**Example 2** *With the notation in Theorem 9, we have:*

*(1) If $\mathcal{T}_0$ is the theory $\mathcal{SL}$ (semilattices) or $\mathcal{L}$ (lattices), the complexity of the universal clause theory of $\mathcal{T}_1$ is in co-NP; that of the universal Horn theory is in* PTIME.[3]

*(2) If $\mathcal{T}_0$ is the theory $\mathcal{DL}$ (distributive lattices) or $\mathcal{B}$ (Boolean algebras) the complexity of the universal clause theory of $\mathcal{T}_1$ is in co-NP.*

*(3) If $\mathcal{T}_0$ is the theory* DLO *or* BAO *(distributive lattices resp. Boolean algebras with join/meet hemimorphisms) the complexity of the universal clause theory of $\mathcal{T}_1$ is in EXPTIME.*

## 6 Boundedness conditions

We now consider extensions with functions satisfying boundedness conditions and possibly also monotonicity.

**Theorem 10** *Let $\mathcal{T}_0$ be a $\Pi_0$-theory with a reflexive binary predicate symbol $\leq$, and $\Sigma_1$ be a set of operation symbols. The extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \{\mathsf{GBound}(f) \mid f \in \Sigma_1\}$ is local, where $(\mathsf{GBound}(f))$ specifies piecewise boundedness of $f$:*

$$(\mathsf{GBound}(f)) \quad \bigwedge_{i=1}^{k} \forall \overline{x}(\phi_i(\overline{x}) \rightarrow t_i(\overline{x}) \leq f(\overline{x}) \leq t'_i(\overline{x}))$$

*where $t_i, t'_i$ are $\Sigma_0$-terms and $\phi_i$ are $\Pi_0$-clauses such that if $i \neq j$ then $\phi_i \wedge \phi_j$ is unsatisfiable w.r.t. $\mathcal{T}_0$.*

**Theorem 11** *Let $\mathcal{T}_0$ be a $\Sigma_0$-theory of bounded $\vee$-semilattice-ordered (possibly many-sorted) structures, and let $\Sigma_1$ be a set of new function symbols. Then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathsf{Mon}_f^{\sigma} \cup \mathsf{Bound}_f^{\sigma}$ is finitely local.*

$$(\mathsf{Bound}_f^t) \quad \forall x_1, \dots, x_n (f(x_1, \dots, x_n) \leq t(x_1, \dots, x_n)),$$

*where $t(x_1, \dots, x_n)$ is a term in the base signature $\Pi_0$ with the same monotonicity as $f$, i.e. satisfying*

$$\forall x (\bigwedge_{i=1}^{n} x_i \leq^{\sigma_i} y_i \rightarrow t(x_1, \dots, x_n) \leq t(y_1, \dots, y_n)).$$

**Example 3** *Let $\mathcal{T}_0$ be one of the theories in Example 1. By using finite chains of theory extensions, we can devise stepwise methods for reasoning in extensions of $\mathcal{T}_0$ with function symbols satisfying a set $\mathcal{K}$ of axioms consisting of monotonicity axioms and axioms of one of the forms:*

$$\forall x(f(x) = g(x)) \qquad\qquad \forall x(h(x) \leq k(x)).$$

[3] This explains why checking subsumption w.r.t. TBOXES in the description logic $\mathcal{EL}$ [1] (having as algebraic models semilattices with monotone operators) is decidable in PTIME. An alternative proof is given in [1].

### 6.1 Example

Let $\mathcal{T}_1 = \mathcal{MV}$ be the theory of MV-algebras, and $\mathcal{T}_2$ be the extension of $\mathcal{T}_1$ with a binary function $f$, decreasing in the first and increasing in the second argument, and bounded by $\Rightarrow$, i.e. satisfying:

$$(\mathsf{Mon}_f^{-+}) \quad x_1 \geq x_2 \wedge y_1 \leq y_2 \rightarrow f(x_1, y_1) \leq f(x_2, y_2)$$
$$(\mathsf{Bound}_f^{\Rightarrow}) \quad f(x, y) \leq (x \Rightarrow y).$$

We want to prove that

$$\mathcal{T}_2 \models \forall x, x', y, y', z (z \leq f(x, y) \wedge x' \leq x \wedge y \leq y' \rightarrow x' \circ z \leq y'),$$

or equivalently, that the (skolemized, i.e. ground) negation of the formula above is unsatisfiable w.r.t. $\mathcal{T}_2$:

$$G: \quad c \leq f(a, b) \ \wedge \ a' \leq a \ \wedge \ b \leq b' \ \wedge \ a' \circ c \not\leq b'.$$

