Duality and Canonical Extensions of Bounded Distributive Lattices with Operators, and Applications to the Semantics of Non-Classical Logics I

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Abstract

The main goal of this paper is to explain the link between the algebraic and the Kripke-style models for certain classes of propositional logics. We start by presenting a Priestley-type duality for distributive lattices endowed with a general class of well-behaved operators. We then show that finitely-generated varieties of distributive lattices with operators are closed under canonical embedding algebras. The results are used in the second part of the paper to construct topological and non-topological Kripke-style models for logics that are sound and complete with respect to varieties of distributive lattices with operators in the above-mentioned classes.

Introduction

In the study of non-classical propositional logics (and especially of modal logics) there are two main ways of defining interpretations or models. One possibility is to use *algebras* – usually lattices with operators – as models. Propositional variables are interpreted over elements of these algebraic models, and the logical connectives correspond to the algebra operators. The other way is to use Kripke-style semantics. Here the models are known as *frames*, and usually carry a relational rather than an algebraic structure. Formulae are interpreted as subsets of the frame in a way that depends on the particular structure on these frames. Usually, the algebraic models have the advantage that soundness and completeness results are easier to prove, which is not always the case with the Kripke-style models. Kripke-style models are, on the other hand, more comfortable to use because of their often much simpler structure. This is one of the reasons why the most substantial progress in the study of modal logics took place since the introduction of relational structures in the study of the semantics of modal logics by Kripke [Kri63].

The ultimate goal of this paper is to give a better understanding of the link between the algebraic models and the Kripke-style models for certain classes of propositional logics based on *distributive lattices with operators* (thus, we go one step beyond the modal case). Our choice is motivated, on the one hand, by the fact that many non-classical logics that occur in a natural way in practical applications can be proved to be sound and complete with respect to certain classes of distributive lattices with operators, and, on the other hand, by the fact that distributive lattices (with well-behaved operators) usually have good and economical representation theorems. In this first part of the paper we present the algebraic results by which we can reach this goal: We start by extending the Priestley duality for distributive lattices to distributive lattices with operators satisfying certain preservation properties, in particular various types of "antimorphisms", thus extending the duality given by Goldblatt in [Gol89]. We then show that finitely-generated varieties of distributive lattices with operators (lattice homomorphisms and antimorphisms; join- and meet-hemimorphisms, and join- and meet-hemiantimorphisms) have the closure property under canonical embedding algebras (which will be formally defined in Section 4). In the second part of the paper [SS98] we use these results in analyzing the link between the algebraic and Kripke-style models of certain non-classical logics.

The idea of using representation theorems for establishing a link between the algebraic and relational semantics of non-classical logics goes back to Jónsson and Tarski [JT51, JT52], who for this purpose used an extension of Stone's representation theorem for Boolean algebras with operators. Every relational structure $X = (X, \{R\}_{R\in\mathcal{R}})$ comprising a collection $\mathcal{R}$ of finitary relations on a set X gives rise to a Boolean algebra with operators $X^+ = (\mathcal{P}(X), \{f_R\}_{R\in\mathcal{R}})$ in a canonical way. Jónsson and Tarski showed that for every Boolean algebra with operators A there exists a relational structure $\mathcal{C}(A)$ of appropriate type (the *canonical structure* of A), and an embedding of A into $\mathcal{C}(A)^+$. The algebra $\mathcal{C}(A)^+$ is, by definition, the *canonical extension* or *canonical embedding algebra* of A.

Similar results are used in the study of modal systems by Lemmon [Lem66a, Lem66b] and Goldblatt [Gol89]. More general duality theorems (for partial orders, semilattices, Galois connections and lattices) are used by Hartonas and Dunn in [HD93]. Representation theorems are also used by Allwein for giving a Kripke-style semantics for linear logics [All93]. Also, similar ideas occur in correspondence theory [vB84], where the link between axioms and theorems of a given logic and properties of the corresponding Kripke models of the logic is investigated.

Our work is influenced by the results of Goldblatt [Gol89], who showed that the "modal case" can be seen as a simple illustration of more general results from universal algebra. Goldblatt establishes various relations between properties of a given class K of relational structures and properties of the class SK^+ of all algebras which are isomorphic to subalgebras of X^+ for some relational structure X in K, and studies the general structure of so-called *complex varieties*, i.e. varieties of Boolean algebras with operators that are representable in the form SK^+ for some class K of relational spaces. One of the properties of varieties of Boolean algebras with operators investigated in [Gol89] is the *closure under canonical embedding algebras*. One of its possible applications is in giving an algebraic formulation and an alternative proof of the fact that if a modal logic is characterized by an elementary class of Kripke structures (i.e. defined by a set of first-order sentences) then it is validated by its Henkin structures [Gol89]. This result was first proved by van Benthem in [vB80], by model-theoretical arguments. Although in [Gol89] only the case of *Boolean algebras* and the corresponding relational structures are considered in detail, the results presented there suggest that similar methods can also be applied to more general classes of non-classical propositional logics, whose algebraic models are bounded distributive lattices endowed with (well-behaved) operators. The closure under canonical embedding algebras for varieties of *distributive lattices with additive operators* was studied by Gehrke and Jónsson in [GJ94]. In this paper we extend both the results in [Gol89], by considering distributive lattices with operators instead of Boolean algebras with operators, and those in [GJ94], in that we also consider certain classes of antimorphisms. In the second part of this paper [SS98] we will show that, under certain circumstances, closure under canonical embedding algebras can be used to establish, by purely algebraic means, soundness and completeness results for certain logics with respect to classes of Kripke-style models obtained in a canonical way from their algebraic models.

