Automated reasoning in some local extensions of ordered structures

VIORICA SOFRONIE-STOKKERMANS*, CARSTEN IHLEMANN†

Max-Planck-Institut für Informatik, Saarbrücken, Germany

Received 15 April 2007

We give a uniform method for automated reasoning in several types of extensions of ordered algebraic structures (definitional extensions, extensions with boundedness axioms or with monotonicity axioms). We show that such extensions are local and, hence, efficient methods for hierarchical reasoning exist in all these cases.

Key words: Automated reasoning, Hierarchical reasoning, Lattice-ordered structures, Monotone functions, MV-algebras, Gödel algebras

1 INTRODUCTION

We present a uniform method for automated reasoning in (i) extensions with functions which can be uniquely defined in terms of existing operations (definitional extensions), (ii) extensions with functions which satisfy general monotonicity laws and (iii) extensions with functions satisfying boundedness conditions (and possibly also monotonicity).

In previous work [24] we gave a resolution-based decision procedure for the universal fragment of the class of distributive lattices (or Boolean algebras) with join/meet hemimorphisms satisfying certain residuation conditions. For this, we used extensions of representation theorems for Boolean algebras [29] or distributive lattices [20] (cf. e.g. [11, 22]). This allowed us

* email: sofronie@mpi-inf.mpg.de
† email: ihlemann@mpi-inf.mpg.de

to use, instead of algebraic models, relational structures (usually much simpler). The main obstacle we faced when attempting to extend these methods to lattice-based structures or to monotone operators is that, although representation theorems exist, the dual spaces are quite complex: in the case of representation theorems for lattices [30, 16] they are doubly-ordered spaces (with a closure operator); in representation theorems for distributive lattices with monotone operators [7] they are endowed with maps having as values sets of elements.

In this paper we use a different approach: We show that extensions with monotone functions (and also several other theory extensions) satisfy an embeddability property of partial into total algebras, and thus are local (cf. [25]). In [25] we showed that in a local extension $\mathcal{T}_1$ of a theory $\mathcal{T}_0$ testing satisfiability of ground clauses w.r.t. $\mathcal{T}_1$ can be reduced to testing satisfiability of certain types of formulae w.r.t. $\mathcal{T}_0$. This also allows us to express, parametrically, the decidability and complexity of the universal theory of $\mathcal{T}_1$ in terms of the decidability (complexity) of a certain fragment of $\mathcal{T}_0$.

For this, we use a generalization of ideas of Burris [2] and Ganzinger [8] which we proposed in [25]. Burris [2] proved that if a quasi-variety axiomatized by a set $\mathcal{K}$ of Horn clauses has the property that every finite partial algebra which is a partial model of the axioms in $\mathcal{K}$ can be extended to a total algebra model of $\mathcal{K}$, then the uniform word problem for $\mathcal{K}$ is decidable in polynomial time (this generalizes previous results by Skolem [21] and Evans [6]). A link between these embeddability properties and a proof theoretic concept (local theories) was established by Ganzinger [8]. In [25], we extended these results to special types of *theory extensions*, which we call *local*.

In this paper we analyze the applications of the results on hierarchical reasoning in local theory extensions presented in [25] to automated reasoning in algebraic structures related to non-classical logics. The main contributions of the paper are summarized below:

(1) We give a new criterion for recognizing locality of a theory extension (Sect. 3.3, Thm. 3.4).
(2) We show that extensions with functions defined in terms of existing operators are local. As an illustration, we give a new decision procedure for the universal fragment of the class $\mathcal{MV}$ of MV-algebras, and show that the problem of deciding validity of sets of clauses w.r.t. $\mathcal{MV}$ is co-NP complete (Sect. 4). We show that similar ideas can be used also for other classes of algebras, including the class of Gödel algebras.
(3) We prove that certain extensions with functions satisfying generalized

monotonicity laws and/or boundedness conditions are also local (Sect. 5 and Sect. 6).

(4) We illustrate these ideas (and, in particular, the hierarchical reasoning method) for the case of MV-algebras with a monotone operator satisfying a boundedness condition (Sect. 6.1).

2 PRELIMINARIES

Posets and lattices. We assume that standard notions, such as partially ordered set (poset) and lattice are known. For further information we refer to [4]. The dual of a poset $P = (X, \leq)$ is $P^\partial = (X, \leq^\partial) = (X, \geq)$. For $x \in P$, $x^\downarrow := \{y \in X \mid y \leq x\}$ and $x^\uparrow := \{y \in X \mid y \geq x\}$. For $\mathrm{A} \subseteq X$, $\mathrm{A}^u = \{x \mid \forall a \in A(x \geq a)\}$ and $\mathrm{A}^l = \{x \mid \forall a \in A(x \leq a)\}$ denote the set of upper resp. lower bounds of A. The *Dedekind-MacNeille* completion $\mathrm{DM}(P)$ of a poset P consists of all subsets of P satisfying $A^{ul} = A$, ordered by inclusion. The map $x \mapsto x^\downarrow$ embeds P into $\mathrm{DM}(P)$ and preserves infima and suprema if they exist. The map ${}^l : \mathrm{DM}(P^\partial) \to \mathrm{DM}(P)^\partial$, sending a set to the set of its lower bounds, is an isomorphism; its inverse is the map ^{u}.

Theories and models. Let $\Pi = (S, \Sigma, \mathsf{Pred})$ be a signature where S is a set of sorts, Σ is a set of function symbols and Pred a set of predicate symbols. A Π-structure is a tuple

$$\mathcal{M} = (\{M_s\}_{s\in S}, \{f_\mathcal{M}\}_{f\in\Sigma}, \{P_\mathcal{M}\}_{P\in\mathsf{Pred}}),$$

where for every $s \in S$, $M_s \neq \emptyset$, for all $f \in \Sigma$ with arity $a(f){=}s_1\times\ldots\times s_n{\to}s$, $f_\mathcal{M} : \prod_{i=1}^n M_{s_i} \to M_s$ and for all $P \in \mathsf{Pred}$ with arity $a(P) = s_1\times\ldots\times s_n$, $P_\mathcal{M} \subseteq \prod_{i=1}^n M_{s_i}$. We consider formulae over variables in a (many-sorted) family $X = \{X_s \mid s \in S\}$, where for every $s \in S$, X_s is a set of variables of sort s. Theories can be regarded as sets of formulae or as sets of models. A model of a set $\mathcal{T}$ of Π-formulae is a Π-structure satisfying all formulae of $\mathcal{T}$. In this paper, whenever we speak of a theory $\mathcal{T}$ we implicitly refer to the set of all models of $\mathcal{T}$.

Definition 2.1 *A* partial Π-structure $(\{M_s\}_{s\in S}, \{f_\mathcal{M}\}_{f\in\Sigma}, \{P_\mathcal{M}\}_{P\in\mathsf{Pred}})$ *is a structure where for every* $s \in S$, $M_s \neq \emptyset$ *and for every* $f \in \Sigma$ *with arity* $s_1\times\ldots\times s_n{\to}s$, $f_\mathcal{M}$ *is a partial function from* $\prod_{i=1}^n M_{s_i}$ *to* M_s.

The notion of evaluating a term t with variables $X = \{X_s \mid s \in S\}$ w.r.t. an assignment $\{\beta_s{:}X_s \to M_s \mid s \in S\}$ (which we denote by $\beta : X \to \mathcal{M}$)

in a partial structure $\mathcal{M}$ is the same as for total structures, except that the evaluation is undefined if $t = f(t_1, \ldots, t_n)$ with $a(f){=}s_1{\times}\ldots{\times}s_n{\rightarrow}s$, and at least one of $\beta_{s_i}(t_i)$ is undefined, or else $(\beta_{s_1}(t_1), \ldots, \beta_{s_n}(t_n))$ is not in the domain of $f_{\mathcal{M}}$.