As $f$ and $\Rightarrow$ satisfy the same type of monotonicity, the extension $\mathcal{T}_1 = \mathcal{MV} \subseteq \mathcal{MV} \cup (\mathsf{Mon}_f^{-+}) \cup (\mathsf{Bound}_f^{\Rightarrow}) = \mathcal{T}_2$ is local. Therefore we only need to consider those instances of $(\mathsf{Mon}_f^{-+}) \cup (\mathsf{Bound}_f^{\Rightarrow})$ which only contain the ground terms occurring in $G$. These are trivial instances of monotonicity of $f$ and the following instance of $(\mathsf{Bound}_f^{\Rightarrow})$:

$$(\mathsf{Bound}_f^{\Rightarrow})[G] \qquad f(a, b) \leq a \Rightarrow b.$$

By Theorem 2, it is sufficient to check the satisfiability of $\mathcal{T}_1 \wedge G \wedge (\mathsf{Bound}_f^{\Rightarrow})[G]$. We flatten $G \wedge (\mathsf{Bound}_f^{\Rightarrow})[G]$ by introducing a new constant $e$ for the extension term $f(a, b)$, together with its definition $e = f(a, b)$. We thus obtain a conjunction of a formula in the base theory $G_0 \wedge (\mathsf{Bound}_f^{\Rightarrow})[G]_0$ and a formula $D$, containing the definitions of extension terms (this conjunction is, by Theorem 2, equisatisfiable w.r.t. partial models in $\mathsf{PMod_w}(\{f\}, \mathcal{T}_2)$).

| $D$ | $G_0 \wedge (\mathsf{Bound}_f^{\Rightarrow})[G]_0$ |
|---|---|
| $e = f(a, b)$ | $c \leq e \wedge a' \leq a \wedge b \leq b' \wedge a' \circ c \not\leq b'$<br>$e \leq (a \Rightarrow b)$ |

$D$ is now replaced by the set $\mathsf{Con}[D]$ of functionality axioms corresponding to the instances $f(c_1, \dots, c_n) = c$ in $D$. As only one extension term occurs in $D$, $\mathsf{Con}[D]$ contains only redundant clauses. By Theorem 2 it is sufficient to check that $G_0 \wedge (\mathsf{Bound}_f^{\Rightarrow})[G]_0$ is satisfiable in the theory of $MV$-algebras. For this, we use the method presented in Section 4. Note that checking the satisfiability of $G_0 \wedge (\mathsf{Bound}_f^{\Rightarrow})[G]_0$ w.r.t. the theory of $MV$-algebras is equivalent to checking whether

$$\mathcal{MV} \models z \leq u \ \wedge \ x' \leq x \ \wedge \ y \leq y' \ \wedge \ u \leq (x \Rightarrow y) \rightarrow x' \circ z \leq y'.$$

As in Section 4, we can check this by checking whether

$$\mathcal{T}_0 \cup \mathsf{Def}_L \wedge G_0 \wedge (\mathsf{Bound}_f^{\Rightarrow})[G]_0 \models \perp,$$

where $\mathcal{T}_0$ is the theory of the unit interval $[0, 1]$ with the operation $+$ and the predicate $\leq$ inherited from the real numbers, and $\mathsf{Def}_L$ is the set of definitions for the Lukasiewicz connectives. We introduce new constants denoting the terms starting with the Lukasiewicz connectives, and add the appropriate (flattened and purified) instances $\mathsf{Def}_{L0}$ of $\mathsf{Def}_L$ and functionality axioms:

| $\mathsf{D}_L$ | $(G_0 \wedge (\mathsf{Bound}_f^{\Rightarrow})[G]_0)_0 \wedge \mathsf{Def}_{L0}$ |
|---|---|
| $p = a' \circ c$ | $c \leq e \wedge a' \leq a \wedge b \leq b' \wedge p \not\leq b'$ |
| $q = (a \Rightarrow b)$ | $e \leq q \wedge (a' + c < 1 \rightarrow p = 0)$ |
| | $(a' + c \geq 1 \rightarrow p = a' + c - 1)$ |
| | $(a \leq b \rightarrow q = 1)$ |
| | $(a > b \rightarrow q = 1 - a + b)$ |

The satisfiability of $(G_0 \wedge (\mathsf{Bound}_f^{\Rightarrow})[G]_0)_0 \wedge \mathsf{Def}_{L0}$ w.r.t. $\mathcal{T}_0$ can be checked, e.g., with a $DPLL(T)$ method for SAT-solving modulo the theory of reals.

## 7 Conclusion

We presented a uniform method for automated reasoning in local extensions of theories of ordered structures. We analyzed definitional extensions, extensions with boundedness axioms and extensions with (generalized) monotonicity axioms and showed that they are local. This allowed us to use efficient methods for hierarchical reasoning in all these cases. We illustrated these methods by presenting a decision procedure for the universal theory of MV-algebras, and for an extension of this theory with monotone functions satisfying a boundedness condition.

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