The paper is structured as follows: In Section 1 we briefly introduce the notations and the main notions that will be used later. In Section 2 we present an extension of the Priestley duality to distributive lattices with operators in the following classes: lattice homomorphisms and antimorphisms; join- and meet-hemimorphisms; and join- and meet-hemiantimorphisms. In Section 3 ordered relational structures are defined and some of their properties are studied. In Section 4 we present some cases of varieties that are closed under embedding algebras. By a result of [GJ94], all varieties of distributive algebras with additive operators are closed under canonical embedding algebras; an essential assumption in the proof is the fact that all operators are order-preserving. We prove that this also holds for certain classes of distributive algebras with operators which are not necessarily order-preserving, namely we show that such a closure condition holds for finitely generated varieties of distributive lattices with operators in the classes mentioned in Section 2. In Section 5 we present some conclusions and plans for future work. In the second part of this paper [SS98] we will show how the results presented here can be applied in studying the link between algebraic and Kripke-style models, for certain classes of propositional logics.

1 Preliminaries

We will now briefly review the main concepts and notations that will be used in our work.

Order theory. A partially-ordered set is pair $(P, \leq)$, where P is a set and $\leq\, \subseteq P \times P$ is a reflexive, transitive and antisymmetric relation (i.e. a partial order on P). Given a partially-ordered set $(P, \leq)$, a subset J of P is an *order-ideal* if for every $x, y \in P$, if $x \leq y$ and $y \in J$ then $x \in J$ (i.e. it is downwards-closed with respect to $\leq$). A subset $F \subseteq P$ is an *order-filter* if for every $x, y \in P$, if $x \leq y$ and $x \in F$

then $y \in F$ (i.e. it is upwards-closed with respect to $\leq$). In what follows we will denote by $\mathcal{O}(P)$ the set of order-filters of a partially-ordered set P; for every subset A of P we will use the following notations: $\uparrow A = \{x \in P \mid \exists a \in A, x \geq a\}$, and $\downarrow A = \{x \in P \mid \exists a \in A, a \geq x\}$. In particular, if $a \in P$ then $\uparrow a = \{x \in P \mid x \geq a\}$ and $\downarrow a = \{x \in P \mid a \geq x\}$. For further informations on partially-ordered sets we refer to [DP90].

Lattices. A *lattice* is a partially-ordered set $(L, \leq)$ with the property that every two elements $x, y \in L$ have a supremum (denoted $x \vee y$) and an infimum (denoted by $x \wedge y$) in L. Alternatively, a non-empty set L together with two binary operations $\vee$ and $\wedge$ on L is called *lattice* if $\vee, \wedge$ are commutative, associative, idempotent, and satisfy the absorption law. A *distributive lattice* is a lattice that satisfies either of the distributive laws (which are equivalent in a lattice).

We say that a lattice L has a *first (smallest) element* if there is an element $0 \in L$ such that $0 \leq x$ for every $x \in L$. A lattice L has a *last (greatest) element* if there is an element $1 \in L$ such that $x \leq 1$ for every $x \in L$. A lattice having both a first and a last element is called *bounded*. If not specified otherwise, in what follows we will denote the first element in a lattice (if any) by 0 and the last element (if any) by 1. Bounded lattices will be represented as algebras $(L, \vee, \wedge, 0, 1)$. (For instance, for every partially-ordered set $(P, \leq)$, $(\mathcal{O}(P), \cup, \cap, \emptyset, P)$ is a bounded distributive lattice with $0 = \emptyset$ and $1 = P$.) An element $x \in L$ is called *join irreducible* if $x \neq 0$ (in case L has a 0) and x cannot be expressed as the join of two other elements in L, i.e. if it has the property that $x = y \vee z$ implies $y = x$ or $z = x$. An element $x \in L$ is called *meet irreducible* if $x \neq 1$ (in case L has a 1) and it has the property that $x = y \wedge z$ implies $y = x$ or $z = x$. Let L be a lattice. A non-empty subset J of L is called an *ideal* if for every $x, y \in J$, $x \vee y \in J$, and for every $x, y \in L$, if $y \in J$ and $x \leq y$ then $x \in J$. Dually, a non-empty subset F of L is called a *filter* if for every $x, y \in F$, $x \wedge y \in F$, and for every $x, y \in L$, if $x \in F$ and $x \leq y$ then $y \in F$. Thus, an ideal is a non-empty downwards-closed set closed under join, and a filter is a non-empty upwards-closed set closed under meet. An ideal or filter is called *proper* if it does not coincide with L. Let L be a lattice and J a proper ideal in L. The ideal J is said to be *prime* if for every $x, y \in L$, if $x \wedge y \in J$ then $x \in J$ or $y \in J$. The set of prime ideals of L is denoted by $\mathcal{I}_p(L)$. A prime filter is defined dually, i.e. it is a proper filter F with the property that if $x, y \in L$ and $x \vee y \in F$ then $x \in F$ or $y \in F$. The set of prime filters is denoted by $\mathcal{F}_p(L)$.