Definition 2.2 *Let $\mathcal{M}$ be a partial Π-structure, C a clause and $\beta : X \rightarrow \mathcal{M}$. Then $(\mathcal{M}, \beta) \models_w C$ if and only if either*

(i) for some term t in C, $\beta(t)$ is undefined, or else

(ii) $\beta(t)$ is defined for all terms t of C, and there exists a literal L in C s.t. $\beta(L)$ is true in $\mathcal{M}$.

$\mathcal{M}$ weakly satisfies C *(notation $\mathcal{M}{\models_w}C$) if $(\mathcal{M}, \beta){\models_w}C$ for all $\beta : X \rightarrow \mathcal{M}$.* $\mathcal{M}$ is a weak partial model of a set of clauses $\mathcal{K}$ *(notation $\mathcal{M}{\models_w}\mathcal{K}$) if $\mathcal{M}{\models_w}C$ for all $C \in \mathcal{K}$.*

Definition 2.3 *A partial Π-structure A weakly embeds into a Π-structure B if there exists an injective total map $h : A \rightarrow B$ which is an embedding with respect to* Pred *and has the property that whenever $f_A(a_1, \ldots, a_n)$ is defined in A, then also $f_B(h(a_1), \ldots, h(a_n))$ is defined in B and $h(f_A(a_1, \ldots, a_n)) = f_B(h(a_1), \ldots, h(a_n))$ (i.e. it is a weak homomorphism).*

3 LOCAL THEORY EXTENSIONS

Let $\mathcal{T}_0$ be a theory with signature $\Pi_0 = (S_0, \Sigma_0, \mathsf{Pred})$. We consider extensions $\mathcal{T}_1$ of $\mathcal{T}_0$ with signature $\Pi = (S, \Sigma, \mathsf{Pred})$, where $S = S_0 \cup S_1$, $\Sigma = \Sigma_0 \cup \Sigma_1$ (i.e. the signature is extended by new sorts and function symbols) obtained by adding a set $\mathcal{K}$ of (universally quantified) clauses. Let $\mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ be the class of all weak partial models P of $\mathcal{K}$, in which the Σ_1-functions are partial and such that $P_{|\Pi_0}$ is a total model of $\mathcal{T}_0$. In what follows, when referring to sets G of ground clauses we assume that they are in the signature $\Pi^c = (S, \Sigma \cup \Sigma_c, \mathsf{Pred})$ where Σ_c is a set of new constants.

An extension $\mathcal{T}_0 \subseteq \mathcal{T}_1{=}\mathcal{T}_0{\cup}\mathcal{K}$ is *local* if satisfiability of any set G of ground clauses w.r.t. $\mathcal{T}_0 \cup \mathcal{K}$ only depends on $\mathcal{T}_0$ and those instances $\mathcal{K}[G]$ of $\mathcal{K}$ in which the terms starting with extension functions are in the set $\mathsf{st}(\mathcal{K}, G)$ of ground terms which already occur in G or $\mathcal{K}$.

Definition 3.1 *$\mathcal{T}_0 \subseteq \mathcal{T}_1{=}\mathcal{T}_0 \cup \mathcal{K}$ is a local extension if it satisfies* (Loc)*:*

(Loc) *For every set G of ground clauses $G \models_{\mathcal{T}_1} \perp$ iff there is no partial Π^c-structure P such that $P_{|\Pi_0}$ is a total model of $\mathcal{T}_0$, all terms in $\mathsf{st}(\mathcal{K}, G)$ are defined in P, and P weakly satisfies $\mathcal{K}[G] \wedge G$.*

3.1 Embeddability and locality

In [25] we showed that embeddability of weak partial models into total models ($\mathsf{Emb_w}$) implies locality of an extension.

($\mathsf{Emb_w}$) Every $A \in \mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ weakly embeds into a total model of $\mathcal{T}_1$.

A non-ground clause is Σ_1-*flat* if function symbols (including constants) do not occur as arguments of functions in Σ_1. A Σ_1-flat non-ground clause is called Σ_1-*linear* if whenever a variable occurs in two terms in the clause which start with a function symbol in Σ_1, the two terms are identical, and if no term which starts with a function in Σ_1 contains two occurrences of the same variable.

Theorem 3.2 ([25]) *Let $\mathcal{K}$ be a set of Σ_1-flat and Σ_1-linear clauses, and let $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$. If $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies* ($\mathsf{Emb_w}$) *then it satisfies* (Loc).

Examples of local theory extensions were given in [25, 26]. In this paper we give other examples, including extensions with various monotonicity axioms.

3.2 Hierarchic reasoning in local theory extensions

Let $\mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ be a local theory extension. To check the satisfiability of a set G of ground clauses w.r.t. $\mathcal{T}_1$ we can proceed as follows (cf. [25]):

Step 1: Use locality. By the locality condition, G is unsatisfiable w.r.t. $\mathcal{T}_1$ iff $\mathcal{K}[G] \wedge G$ has no weak partial model in which all the subterms of $\mathcal{K}[G] \wedge G$ are defined, and whose restriction to Π_0 is a total model of $\mathcal{T}_0$.

Step 2: Flattening and purification. We purify and flatten $\mathcal{K}[G] \wedge G$ by introducing new constants for the arguments of the extension functions as well as for the (sub)terms $t = f(g_1, \dots, g_n)$ starting with extension functions $f \in \Sigma_1$, together with new corresponding definitions $c_t \approx t$. The set of clauses thus obtained has the form $\mathcal{K}_0 \wedge G_0 \wedge D$, where D is a set of ground unit clauses of the form $f(c_1, \dots, c_n) \approx c$ (where $f \in \Sigma_1$ and $c_1, \dots, c_n, c$ are constants) and $\mathcal{K}_0, G_0$ are clause sets without function symbols in Σ_1.

Step 3: Reduction to testing satisfiability in $\mathcal{T}_0$. We reduce the problem to testing satisfiability in $\mathcal{T}_0$ by replacing D with the following set of clauses:

$$\mathsf{Con}[D] = \bigwedge \{ \bigwedge_{i=1}^{n} c_i = d_i \rightarrow c = d \mid f(c_1, \dots, c_n) = c, f(d_1, \dots, d_n) = d \in D \}.$$

Theorem 3.3 ([25]) *If $\mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ satisfies* (Loc) *then (with the notations above) the following are equivalent:*

(1) $\mathcal{T}_0 \wedge \mathcal{K} \wedge G$ has a model.

(2) $\mathcal{T}_0 \wedge \mathcal{K}[G] \wedge G$ *has a partial model in* $\mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ *(where all terms in* $\mathsf{st}(\mathcal{K}, G)$ *are defined).*

(3) $\mathcal{T}_0 \wedge \mathcal{K}_0 \wedge G_0 \wedge D$ *has a partial model in* $\mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ *(where all terms in* $\mathsf{st}(\mathcal{K}, G)$ *are defined).*

(4) $\mathcal{T}_0 \wedge \mathcal{K}_0 \wedge G_0 \wedge \mathsf{Con}[D]$ *has a (total)* Π_0*-model.*

3.3 A finite locality criterion

Note that (Loc) refers to satisfiability of *arbitrary* sets G of ground clauses. Here, we also consider *finite locality* $(\mathsf{Loc_f})$ (the same as (Loc), except it has to hold for *finite* sets G of clauses) [25]. We say that *the extension* $\mathcal{T}_0 \subseteq \mathcal{T}_1$ *is finitely local* ($\mathcal{T}_1$ *is a finitely local extension of* $\mathcal{T}_0$) if $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies condition $(\mathsf{Loc_f})$. In [25] we showed that finite locality is guaranteed if (i) $\mathcal{T}_0$ is universal and locally finite, (ii) $\mathcal{K}$ contains finitely many ground terms, and (iii) all *finite* models in $\mathsf{PMod}(\Sigma_1, \mathcal{T}_1)$ embed into total models.

An easy change in the proof of Theorem 3.2 yields an alternative criterion for recognizing finite locality. Let $\mathsf{PMod_w^{fd}}(\Sigma_1, \mathcal{T}_1)$ be the subclass of $\mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ in which the Σ_1-functions are defined on a finite set.

$(\mathsf{Emb_w^{fd}})$ Every $A \in \mathsf{PMod_w^{fd}}(\Sigma_1, \mathcal{T}_1)$ weakly embeds into a total model of $\mathcal{T}_1$.

Theorem 3.4 *Let* $\mathcal{K}$ *be a set of clauses which are* Σ_1*-flat and* Σ_1*-linear, and let* $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$*. Assume that* $\mathcal{T}_0 \subseteq \mathcal{T}_1$ *satisfies* $(\mathsf{Emb_w^{fd}})$*. Then* $\mathcal{T}_0 \subseteq \mathcal{T}_1$ *satisfies* $(\mathsf{Loc_f})$.

Proof. Assume that $\mathcal{T}_0 \cup \mathcal{K}$ is not a local extension of $\mathcal{T}_0$. Then there exists a finite set G of ground clauses (with additional constants) such that $\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$ but $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G$ has a weak partial model in which all terms in $\mathsf{st}(\mathcal{K}, G)$ are defined. By results in [25] we can assume w.l.o.g. that $G = G_0 \cup G_1$, where G_0 contains no function symbols in Σ_1 and G_1 consists of ground unit clauses of the form $f(c_1, \dots, c_n) \approx c$, where $c_1, \dots, c_n, c$ are constants in $\Sigma_0 \cup \Sigma_c$ and $f \in \Sigma_1$.

Let P be a weak partial model of $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G$ in which all terms in $\mathsf{st}(\mathcal{K}, G)$ are defined. We construct a structure A having the same support as P, which inherits all relations in Pred and all maps in $\Sigma_0 \cup \Sigma_c$ from P, but on which the domains of definition of the Σ_1-functions are restricted as follows: for every $f \in \Sigma_1$, $f_A(a_1, \dots, a_n)$ is defined if and only if there exist constants $c^1, \dots, c^n$ such that $f(c^1, \dots, c^n)$ is in $\mathsf{st}(\mathcal{K}, G)$ and $a^i = c^i_P$ for all $i \in \{1, \dots, n\}$. In this case we define $f_A(a_1, \dots, a_n) := f_P(c^1_P, \dots, c^n_P)$. As $\mathcal{K}$ is flat, it contains no Σ_1-terms $f(c_1, ..., c_n)$ for $n \geq 1$, and since G was finite, the definition domains of the functions on Σ_1 are finite in A. The reduct of A to $\Sigma_0 \cup \Sigma_c$ coincides with that of P. Thus, A is a model of $\mathcal{T}_0 \cup G_0$. By the way the operations in Σ_1 are defined in A it is clear that A satisfies G_1, so A satisfies G.

We prove that $A \models_w \mathcal{K}$. Let $D \in \mathcal{K}$, and $\beta : X \to A$. If for some term t occurring in D, $\beta(t)$ is undefined then, by the definition of weak satisfiability, $(A, \beta) \models_w D$. Assume that for every term t occurring in D, $\beta(t)$ is defined. As all terms in D starting with a function symbol in Σ_1 are flat and linear, every variable x in D either does not occur below a function symbol in Σ_1 or it occurs in a (unique) term of the form $f(x_1, \dots, x_m)$ with $f \in \Sigma_1$ (as, say, $x = x_i$). In the latter case, as $\beta(f(x_1, \dots, x_m))$ is defined, there exist constants $c^1, \dots, c^n$ with $\beta(x_1) = c^1_P, \dots, \beta(x_m) = c^m_P$, and $f(c^1, \dots, c^m) \in \mathsf{st}(\mathcal{K}, G)$. We define a substitution $\sigma : X \to T_\Sigma(X)$ by:

$$\sigma(x) = \begin{cases} x \text{ if } x \text{ does not occur below a function symbol in } \Sigma_1 \text{ in } D \\ c^i \text{ if } x = x_i \text{ occurs in } f(x_1, \dots, x_m) \text{ with } f \in \Sigma_1, f(c^1, \dots, c^m) \\ \quad \in \mathsf{st}(\mathcal{K}, G) \text{ and } \beta(f(x_1, \dots, x_m)) = f_P(c^1_P, \dots, c^m_P). \end{cases}$$

All terms in $\sigma(D)$ starting with a Σ_1-function are (flat and linear) ground subterms of G or $\mathcal{K}$, so $\sigma(D) \in \mathcal{K}[G]$. Therefore $(P, \beta) \models_w \sigma(D)$, and as, by construction $\beta \circ \sigma = \beta$, $(A, \beta) \models_w D$.

As A weakly satisfies $\mathcal{K}$ and the domain of definition of all functions in Σ_1 in A is finite, by $(\mathsf{Emb}^{\mathsf{fd}}_{\mathsf{w}})$, A weakly embeds into a total algebra B satisfying $\mathcal{T}_0 \cup \mathcal{K}$. But then $B \models G$, so $B \models \mathcal{T}_0 \cup \mathcal{K} \cup G$, which is a contradiction. $\square$

Theorem 3.5 (Decidability [25]) *Assume that the theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies condition $(\mathsf{Loc_f})$. Then:*

(a) If all variables in the clauses in $\mathcal{K}$ occur below some Σ_1-function symbol and if the universal theory of $\mathcal{T}_0$ is decidable, then the universal theory of $\mathcal{T}_1$ is decidable.

(b) If the $\forall\exists$ theory of $\mathcal{T}_0$ is decidable then the universal theory of $\mathcal{T}_1$ is decidable.

Corollary 3.6 (Complexity [25]) *Let $\mathcal{T}_0$ be a theory for which the satisfiability of a set of ground clauses of size n can be checked in time at most $g(n)$, and let $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ be a local theory extension where in every clause in $\mathcal{K}$ each variable occurs below some extension function. The validity of a set of clauses in the extension can be checked in time $g(c \cdot n^k)$, where c is a constant and k is the maximum number of extension terms in a clause in $\mathcal{K}$ or in a congruence axiom.*

4 DEFINITIONAL EXTENSIONS

We start by studying extensions of a Π_0-theory $\mathcal{T}_0$, where $\Pi_0 = (\Sigma_0, \mathsf{Pred})$, with operators in a set Σ which are defined in terms of the operations in Σ_0.

Theorem 4.1 *Let $\mathcal{T}_0$ be a Π_0-theory and Σ_1 be a set of operation symbols. Let $\mathcal{K} = \{\mathsf{Def}_f \mid f \in \Sigma_1\}$, where for every $f \in \Sigma_1$, Def_f is a conjunction of formulae (which can be seen as a definition of f) of the form:*

$$\bigwedge_{i=1}^{k} \forall \overline{x}(\phi_i(x_1, \dots, x_n) \rightarrow f(x_1, \dots, x_n) = t_i(x_1, \dots, x_n))$$

where t_i are Σ_0-terms, and ϕ_i are Π_0-clauses such that for $i \neq j$, $\phi_i \wedge \phi_j$ is unsatisfiable w.r.t. $\mathcal{T}_0$. Then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ is local.