Ordered topological spaces. An *ordered topological space* is a triple $(X, \leq, \tau)$ where $\leq$ is a partial order on X and τ is a topology on X. The set of clopen (i.e. closed and open) order-filters of such a space will be denoted $\mathsf{ClopenOF}(X, \leq, \tau)$, or simply by $\mathsf{ClopenOF}(X)$ if the order and topology are supposed to be known. $\mathsf{ClopenOF}(X, \leq, \tau)$ is a sublattice of the lattice of order-filters of the partially-ordered space $(X, \leq)$, where the join is the union and the meet is the intersection. An ordered topological space is *totally order-disconnected* if, whenever $x \not\leq y$, there exists a clopen order-filter $U \in \mathcal{O}(X)$ with $x \in U$ and $y \notin U$. A *Priestley space* is a totally order-disconnected space that is also *compact*. We note that in a Priestley space the clopen order-filters and their complements form a subbasis for the topology. It is easy to see that this topology is the join of two topologies, namely: (i) the *upper topology*, generated by the set of clopen order-filters as a basis, and (ii) the *lower topology*, generated by the set of complements of clopen order-filters as a basis.

Priestley Duality. In what follows we will often refer to the Priestley duality theorem for distributive lattices; for details see e.g. [Pri70, Pri72, DP90, Gol89]. For the convenience of the reader we mention the main results. The Priestley representation theorem states that every distributive lattice L is isomorphic to the lattice of clopen order filters of the ordered topological space $D(L) = (\mathcal{F}_p(L), \leq, \tau)$ having as points the prime filters of L, ordered by inclusion, and the topology τ generated by the sets of the form $X_a = \{F \mid F \text{ prime filter } a \in F\}$ and their complements as a subbasis. Moreover, it was shown that a dual equivalence exists between the category $\mathsf{D_{01}}$ of bounded distributive lattices (with homomorphisms of bounded distributive lattices as morphisms) and the category P of Priestley spaces (with continuous, order-preserving maps as morphisms). This dual equivalence is defined by two contravariant functors $D : \mathsf{D_{01}} \to \mathsf{P}$ and $E : \mathsf{P} \to \mathsf{D_{01}}$. These functors are defined on objects as follows[1]: for every bounded distributive lattice L, $D(L) = (\mathcal{F}_p(L), \leq, \tau)$ (which will be called in what follows the Priestley dual of L) is the ordered topological space defined as explained above, and for every Priestley space X, $E(X)$ is its set of clopen order-filters, on which a bounded distributive lattice structure is defined by taking

[1]Actually, in [Pri70, Pri72] the functors are defined, equivalently, as follows: for every bounded distributive lattice L, $D(L) = \{h : L \to \{0,1\} \mid h \text{ 0,1-lattice homomorphism}\}$ and for every Priestley space X, $E(X) = \{f : X \to \{0,1\} \mid f \text{ continuous, order-preserving}\}$, where $\{0,1\}$ is regarded as a bounded lattice in the first case, and as an ordered set (with $0 < 1$) endowed with the discrete topology in the second case.

$0 = \emptyset, 1 = X$ and defining joins and meets as unions resp. intersections of clopen order filters. (The set of clopen order-filters of $D(L)$ is the set $\{X_a \mid a \in L\}$.) The functors D and E are defined on morphisms as follows: for every morphism of bounded distributive lattices $f : L_1 \to L_2$, $D(f) : D(L_2) \to D(L_1)$ is defined by $D(f)(F) = f^{-1}(F)$ for every prime filter F of L_2, and for every continuous, order-preserving map $h : X_1 \to X_2$, $E(h) : E(X_2) \to E(X_1)$ is defined by $E(h)(U) = h^{-1}(U)$ for every clopen order-filter U of X_2. The functors D and E establish a dual equivalence between the categories $\mathsf{D_{01}}$ and P: for every bounded distributive lattice L there is a 0,1-lattice isomorphism $\eta_L : L \simeq E(D(L))$, and for every Priestley space X, there exists an order-preserving homeomorphism $\epsilon_X : X \simeq D(E(X))$. In particular, for every finite lattice L, $D(L) = \{\uparrow i \mid i \text{ join-irreducible in } L\}$, the order is defined by $\uparrow i \leq \uparrow j$ if and only if $i \geq j$ in L, and the Priestley topology on $D(L)$ is discrete. Thus, if L is finite, then $L \simeq \mathcal{O}(D(L))$.