Proof. Any $(A, \{f_A\}_{f \in \Sigma_0 \cup \Sigma_1}) \in \mathsf{PMod_w}(\Sigma_1, \mathcal{T}_1)$ can be completed to a total model of $\mathcal{T}_1$ by fixing an element $a_0 \in A$ and defining (if $f(a_1, \dots, a_n)$ is undefined) $f(a_1, \dots, a_n) = t_i(a_1, \dots, a_n)$ whenever $\phi_i(a_1, \dots, a_n)$ is true, and $f(a_1, \dots, a_n) = a_0$ if for every $i \in \{1, \dots, k\}$ $\phi_i(a_1, \dots, a_n)$ is false. $\square$

4.1 Examples

Methods for automatically testing validity of formulae in (in)finite-valued propositional Łukasiewicz and Gödel logics, and in propositional product logic are known (cf. e.g. [14]). It has been proved that in all cases above the problem of testing validity is co-NP complete [19, 12, 13, 15].

We now show that Theorems 3.3 and 4.1 yield a simple and easy to implement alternative decision procedure for the *universal theory* of the class $\mathcal{MV}$ of MV-algebras. As a consequence, we show that the problem of testing the validity of (sets of) clauses w.r.t. the class $\mathcal{MV}$ of all $\mathcal{MV}$-algebras is co-NP complete. We argue that similar results hold for a wider class of algebras including e.g. the class of Gödel algebras.

Decision procedure for the universal theory of MV-algebras. An MV-algebra (as originally defined by Chang [3]) $(A, 0, \neg, \oplus)$ is an abelian monoid $(A, 0, \oplus)$ equipped with a unary operation $\neg$ such that $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$ and $y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus \neg y)$.

The structure of every MV-algebra $(A, 0, \neg, \oplus)$ can be enriched as follows: Let 1 denote $\neg 0$. We define $x \leq y$ iff $\neg x \oplus y = 1$. Then $\leq$ induces a partial order relation on A which endows A with a bounded distributive lattice structure, where $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x \wedge y = \neg(\neg x \vee \neg y)$. Let

$x{\circ}y = \neg(\neg x{\oplus}\neg y)$ and $x \Rightarrow y = \neg x{\oplus}y$. We thus can regard any MV-algebra as a structure $(A, \vee, \wedge, 0, 1, \circ, \Rightarrow)$ where: (M1) $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice; (M2) $(A, \circ, 1)$ is a commutative semigroup with 1; (M3) $\circ$ is monotone in both arguments; and for all $x, y, z \in A$: (M4) $x \circ z \leq y$ iff $z \leq (x \Rightarrow y)$, (M5) $x \wedge y = x \circ (x \Rightarrow y)$, and (M6) $(x \Rightarrow 0) \Rightarrow 0$. A converse correspondence between structures satisfying conditions (M1)-(M6) and MV-algebras exists cf. e.g. [15]. Accordingly, following [15], we will regard the class $\mathcal{MV}$ of all MV-algebras as the class of all structures $(A, \vee, \wedge, 0, 1, \circ, \Rightarrow)$ which satisfy conditions (M1)-(M6) above.

As $\mathcal{MV}$ is closed under products, it is sufficient to give a decision procedure for the universal Horn theory of $\mathcal{MV}$ (then its clause theory and, hence, the universal theory is decidable [18]). For giving a decision procedure for the universal Horn theory of $\mathcal{MV}$, note that $\mathcal{MV}$ is the quasi-variety generated by the real unit interval $[0, 1]$ with the Łukasiewicz connectives $\{\vee, \wedge, \circ, \Rightarrow\}$, i.e. the algebra $[0, 1]_{Ł} = ([0, 1], \vee, \wedge, \circ, \Rightarrow)$ (cf. [10], Corollary 7.2). In particular, $[0, 1]_{Ł}$ and $\mathcal{MV}$ satisfy the same universal Horn sentences. Therefore, the following are equivalent:

(1) $\mathcal{MV} \models \forall \overline{x} \bigwedge_{i=1}^n s_i(\overline{x}) = t_i(\overline{x}) \rightarrow s(\overline{x}) = t(\overline{x})$
(2) $[0, 1]_{Ł} \models \forall \overline{x} \bigwedge_{i=1}^n s_i(\overline{x}) = t_i(\overline{x}) \rightarrow s(\overline{x}) = t(\overline{x})$
(3) $\mathcal{T}_0 \cup \mathsf{Def}_{Ł} \wedge \bigwedge_{i=1}^n s_i(\overline{c}) = t_i(\overline{c}) \wedge s(\overline{c}) \neq t(\overline{c}) \models \perp$,

where $\mathcal{T}_0$ consists of the real unit interval $[0, 1]$ with the operations $+, -$ and predicate symbol $\leq$, and $\mathsf{Def}_{Ł}$ is the following set of definitions for the Łukasiewicz connectives:

$(\mathsf{Def}_\vee)$	$x{\leq}y \rightarrow x{\vee}y{=}y$	$x{>}y \rightarrow x{\vee}y{=}x$
$(\mathsf{Def}_\wedge)$	$x{\leq}y \rightarrow x{\wedge}y = x$	$x{>}y \rightarrow x{\wedge}y = y$
$(\mathsf{Def}_{\circ_{Ł}})$	$x{+}y{<}1 \rightarrow x{\circ}y{=}0$	$x{+}y{\geq}1 \rightarrow x{\circ}y{=}x{+}y{-}1$
$(\mathsf{Def}_{\Rightarrow_{Ł}})$	$x{\leq}y \rightarrow x{\Rightarrow}y{=}1$	$x{>}y \rightarrow x{\Rightarrow}y = 1{-}x{+}y$

To check (3), we proceed as follows. Let G be the set of clauses $\bigwedge_{i=1}^n s_i(\overline{c}) = t_i(\overline{c}) \wedge s(\overline{c}) \neq t(\overline{c})$.

Step 1: By the locality of the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathsf{Def}_{Ł}$, we only need to consider those instances $\mathsf{Def}_{Ł}[G]$ of $\mathsf{Def}_{Ł}$ which correspond to the ground instances occurring in G.

Step 2: We flatten $\mathsf{Def}_{Ł}[G] \wedge G$ by introducing new constants for the arguments of Łukasiewicz connectives as well as for the subterms starting with such connectives, together with corresponding definitions $c_t = t$ (stored in a set D). We thus obtain a set $(\mathsf{Def}_{Ł})_0 \wedge G_0 \wedge D$ of ground clauses.

Step 3: We replace D by the set $\mathsf{Con}[D]$ corresponding to the instances $f(c_1, \dots, c_n) = c$ in D. By Theorem 3.3 it is sufficient to check that $(\mathsf{Def}_{Ł})_0 \wedge G_0 \wedge \mathsf{Con}[D]$ (a conjunction of ground Horn clauses in linear arithmetic over $[0, 1]$) has a $\mathcal{T}_0$-model. For this, one can use, for instance, a $DPLL(T)$ method for SAT-solving modulo the theory of reals or rationals [9].

As the problem of testing satisfiability of arbitrary disjunctions of linear constraints over the reals (or rationals) is NP-complete [28], testing the validity of (sets of) clauses w.r.t. the class $\mathcal{MV}$ of all $\mathcal{MV}$-algebras is co-NP complete.

Decision procedure for the universal theory of Gödel algebras. A Gödel algebra is a Heyting algebra $(A, \wedge, \vee, 0, 1, \Rightarrow)$ satisfying the linearity axiom $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$.