Universal algebra. For the basic notions of universal algebra needed in what follows we refer for instance to [BS81]. For every signature Σ and every arity function $a : \Sigma \to N$, a Σ-algebra is a structure $(A, \{\sigma_A\}_{\sigma \in \Sigma})$, where for every $\sigma \in \Sigma$, $\sigma_A : A^{a(\sigma)} \to A$. If the signature Σ is assumed to be known we will sometimes use the notation A for the Σ-algebra $(A, \{\sigma_A\}_{\sigma \in \Sigma})$. We say that a Σ-algebra has a *bounded distributive lattice reduct* if there exist operation symbols $\vee, \wedge, 0, 1$ in Σ such that $(A, \vee_A, \wedge_A, 0_A, 1_A)$ is a bounded distributive lattice. A variety is a class of algebras which is closed under homomorphic images, subalgebras and direct products. The variety generated by a class $\mathcal{K}$ of algebras is the class of all algebras that are homomorphic images of a subalgebra of a product of elements in $\mathcal{K}$; it will be denoted $HSP(\mathcal{K})$.

2 Priestley duality for distributive lattices with operators

In [Gol89] the Priestley duality for distributive lattices is extended to a dual equivalence between the category DLO of distributive lattices having as operators join- and meet-hemimorphisms and a suitable category RPS of relational Priestley spaces. In what follows we extend the results of [Gol89] in that besides considering distributive lattices endowed with join- and meet-hemimorphisms we also consider separately operators that are homomorphisms, antimorphisms, join- and meet-hemiantimorphisms. These results are then used in [SS98] in order to define a general notion of relational (not necessarily topological) models for certain logics, and a notion of satisfiability in such models. This type of results was used in [SS97] in the study of (first-order) finitely-valued logics. The Priestley dual of the algebra of truth values was used for obtaining efficient translations to clause forms and an automated theorem proving procedure.

Definition 1 *Let A be an algebra with a bounded lattice reduct.*
A lattice antimorphism *on A is a function $k : A \to A$ with $k(0) = 1, k(1) = 0, k(a_1 \vee a_2) = k(a_1) \wedge k(a_2)$ and $k(a_1 \wedge a_2) = k(a_1) \vee k(a_2)$.*
A join-hemimorphism[2] *on A is a function $f : A^n \to A$ such that for every $i, 1 \leq i \leq n$,*

$$\begin{array}{ll} (Jh1) & f(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) = 0, \\ (Jh2) & f(a_1, \dots, a_{i-1}, b_1 \vee b_2, a_{i+1}, \dots, a_n) = \\ & = f(a_1, \dots, a_{i-1}, b_1, a_{i+1}, \dots, a_n) \vee f(a_1, \dots, a_{i-1}, b_2, a_{i+1}, \dots, a_n). \end{array}$$

A meet-hemimorphism *on A is a function $g : A^n \to A$ such that for every $i, 1 \leq i \leq n$,*

$$\begin{array}{ll} (Mh1) & g(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = 1, \\ (Mh2) & g(a_1, \dots, a_{i-1}, b_1 \wedge b_2, a_{i+1}, \dots, a_n) = \\ & = g(a_1, \dots, a_{i-1}, b_1, a_{i+1}, \dots, a_n) \wedge g(a_1, \dots, a_{i-1}, b_2, a_{i+1}, \dots, a_n). \end{array}$$

A join-hemiantimorphism *on A is a function $f' : A^n \to A$ such that for every $i, 1 \leq i \leq n$,*

$$\begin{array}{ll} (Ja1) & f'(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = 0, \\ (Ja2) & f'(a_1, \dots, a_{i-1}, b_1 \wedge b_2, a_{i+1}, \dots, a_n) = \\ & = f'(a_1, \dots, a_{i-1}, b_1, a_{i+1}, \dots, a_n) \vee f'(a_1, \dots, a_{i-1}, b_2, a_{i+1}, \dots, a_n). \end{array}$$

A meet-hemiantimorphism *on A is a function $g' : A^n \to A$ such that for every $i, 1 \leq i \leq n$,*

$$\begin{array}{ll} (Ma1) & g'(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) = 1, \\ (Ma2) & g'(a_1, \dots, a_{i-1}, b_1 \vee b_2, a_{i+1}, \dots, a_n) = \\ & = g'(a_1, \dots, a_{i-1}, b_1, a_{i+1}, \dots, a_n) \wedge g'(a_1, \dots, a_{i-1}, b_2, a_{i+1}, \dots, a_n). \end{array}$$

[2] The term "hemimorphism" was introduced by Halmos in [Hal55]; the concept was used for the representation of the necessity operator of modal logic.