By Herbrand's theorem, a Horn formula is true in the class of Gödel algebras iff it is true in the class of all finite or countable Gödel algebras. Every (finite, countable) Gödel algebra is isomorphic to a subdirect product of (finite, countable) subdirectly irreducible Gödel algebras. It is well-known that every subdirectly irreducible Gödel algebra is linearly ordered. Thus, a Horn formula is true in the class of all Gödel algebras iff it is true in the class of all *finite or countable* linearly ordered Gödel algebras. On the other hand, every finite or countable linearly ordered Gödel algebra embeds into an ultrapower of the Gödel t-norm algebra on the unit interval, $[0, 1]_G = ([0, 1], \wedge, \vee, \circ_G, \Rightarrow_G)$, where $\circ_G, \Rightarrow_G$ are the Gödel connectives [5]. Thus, a universal formula is true in the class of linearly ordered Gödel algebras iff it is true in the Gödel t-norm algebra on the unit interval $[0, 1]_G$.

Therefore a reduction similar to that used in the case of MV-algebras can also be used for Gödel algebras. Reasoning in $[0, 1]_G$ is similar to reasoning in $[0, 1]_Ł$. In this case, the signature of $\mathcal{T}_0$ only needs to contain $\leq$; we obtain a reduction to testing the satisfiability of a set of ground Horn clauses in a restricted fragment of linear arithmetic over $[0, 1]$, where the atoms have the form $c \leq d$ or $c = d$.

Further examples. For $[0, 1]_\Pi = ([0, 1], \wedge, \vee, \circ_\Pi, \Rightarrow_\Pi)$ where $\circ_\Pi, \Rightarrow_\Pi$ are the product logic operations one can proceed similarly. In this case, the signature of $\mathcal{T}_0$ needs to contain $\{\leq, *, /\}$. Similar methods can be used for extensions with projection operators Δ, ∇, defined by $(\Delta(1) = 1) \wedge \forall x(x < 1 \rightarrow \Delta(x) = 0)$; resp. $(\nabla(0) = 0) \wedge \forall x(x > 0 \rightarrow \nabla(x) = 1)$, and can in principle be used for other subclasses of BL-logic [17] whose connectives are definable using terms in real arithmetic.

5 EXTENSIONS WITH MONOTONE FUNCTIONS

We are interested in extensions of a theory with additional functions, subject to monotonicity axioms w.r.t. a subset $I \subseteq \{1, \dots, n\}$ of their arguments:

$$(\mathsf{Mon}^I_f) \quad \bigwedge_{i \in I} x_i \leq_i y_i \ \wedge \ \bigwedge_{i \notin I} x_i = y_i \ \longrightarrow \ f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n).$$

Remarks and notation: $\mathsf{Mon}^{\emptyset}_f$ is equivalent to the congruence axiom for f. If $I = \{1, \dots, n\}$ we speak of monotonicity in all arguments; we denote $\mathsf{Mon}^{\{1,\dots,n\}}_f$ by Mon_f. Monotonicity in some arguments and antitonicity in other arguments is modeled by considering functions $f : \prod_{i \in I} P_i^{\sigma_i} \times \prod_{j \notin I} P_j \rightarrow P$ with $\sigma_i \in \{-, +\}$, where $P_i^+ = P_i$ and $P_i^- = P_i^\partial$, the dual of the poset P_i. The corresponding axioms are denoted by Mon^σ_f, where for $i \in I$, $\sigma(i) = \sigma_i \in \{-, +\}$, and for $i \notin I$, $\sigma(i) = 0$.

We show that in numerous cases, theory extensions $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \bigwedge_{f \in \Sigma_1} \mathsf{Mon}^{\sigma_f}_f$ satisfy the embeddability condition $(\mathsf{Emb_w})$ or $(\mathsf{Emb^{fd}_w})$ and thus are local.

Theorem 5.1 *Let $(P_1, P_2, \dots, P_n, P, f)$ be a weak partial model of Mon^σ_f, i.e. such that $P_1, \dots, P_n, P$ are structures over a signature containing a binary predicate $\leq$, and $f\colon \prod_{i \in I} P_i^{\sigma_i} \times \prod_{i \notin I} P_i \rightarrow P$ is a partial function weakly satisfying Mon^I_f.*

(1) If P is a $\vee$-semilattice with 0 (or dually) or a totally ordered set and the definition domain of f is finite, then f has a total extension $\overline{f} : \prod_{i \in I} P_i^{\sigma_i} \times \prod_{i \notin I} P_i \rightarrow P$ satisfying Mon^I_f.

(2) If P is a poset (not necessarily a semilattice) then there exists a total function $\overline{f} : \prod_{i \in I} \mathrm{DM}(P_i)^{\sigma_i} \times \prod_{i \notin I} \mathrm{DM}(P_i) \rightarrow \mathrm{DM}(P)$ satisfying Mon^I_f and a many-sorted weak embedding

$$\iota : (P_1, \dots, P_n, P, f) \hookrightarrow (\mathrm{DM}(P_1), \dots, \mathrm{DM}(P_n), \mathrm{DM}(P), \overline{f}).$$

Proof. (1) Assume that P is a $\vee$-semilattice with 0. Then the extension $\overline{f} : P_1 \times \cdots \times P_n \rightarrow P$ of f defined by

$$\overline{f}(x_1, \dots, x_n) = \bigvee \{ f(y_1, \dots, y_n) | y_i \leq^{\sigma_i} x_i, f(y_1, \dots, y_n) \text{ defined} \}$$

(where $y_i \leq^+ x_i$ means $y_i \leq x_i$, $y_i \leq^- x_i$ means $y_i \geq x_i$, and $y_i \leq^0 x_i$ means $y_i = x_i$) has the desired properties. If P is a $\vee$-semilattice with 0 the supremum always exists (it is 0 if $f(y_1, \dots, y_n)$ is undefined for all $y_i \leq^{\sigma_i} x_i$). If P is totally ordered, let $c \in P$ be arbitrary (but fixed) if f

is nowhere defined and $c = \mathsf{min}\{f(x_1,\ldots,x_n) \mid f(x_1,\ldots,x_n) \text{ defined}\}$ otherwise. We define $\overline{f}(x_1,\ldots,x_n) = c$ in case $f(y_1,\ldots,y_n)$ is undefined for all $y_i \leq^{\sigma_i} x_i$. The identity $id : (P_1,\ldots,P_n,P,f) \to (P_1,\ldots,P_n,P,\overline{f})$ is a weak embedding. Indeed, if $f(x_1,...,x_n)$ is defined then, by monotonicity, $\overline{f}(x_1,\ldots,x_n) = f(x_1,...,x_n)$.

The case when P is a $\wedge$-semilattice with 1 is dual; we define:

$$\overline{f}(x_1,\ldots,x_n) = \bigwedge\{f(y_1,\ldots,y_n) | y_i \leq^{-\sigma_i} x_i, f(y_1,\ldots,y_n) \text{ defined}\}.$$

(2) Let $\widehat{f} : \prod_{i\in I}^n \mathrm{DM}(P_i^{\sigma_i}) \times \prod_{j\notin I} P_j \to \mathrm{DM}(P)$ be defined by

$$\widehat{f}((C_i)_{i\in I},(x_j)_{j\notin I}) = f((C_i)_{i\in I},(x_j)_{j\notin I})^{ul}.$$