Definition 2 *Let $(X, \leq)$ be a partially ordered set and let $R \subseteq X^{n+1}$ be a $(n+1)$-ary relation on X.*

(1) R is an increasing relation *if it has the property that for all $x \in X^n$ and every $y, z \in X$, if $R(x, y)$ and $y \leq z$ then $R(x, z)$.*

(2) R is a decreasing relation *if it has the property that for all $x \in X^n$ and every $y, z \in X$, if $R(x, y)$ and $z \leq y$ then $R(x, z)$.*

Definition 3 *Let $R \subseteq X^{n+1}$ be a $(n+1)$-ary relation, and let $y \in X$. The* inverse image *of y with respect to R is the set $R^{-1}(y) = \{(x_1, \dots, x_n) \mid R(x_1, \dots, x_n, y)\}$.*

We will start by defining the category of distributive lattices with operators which will be studied in what follows, as well as a corresponding category of Priestley spaces with additional functions and relations. Then, based on previous papers in which the link between the algebraic and the relational semantics for modal logics is analyzed (as [JT51, JT52, Lem66a, Lem66b], as well as the general study in [Gol89]) we will indicate a canonical way in which one can associate functions or relations in the dual space and vice-versa, and will show that a dual equivalence between these categories can be thus defined.

2.1 The categories DLO_Σ and RPS_Σ

Let Σ be a signature containing function symbols in the classes mentioned above. In order to distinguish these classes, we will write $\Sigma = Lh \cup La \cup Jh \cup Mh \cup Ja \cup Ma$, where Lh, La, Jh, Mh, Ja, and Ma can possibly be empty. As a convention, in what follows (if not explicitly specified otherwise) the name h will be used for an operation symbol in Lh, k for an operation symbol in La, f for one in Jh, g for one in Mh, f' for one in Ja, and g' for one in Ma. Similarly, on the corresponding Priestley spaces, we will denote by H (resp. K) the map corresponding to operation symbols in Lh (resp. La), by R (resp. R') the relation corresponding to a symbol in Jh (resp. Ja), and by Q (resp. Q') the relation corresponding to a symbol in Mh (resp. Ma).

Let DLO_Σ be the category of bounded distributive lattices with operators in Σ having:

Objects: $(A, \vee, \wedge, 0, 1, \{h_A\}_{h \in Lh}, \{k_A\}_{k \in La}, \{f_A\}_{f \in Jh}, \{g_A\}_{g \in Mh}, \{f'_A\}_{f' \in Ja}, \{g'_A\}_{g' \in Ma})$ where $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice, and for every $h \in Lh$, $k \in La$, $f \in Jh$, $g \in Mh$, $f' \in Ja$, $g' \in Ma$: h_A is a lattice homomorphism, k_A is a lattice antimorphism, f_A is a join-hemimorphism, g_A is a meet-hemimorphism, f'_A is a join-hemiantimorphism, and g'_A is a meet-hemiantimorphism.

Morphisms: Lattice morphisms that preserve $0, 1$ and the operators in Lh, La, Jh, Mh, Ja, and Ma.

Let RPS_Σ be the category of relational Priestley spaces, having:

Objects: Relational Priestley spaces, i.e. ordered topological spaces endowed with relations, $(X, \leq, \tau, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma})$, where:
(1) $(X, \leq \tau)$ is a Priestley space,
(2) for every $H \in Lh$, $H_X : X \to X$ is a continuous order-preserving map,
(3) for every $K \in La$, $K_X : X \to X$ is a continuous order-reversing map,
(4) for every $R \in Jh$ with arity n, $R_X \subseteq X^{n+1}$ is an increasing relation such that:
  (4a) $\forall y \in D(A)$, $R_X^{-1}(y)$ is closed in the product on $D(A)^n$ of the upper topology,
  (4b) $\forall Y_1, \dots, Y_n \subseteq X$ clopen, $\{y \mid \exists x_1 \in Y_1, \dots, x_n \in Y_n : R_X(x_1, \dots, x_n, y)\}$ is clopen.
(5) for every $Q \in Mh$ with arity n, $Q_X \subseteq X^{n+1}$ is a decreasing relation such that:
  (5a) $\forall y \in D(A)$, $Q_X^{-1}(y)$ is closed in the product on $D(A)^n$ of the lower topology,
  (5b) $\forall Y_1, \dots, Y_n \subseteq X$ clopen, $\{y \mid \forall x_1, \dots, x_n, Q_X(x_1, \dots, x_n) \Rightarrow \exists i : x_i \in Y_i\}$ is clopen.
(6) for every $R' \in Ja$ with arity n, $R'_X \subseteq X^{n+1}$ is an increasing relation such that:
  (6a) $\forall y \in D(A)$, $R'^{-1}_X(y)$ is closed in the product on $D(A)^n$ of the lower topology,
  (6b) $\forall Y_1, \dots, Y_n \subseteq X$ clopen, $\{y \mid \exists x_1 \notin Y_1, \dots, x_n \notin Y_n : R'_X(x_1, \dots, x_n, y)\}$ is clopen.
(7) for every $Q' \in Ma$ with arity n, $Q'_X \subseteq X^{n+1}$ is a decreasing relation such that:
  (7a) $\forall y \in D(A)$, $Q'^{-1}_X(y)$ is closed in the product on $D(A)^n$ of the upper topology,
  (7b) $\forall Y_1, \dots, Y_n \subseteq X$ clopen, $\{y \mid \forall x_1, \dots, x_n, Q'_X(x_1, \dots, x_n) \Rightarrow \exists i : x_i \notin Y_i\}$ is clopen.

Morphisms continuous bounded morphisms, see the following definition.

Definition 4 (Bounded morphism) *Let $\mathbf{X_1}$ and $\mathbf{X_2}$ be two relational Priestley spaces,*

$$\mathbf{X_1} = (X_1, \leq, \tau_1, \{H_{X_1}\}_{H\in Lh}, \{K_{X_1}\}_{K\in La}, \{R_{X_1}\}_{R\in Jh}, \{Q_{X_1}\}_{Q\in Mh}, \{R'_{X_1}\}_{R'\in Ja}, \{Q'_{X_1}\}_{Q'\in Ma}),$$

$$\mathbf{X_2} = (X_2, \leq, \tau_2, \{H_{X_2}\}_{H\in Lh}, \{K_{X_2}\}_{K\in La}, \{R_{X_2}\}_{R\in Jh}, \{Q_{X_2}\}_{Q\in Mh}, \{R'_{X_2}\}_{R'\in Ja}, \{Q'_{X_2}\}_{Q'\in Ma}).$$

(1) A map $\phi : X_1 \to X_2$ is a morphism *if it preserves the order $\leq$, the operations in Lh, La and the relations in Jh, Mh, Ja, and Ma in passing from X_1 to X_2, i.e. it satisfies the following conditions:*

(M1) $x \leq y$ implies $\phi(x) \leq \phi(y)$,

(M2) for every $H \in Lh \cup La$, $x = H_{X_1}(y)$ implies $\phi(x) = H_{X_2}(\phi(y))$

(M3) for every $R \in Jh \cup Mh \cup Ja \cup Ma$, $R_{X_1}(x_1, \dots, x_n, x)$ implies $R_{X_2}(\phi(x_1), \dots, \phi(x_n), \phi(x))$.

(2) A morphism $\phi : X_1 \to X_2$ is bounded *if for all $z \in X_1$ it satisfies:*

(B1) $R_{X_2}(y_1, \dots, y_n, \phi(z))$ implies $\exists x_1, \dots, x_n \in X_1(R_{X_1}(x_1, \dots, x_n, z)$ and $y_i \leq \phi(x_i)$, for every $1 \leq i \leq n)$, for every $y_1, \dots, y_n \in X_2$, and every $R \in Jh \cup Ma$,

(B2) $R_{X_2}(y_1, \dots, y_n, \phi(z))$ implies $\exists x_1, \dots, x_n \in X_1(R_{X_1}(x_1, \dots, x_n, z)$ and $\phi(x_i) \leq y_i$, for every $1 \leq i \leq n)$, for every $y_1, \dots, y_n \in X_2$, and every $Q \in Mh \cup Ja$.

It is easy to see that the composition of bounded morphisms is again a bounded morphism. Note that the operations in Lh and La can be seen as relations in the usual way, i.e. as follows:

$$\rho_H = \{(x_1, x_2) \mid x_1 = H_X(x_2)\};\ \rho_K = \{(x_1, x_2) \mid x_1 = K_X(x_2)\}.$$

Conditions similar to $(B1)$ and $(B2)$ can be stated also for these relations, namely:

(BLh) $\rho_{H_{X_2}}(y, \phi(x_2))$ implies $\exists x_1 \in X_1$: $\rho_{H_{X_1}}(x_1, x_2)$ and $y = \phi(x_1)$,

(BLa) $\rho_{K_{X_2}}(y, \phi(x_2))$ implies $\exists x_1 \in X_1$: $\rho_{K_{X_1}}(x_1, x_2)$ and $y = \phi(x_1)$.