$\widehat{f}$ is clearly monotone. We construct $\overline{f}$: for every $j \notin I$ let $a_j \in P_j$ be arbitrary but fixed, and let $\mathsf{max}_j : \mathrm{DM}(P_j) \to P_j$ be such that $\mathsf{max}_i(A)$ is a maximal element of A if A has one, and a_i otherwise. Let $\overline{f} : \prod_{i=1}^n \mathrm{DM}(P_i)^{\sigma_i} \to \mathrm{DM}(P)$ be the unique function which makes the following diagram commute:

$$\begin{array}{ccc}
\prod_{i\in I}\mathrm{DM}(P_i^{\sigma_i}) \times \prod_{j\notin I} P_j & \xrightarrow{\ \widehat{f}\ } & DM(P) \\
\uparrow{\scriptstyle (u_1,...,u_n)} & & \downarrow{\scriptstyle \mathrm{id}} \\
\prod_{i\in I}\mathrm{DM}(P_i)^{\sigma_i} \times \prod_{j\notin I}\mathrm{DM}(P_j) & \xrightarrow{\ \overline{f}\ } & DM(P)
\end{array}$$

Here $u_i(A) = \begin{cases} A^u & \text{if } i\in I \text{ and } \sigma_i = - \\ A & \text{if } i\in I \text{ and } \sigma_i = + \\ \mathsf{max}_i(A) & \text{if } i\notin I. \end{cases}$. We show that $\overline{f}$ is monotone. Let $A_i \subseteq^{\sigma_j} A'_i$ for $i \in \{1,\ldots,n\}$. As ^{u} is antitone, $u_i(A_i) \subseteq u_i(A'_i)$ for all $i \in \{1,\ldots,n\}$. Hence, $f(u_1(A_1),...,u_n(A_n)) \subseteq f(u_1(A'_1),...,u_n(A'_n))$. Moreover, as ^{ul} is monotone:

$$\begin{aligned}\overline{f}(A_1,...,A_n) = [(f(u_1(A_1),...,u_n(A_n))]^{ul} &\subseteq [(f(u_1(A'_1),...,u_n(A'_n))]^{ul} \\ &= \overline{f}(A'_1,...,A'_n).\end{aligned}$$

Let $\iota_i{:}P_i \hookrightarrow \mathrm{DM}(P_i)$, $\iota_0{:}P \to \mathrm{DM}(P)$ be the Dedekind-MacNeille embeddings $\iota_i(x) = x^{\downarrow}$. Let $\iota : (P_1,\ldots,P_n,P) \hookrightarrow (\mathrm{DM}(P_1),\ldots,\mathrm{DM}(P_n),\mathrm{DM}(P))$ be the many-sorted map agreeing with ι_i on the corresponding sorts. We show that ι is a weak embedding. Assume that $f(x_1,...,x_n)$ is defined. We show that $f(x_1,...,x_n)^{\downarrow} = \overline{f}(x_1^{\downarrow},...,x_n^{\downarrow})$.

First note that $u_i(x_i^{\downarrow}) = x_i^{\downarrow}$ if $i \in I$ and $\sigma_i = +$; $u_i(x_i^{\downarrow}) = x_i^{\uparrow}$ if $i \in I$ and $\sigma_i = -$; and $u_i(x_i^{\downarrow}) = x_i$ if $i \notin I$. By definition, $\overline{f}(x_1^{\downarrow},\ldots,x_n^{\downarrow}) =$

$\widehat{f}(u_1(x_1^{\downarrow}), \ldots, u_n(x_n^{\downarrow})) = \{f(y_1, \ldots, y_n) \mid y_i \leq^{\sigma_i} x_i\}^{ul}$. It is easy to check that if the supremum of A exists in P then $\mathrm{A}^{ul} = (\bigvee \mathrm{A})^{\downarrow}$. By monotonicity, $\bigvee\{f(y_1, \ldots, y_n) \mid y_i \leq^{\sigma_i} x_i\} = f(x_1, ..., x_n)$ (we assumed that $f(x_1, \ldots, x_n)$ is defined). Therefore, $\overline{f}(x_1^{\downarrow}, \ldots, x_n^{\downarrow}) = f(x_1, \ldots, x_n)^{\downarrow}$. □

Theorem 5.2 *The following hold:*

(1) Let $\mathcal{T}_0$ be a class of (many-sorted) bounded semilattice-ordered Σ_0-structures. Let Σ_1 be disjoint from Σ_0 and $\mathcal{T}_1 = \mathcal{T}_0 \cup \{\mathsf{Mon}_f^{\sigma_f} \mid f \in \Sigma_1\}$. Then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ is finitely local.

(2) Any extension of the theory of posets with functions in a set Σ_1 satisfying $\{\mathsf{Mon}_f^{\sigma_f} \mid f \in \Sigma_1\}$ is local.

Proof. Direct consequence of Theorems 3.4, 3.2, and 5.1. □

This theorem allows us to give a large number of useful examples.

Corollary 5.3 *The extensions with functions satisfying monotonicity axioms of the following (possibly many-sorted) classes of algebras are finitely local:*

(1) Any class of algebras with a bounded (semi)lattice reduct, a bounded distributive lattice reduct, or a Boolean algebra reduct.

(2) Any extension of a class of algebras with a semilattice reduct, a (distributive) lattice reduct, or a Boolean algebra reduct, with monotone functions into an infinite numeric domain* *.*

(3) $\mathcal{T}$, the class of totally-ordered sets; $\mathcal{DO}$, the theory of dense totally-ordered sets.

(4) Any extension of the theory of reals (integers) with monotone functions into a fixed infinite numerical domain† *.*

Proof. (1) and (3) are immediate consequences of Theorem 5.1(1) and of the fact that in this case the support of the algebra does not change when extending a partial monotone function f to a total function.
(2) and (4) are also consequences of Theorem 5.1(1), taking into account that (with the notations in Theorem 5.1) if P is an infinite numerical domain, then it is in particular totally ordered, so there exists an element $c \in P$ such that $c \leq f(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n)$ in the (finite) definition domain of f. Thus, we can choose c instead of 0 in the arguments of Theorem 5.1. □

* Of interest in non-classical logics (e.g. description logics) [23].
† Such extensions may be useful for reasoning about fuzzy notions.

Theorem 5.4 *Assume that in $\mathcal{T}_0$ the satisfiability of a set of ground clauses of size n can be checked in time at most $g(n)$. Let $\mathcal{T}_1 = \mathcal{T}_0 \cup \{\mathsf{Mon}_f^{\sigma_f} \mid f \in \Sigma_1\}$ be an extension of $\mathcal{T}_0$ with monotone functions. The satisfiability of a set of ground clauses of size n w.r.t. $\mathcal{T}_1$ can be checked in time $g(c \cdot n^2)$, where c is a constant.*

Corollary 5.5 *With the notation in Theorem 5.4, we have:*

(1) If $\mathcal{T}_0$ is the theory $\mathcal{SL}$ (bounded semilattices) or $\mathcal{L}$ (bounded lattices), the complexity of the universal clause theory of $\mathcal{T}_1$ is in co-NP; that of the universal Horn theory is in PTIME.‡

(2) If $\mathcal{T}_0$ is the theory of bounded distributive lattices or the theory of Boolean algebras, then the complexity of the universal clause theory of $\mathcal{T}_1$ is in co-NP.

(3) If $\mathcal{T}_0$ is the theory DLO *or* BAO *(bounded distributive lattices resp. Boolean algebras with join/meet hemimorphisms) the complexity of the universal clause theory of $\mathcal{T}_1$ is in EXPTIME.*

6 BOUNDEDNESS CONDITIONS

We now consider extensions with functions satisfying boundedness conditions and possibly also monotonicity.

Theorem 6.1 *Let $\mathcal{T}_0$ be a Π_0-theory with a reflexive binary predicate symbol $\leq$, and Σ_1 be a set of operation symbols. The extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \{\mathsf{GBound}_f \mid f \in \Sigma_1\}$ is local, where (GBound_f) specifies piecewise boundedness of f:*

$$(\mathsf{GBound}_f) \quad \bigwedge_{i=1}^{k} \forall \overline{x} (\phi_i(\overline{x}) \to t_i(\overline{x}) \leq f(\overline{x}) \leq t_i'(\overline{x}))$$

where t_i, t_i' are Σ_0-terms and ϕ_i are Π_0-clauses such that if $i \neq j$ then $\phi_i \wedge \phi_j$ is unsatisfiable w.r.t. $\mathcal{T}_0$.