However, it is not necessary to explicitly specify conditions similar to $(B1)$ and $(B2)$ for the operations in Lh and La because they are already satisfied, as shown by Lemma 1.

Lemma 1 *Let $\phi : \mathbf{X_1} \to \mathbf{X_2}$ be a morphism of relational Priestley spaces. Let $H \in Lh \cup La$. For every $y \in X_2$, if $H_{X_2}(\phi(x_2)) = y$, then there exists a $x_1 \in X_1$ such that $H_{X_1}(x_2) = x_1$ and $y = \phi(x_1)$.*

Proof: It suffices to take $x_1 = H_{X_1}(x_2)$. Then $\phi(x_1) = y$ follows from the fact that ϕ is a morphism.□

In [Gol89] it is proved that there is a dual equivalence between the category DLO of distributive lattices with operators (join- and meet-hemimorphisms) and a corresponding category RPS of relational Priestley spaces. We now show that this correspondence can be extended to a dual equivalence between the category $\mathsf{DLO}_{\mathbf{\Sigma}}$ of distributive lattices with operators in Σ (i.e. including lattice morphisms and antimorphisms, and join- and meet-hemiantimorphisms) and the corresponding category $\mathsf{RPS}_{\mathbf{\Sigma}}$ of relational Priestley spaces. For this, we define two functors, $D : \mathsf{DLO}_{\mathbf{\Sigma}} \to \mathsf{RPS}_{\mathbf{\Sigma}}$ and $E : \mathsf{RPS}_{\mathbf{\Sigma}} \to \mathsf{DLO}_{\mathbf{\Sigma}}$.

2.2 The functor D

In this section we define a contravariant functor $D : \mathsf{DLO}_{\mathbf{\Sigma}} \to \mathsf{RPS}_{\mathbf{\Sigma}}$. We present the definition of D on objects and on morphisms, as well as the main properties of D which are needed later.

2.2.1 Definition of D on objects

For every distributive lattice with operators A in $\mathsf{DLO}_{\mathbf{\Sigma}}$, let $D(A)$ be the space

$$(\mathcal{F}_p(A), \leq, \tau, \{D(h_A)\}_{h\in Lh}, \{D(k_A)\}_{k\in La}, \{D(f_A)\}_{f\in Jh}, \{D(g_A)\}_{g\in Mh}, \{D(f'_A)\}_{f'\in Ja}, \{D(g'_A)\}_{g'\in Ma}),$$

where $(\mathcal{F}_p(A), \leq, \tau)$ is the Priestley space associated with A (consisting of the set of all prime filters of A, ordered by inclusion, endowed with the Priestley topology), and the additional mappings and relations are defined as follows. If $h \in Lh$ (i.e. $h_A : A \to A$ is a lattice homomorphism), $D(h_A) : D(A) \to D(A)$ is defined by

$$D(h_A)(F) = h_A^{-1}(F) = \{a \in A \mid h_A(a) \in F\}.$$

If $k \in La$ (i.e. $k_A : A \to A$ is a lattice antimorphism), $D(k_A) : D(A) \to D(A)$ is defined by

$$D(k_A)(F) = A \backslash k_A^{-1}(F) = \{a \in A \mid k_A(a) \notin F\}.$$

If $f \in Jh$ has arity n (i.e. $f_A : A^n \to A$ is a join-hemimorphism), $D(f_A) \subseteq D(A)^{n+1}$ is defined by

$$D(f_A)(F_1, \dots, F_n, F_{n+1}) \text{ iff } \forall a_1, \dots, a_n \, (a_i \in F_i, \forall i \in \{1, \dots, n\} \text{ implies } f_A(a_1, \dots, a_n) \in F_{n+1}).$$

If $g \in Mh$ has arity n (i.e. $g_A : A^n \to A$ is a meet-hemimorphism), $D(g_A) \subseteq D(A)^{n+1}$ is defined by

$$D(g_A)(F_1, \dots, F_n, F_{n+1}) \text{ iff } \forall a_1, \dots, a_n \, (g_A(a_1, \dots, a_n) \in F_{n+1} \text{ implies } \exists i \in \{1, \dots, n\} \text{ with } a_i \in F_i).$$

If $f' \in Ja$ has arity n (i.e. $f'_A : A^n \to A$ is a join-hemiantimorphism), $D(f'_A) \subseteq D(A)^{n+1}$ is defined by

$$D(f'_A)(F_1, \dots, F_n, F_{n+1}) \text{ iff } \forall a_1, \dots, a_n \, (a_i \notin F_i, \forall i \in \{1, \dots, n\} \text{ implies } f'_A(a_1, \dots, a_n) \in F_{n+1}).$$