Theorem 6.2 *Let $\mathcal{T}_0$ be a Σ_0-theory of bounded $\vee$-semilattice-ordered (possibly many-sorted) structures, and let f be a new function symbol. Then the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathsf{Mon}_f^{\sigma} \cup \mathsf{Bound}_f^{t}$ is finitely local.*

$$(\mathsf{Bound}_f^t) \quad \forall x_1, \dots, x_n (f(x_1, \dots, x_n) \leq t(x_1, \dots, x_n)),$$

‡ This explains why checking subsumption w.r.t. TBOXES in the description logic $\mathcal{EL}$ [1] (having as algebraic models bounded semilattices with monotone operators; see [27]) is decidable in PTIME. An alternative proof is given in [1].

where $t(x_1, \ldots, x_n)$ is a term in the base signature Π_0 with the same monotonicity as f, i.e. satisfying

$$\forall x(\bigwedge_{i=1}^{n} x_i \leq^{\sigma_i} y_i \rightarrow t(x_1, \ldots, x_n) \leq t(y_1, \ldots, y_n)).$$

Proof. By Theorem 3.4, in order to prove locality it is sufficient to prove that condition $(\mathsf{Emb}_\mathsf{w}^\mathsf{fd})$ holds. Let $(P_1, P_2, \ldots, P_n, P, f)$ be a weak partial model of $(\mathsf{Mon}_f^\sigma) \cup (\mathsf{Bound}_f^t)$ such that the definition domain of f is finite.

By Theorem 5.1(1), we can extend f to the total function $\overline{f}$ defined by $\overline{f}(x_1, \ldots, x_n) = \bigvee\{f(u_1, \ldots, u_n) \mid u_i \leq^{\sigma_i} x_i, f(u_1, \ldots, u_n) \text{ defined}\}$.

Let $u_i \leq^{\sigma_i} x_i$, $i = 1, \ldots, n$ with $f(u_1, \ldots, u_n)$ defined. Then, as f weakly satisfies (Bound_f^t), $f(u_1, \ldots, u_n) \leq t(u_1, \ldots, u_n)$. From the monotonicity assumption for t we know that $t(u_1, \ldots, u_n) \leq t(x_1, \ldots, x_n)$. Thus for all $u_i \leq^{\sigma_i} x_i$, $i = 1, \ldots, n$ with $f(u_1, \ldots, u_n)$ defined, $f(u_1, \ldots, u_n) \leq t(x_1, \ldots, x_n)$. Therefore, $\overline{f}(x_1, \ldots, x_n) = \bigvee\{f(u_1, \ldots, u_n) \mid u_i \leq^{\sigma_i} x_i, f(u_1, \ldots, u_n) \text{ defined}\} \leq t(x_1, \ldots, x_n)$. □

Corollary 6.3 *Let $\mathcal{T}_0$ be one of the theories in Example 5.3. By using finite chains of theory extensions, we can devise stepwise methods for reasoning in extensions of $\mathcal{T}_0$ with function symbols satisfying a set $\mathcal{K}$ of axioms consisting of monotonicity axioms and axioms of one of the forms:*

$$\forall x(f(x) = g(x)) \qquad \forall x(h(x) \leq k(x)).$$

6.1 Example

Let $\mathcal{T}_1 = \mathcal{MV}$ be the theory of MV-algebras, and $\mathcal{T}_2$ be the extension of $\mathcal{T}_1$ with a binary function f, decreasing in the first and increasing in the second argument, and bounded by $\Rightarrow$, i.e. satisfying:

$$\begin{array}{ll} (\mathsf{Mon}_f^{-+}) & x_1 \geq x_2 \wedge y_1 \leq y_2 \rightarrow f(x_1, y_1) \leq f(x_2, y_2) \\ (\mathsf{Bound}_f^{\Rightarrow}) & f(x, y) \leq (x \Rightarrow y). \end{array}$$

We prove that $\mathcal{T}_2 \models \forall x, x', y, y', z(z \leq f(x, y) \wedge x' \leq x \wedge y \leq y' \rightarrow x' \circ z \leq y')$, or equivalently, that the (skolemized, i.e. ground) negation of the formula above is unsatisfiable w.r.t. $\mathcal{T}_2$:

$$G: \quad c \leq f(a, b) \ \wedge \ a' \leq a \ \wedge \ b \leq b' \ \wedge \ a' \circ c \not\leq b'.$$

As f and $\Rightarrow$ satisfy the same type of monotonicity, the extension $\mathcal{T}_1 = \mathcal{MV} \subseteq \mathcal{MV} \cup (\mathsf{Mon}_f^{-+}) \cup (\mathsf{Bound}_f^{\Rightarrow}) = \mathcal{T}_2$ is local. Therefore we only need to consider those instances of $(\mathsf{Mon}_f^{-+}) \cup (\mathsf{Bound}_f^{\Rightarrow})$ which only contain the ground terms occurring in G. These are trivial instances of monotonicity of f and the following instance of $(\mathsf{Bound}_f^{\Rightarrow})$:

$$(\mathsf{Bound}_f^{\Rightarrow})[G] \qquad f(a,b) \leq a \Rightarrow b.$$

Thus, it is sufficient to check the satisfiability of $\mathcal{T}_1 \wedge (\mathsf{Bound}_f^{\Rightarrow})[G] \wedge G$. We flatten $(\mathsf{Bound}_f^{\Rightarrow})[G] \wedge G$ by introducing a new constant e for the extension term $f(a,b)$, together with its definition $e = f(a,b)$. We thus obtain a conjunction of a formula in the base theory $(\mathsf{Bound}_f^{\Rightarrow})[G]_0 \wedge G_0$ and a formula D, containing the definitions of extension terms (this conjunction is, by Theorem 3.3, equisatisfiable w.r.t. partial models in $\mathsf{PMod_w}(\{f\}, \mathcal{T}_2)$).

D	$(\mathsf{Bound}_f^{\Rightarrow})[G]_0 \wedge G_0$
$e = f(a,b)$	$e \leq (a \Rightarrow b) \;\wedge\; c \leq e \;\wedge\; a' \leq a \;\wedge\; b \leq b' \;\wedge\; a' \circ c \not\leq b'$

D is now replaced by the set $\mathsf{Con}[D]$ of functionality axioms corresponding to the instances $f(c_1, \dots, c_n) = c$ in D. As only one extension term occurs in D, $\mathsf{Con}[D]$ contains only redundant clauses. By Theorem 3.3 it is sufficient to check that $(\mathsf{Bound}_f^{\Rightarrow})[G]_0 \wedge G_0$ is satisfiable in the theory of MV-algebras. For this, we use the method presented in Section 4. Note that checking the satisfiability of $(\mathsf{Bound}_f^{\Rightarrow})[G]_0 \wedge G_0$ w.r.t. the theory of MV-algebras is equivalent to checking whether

$$\mathcal{MV} \models u \leq (x \Rightarrow y) \;\wedge\; z \leq u \;\wedge\; x' \leq x \;\wedge\; y \leq y' \rightarrow x' \circ z \leq y'.$$

As in Section 4, we can check this by checking whether

$$\mathcal{T}_0 \cup \mathsf{Def}_{Ł} \wedge (\mathsf{Bound}_f^{\Rightarrow})[G]_0 \wedge G_0 \models \perp,$$

where $\mathcal{T}_0$ is the theory of the unit interval $[0,1]$ with the operation $+$ and the predicate $\leq$ inherited from the real numbers, and $\mathsf{Def}_{Ł}$ is the set of definitions for the Łukasiewicz connectives. We introduce new constants denoting the terms starting with the Łukasiewicz connectives, and add the appropriate (flattened and purified) instances $(\mathsf{Def}_{Ł})_0$ of $\mathsf{Def}_{Ł}$ and functionality axioms:

D'	$((\mathsf{Bound}_f^{\Rightarrow})[G]_0 \wedge G_0)_0 \wedge (\mathsf{Def}_{Ł})_0$
$p = a' \circ c$	$e \leq q \;\wedge\; c \leq e \;\wedge\; a' \leq a \;\wedge\; b \leq b' \;\wedge\; p \not\leq b'$
$q = (a \Rightarrow b)$	$(a' + c < 1 \rightarrow p = 0)$
	$(a' + c \geq 1 \rightarrow p = a' + c - 1)$
	$(a \leq b \rightarrow q = 1)$
	$(a > b \rightarrow q = 1 - a + b)$

The satisfiability of $((\mathsf{Bound}_f^{\Rightarrow})[G]_0 \wedge G_0)_0 \wedge (\mathsf{Def}_{Ł})_0$ w.r.t. $\mathcal{T}_0$ can be checked, e.g., with a $DPLL(T)$ method for SAT-solving modulo the theory of reals.

7 CONCLUSION

We presented a uniform method for automated reasoning in local extensions of theories of ordered structures. We analyzed definitional extensions, extensions with boundedness axioms and extensions with (generalized) monotonicity axioms and showed that they are local. This allowed us to use efficient methods for hierarchical reasoning in all these cases. We illustrated these methods by presenting a decision procedure for the universal theory of MV-algebras, and for an extension of this theory with monotone functions satisfying a boundedness condition.

Acknowledgement. This work was partly supported by the German Research Council (DFG) as part of the Transregional Collaborative Research Center "Automatic Verification and Analysis of Complex Systems" (SFB/TR 14 AVACS). See `www.avacs.org` for more information.

REFERENCES

[1] F. Baader. (2003). Terminological cycles in a description logic with existential restrictions. In G. Gottlob and T. Walsh, editors, *Proc. 18th International Joint Conference on Artificial Intelligence*, pages 325–330. Morgan Kaufmann.

[2] S. Burris. (1995). Polynomial time uniform word problems. *Mathematical Logic Quarterly*, 41:173–182.

[3] C. C Chang. (1958). Algebraic analysis of many-valued logics. *Trans. Amer. Math. Soc*, 88:467–490.

[4] B.A. Davey and H.A. Priestley. (1990). *Introduction to Lattices and Order*. Cambridge University Press.

[5] F. Esteva, J. Gispert, L. Godo, and F. Montagna. (2002). On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic. *Studia Logica*, 71:199–226.

[6] T. Evans. (1951). The word problem for abstract algebras. *J. London Math. Soc.*, 28:64–71.

[7] J.D. Farley. (1996). Priestley duality for order-preserving maps into distributive lattices. *Order*, 13:65–98.

[8] H. Ganzinger. (2001). Relating semantic and proof-theoretic concepts for polynomial time decidability of uniform word problems. In *Proc. 16th IEEE Symp. on Logic in Computer Science (LICS'01)*, pages 81–92. IEEE Comp. Soc. Press.

[9] H. Ganzinger, G. Hagen, R. Nieuwenhuis, A. Oliveras, and C. Tinelli. (2004). DPLL(T): Fast decision procedures. In R. Alur and D. Peled, editors, *Proc. 16th International Conference on Computer Aided Verification (CAV'04), LNCS 3114*, pages 175–188. Springer.

[10] J. Gispert and D. Mundici. (2005). MV-algebras: A variety for magnitudes with archimedean units. *Algebra Universalis*, 53:7–43.

[11] R. Goldblatt. (1989). Varieties of complex algebras. *Annals of Pure and Applied Logic*, 44(3):153–301.

[12] R. Hähnle. (1994). Many-valued logic and mixed integer programming. *Annals of Mathematics and Artificial Intelligence*, 12(3,4):231–264.

[13] R. Hähnle. (1997). Proof theory of many-valued logic – linear optimization – logic design. *Soft Computing – A Fusion of Foundations, Methodologies and Applications*, 1(3):107–119.

[14] R. Hähnle. (2003). Complexity of many-valued logics. In M. Fitting and E. Orłowska, editors, *Beyond Two: Theory and Applications of Multiple Valued Logic*, volume 114 of *Studies in Fuzziness and Soft Computing*, chapter 3, pages 211–233. Springer, Berlin.

[15] P. Hájek. (1998). *Metamathematics of Fuzzy Logic*. Trends in Logic, Volume 4. Kluwer Academic Publishers, Dordrecht.

[16] C. Hartonas and J.M. Dunn. (1997). Stone duality for lattices. *Algebra Universalis*, 37(3):391–401.

[17] E. Marchioni and F. Montagna. (To appear). Complexity and definability issues in $Ł\Pi\frac{1}{2}$. *Journal of Logic and Computation*.

[18] J.C.C. McKinsey. (1943). The decision problem for some classes of sentences without quantifiers. *The Journal of Symbolic Logic*, 8(3):61–76.

[19] D. Mundici. (1987). Satisfiability of many-valued sentential logics is NP-complete. *Theoretical Computer Science*, 52:145–153.

[20] H.A. Priestley. (1970). Representation of distributive lattices by means of ordered Stone spaces. *Bull. London Math. Soc.*, 2:186–190.

[21] T. Skolem. (1920). Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit und Beweisbarkeit mathematischer Sätze nebst einem Theorem über dichte Mengen. *Skrifter utgit av Videnskabsselskapet i Kristiania, I. Matematisk-naturvidenskabelig klasse, 4*, pages 1–36.

[22] V. Sofronie-Stokkermans. (2000). Duality and canonical extensions of bounded distributive lattices with operators, and applications to the semantics of non-classical logics I. *Studia Logica*, 64(1):93–132.

[23] V. Sofronie-Stokkermans. (2002). On uniform word problems involving bridging operators on distributive lattices. In U. Egly and Ch. Fermüller, editors, *Proceedings of TABLEAUX 2002. LNAI 2381*, pages 235–250. Springer.

[24] V. Sofronie-Stokkermans. (2003). Resolution-based decision procedures for the universal theory of some classes of distributive lattices with operators. *Journal of Symbolic Computation*, 36(6):891–924.

[25] V. Sofronie-Stokkermans. (2005). Hierarchic reasoning in local theory extensions. In R. Nieuwenhuis, editor, *20th International Conference on Automated Deduction (CADE-20), LNAI 3632*, pages 219–234. Springer.

[26] V. Sofronie-Stokkermans. (2006). Interpolation in local theory extensions. In U. Furbach and N. Shankar, editors, *Proc. 3rd International Joint Conference on Automated Reasoning (IJCAR 2006), LNAI 4130*, pages 235–250. Springer.

[27] Viorica Sofronie-Stokkermans. (2007). Automated theorem proving by resolution in non-classical logics. *Annals of Mathematics and Artificial Intelligence*. To appear.

[28] E.D. Sontag. (1985). Real addition and the polynomial hierarchy. *Information Processing Letters*, 20(3):115–120.

[29] H.M. Stone. (1938). The representation of boolean algebras. *Bull. Amer. Math. Soc*, 44:807–816.

[30] A. Urquhart. (1978). A topological representation theory for lattices. *Algebra Universalis*, 8:45–58.