If $g' \in Ma$ has arity n (i.e. $g'_A : A^n \to A$ is a meet-hemiantimorphism), $D(g'_A) \subseteq D(A)^{n+1}$ is defined by

$$D(g'_A)(F_1, \dots, F_n, F_{n+1}) \text{ iff } \forall a_1, \dots, a_n \, (g'_A(a_1, \dots, a_n) \in F_{n+1} \text{ implies } \exists i \in \{1, \dots, n\} \text{ with } a_i \notin F_i).$$

Lemma 2 *Let A be a distributive lattice with operators in the classes Lh, La, Jh, Mh, Ja, Ma. Let $(D(A), \leq, \tau)$ be the Priestley dual of A. The following holds:*

(1) If $h \in Lh$ then $D(h_A) : D(A) \to D(A)$ is order-preserving, and continuous with respect to τ.

(2) If $k \in La$ then $D(k_A) : D(A) \to D(A)$ is order-reversing, and continuous with respect to τ.

(3) If $f \in Jh$ then $D(f_A) \subseteq D(A)^{n+1}$ is an increasing relation such that for every $F \in D(A)$, $D(f_A)^{-1}(F)$ is closed in the product topology on $D(A)^n$ of the upper topology, and moreover, for every $U_1, \dots, U_n \in \mathsf{ClopenOF}(D(A))$, the set $\{F \mid \exists F_1 \in U_1, \dots, F_n \in U_n : D(f_A)(F_1, \dots, F_n, F)\}$ is clopen.

(4) If $g \in Mh$ then $D(g_A) \subseteq D(A)^{n+1}$ is a decreasing relation such that for every $F \in D(A)$, $D(g_A)^{-1}(F)$ is closed in the product topology on $D(A)^n$ of the lower topology, and moreover, for every $U_1, \dots, U_n \in \mathsf{ClopenOF}(D(A))$, the set $\{F \mid \forall F_1, \dots, F_n, D(g_A)(F_1, \dots, F_n, F) \Rightarrow \exists i : F_i \in U_i\}$ is clopen.

(5) If $f' \in Ja$ then $D(f'_A) \subseteq D(A)^{n+1}$ is an increasing relation such that for every $F \in D(A)$, $D(f'_A)^{-1}(F)$ is closed in the product topology on $D(A)^n$ of the lower topology, and moreover, for every $U_1, \dots, U_n \in \mathsf{ClopenOF}(D(A))$, the set $\{F \mid \exists F_1 \notin U_1 \dots \exists F_n \notin U_n : D(f'_A)(F_1, \dots, F_n, F)\}$ is clopen.

(6) If $g' \in Ma$ then $D(g'_A) \subseteq D(A)^{n+1}$ is a decreasing relation such that for every $F \in D(A)$, $D(g'_A)^{-1}(F)$ is closed in the product topology on $D(A)^n$ of the upper topology, and moreover, for every $U_1, \dots, U_n \in \mathsf{ClopenOF}(D(A))$, the set $\{F \mid \forall F_1, \dots, F_n, D(g'_A)(F_1, \dots, F_n, F) \Rightarrow \exists i : F_i \notin U_i\}$ is clopen.

Proof: (1) has been proved by Priestley [Pri70], see also [Pri72], p.515 and [DP90], p.207ff.

(2) Let $k_A : A \to A$ be a lattice antimorphism. Let $F_1, F_2 \in D(A)$ be such that $F_1 \subseteq F_2$. Then for every $a \in D(k_A)(F_2) = A \backslash k_A^{-1}(F_2)$, we have $k_A(a) \notin F_2$, hence, since $F_1 \subseteq F_2$, $k_A(a) \notin F_1$. Hence, $a \in D(k_A)(F_1) = A \backslash k_A^{-1}(F_1)$. Thus, $D(k_A)(F_2) \subseteq D(k_A)(F_1)$.

In order to show that $D(k_A)$ is continuous it is sufficient to show that $D(k_A)^{-1}(X_a)$ and $D(k_A)^{-1}(D(A) \backslash X_a)$ are open. This holds, because $D(k_A)^{-1}(X_a) = \{F \mid D(k_A)(F) \in X_a\} = \{F \mid a \notin k_A^{-1}(F)\} = \{F \mid k_A(a) \notin F\} = D(A) \backslash X_{k_A(a)}$ and $D(k_A)^{-1}(D(A) \backslash X_a) = \{F \mid D(k_A)(F) \notin X_a\} = \{F \mid a \in k_A^{-1}(F)\} = \{F \mid k_A(a) \notin F\} = X_{k_A(a)}$.

The properties (3) and (4) are analyzed in [Gol89]. For details concerning these proofs we refer to Theorem 2.2.1 and Theorem 2.2.2 in [Gol89], pp.187-190.

(5) Let $f \in Ja$. By the definition of $D(f_A)$ we know that for every $F_1, \dots, F_n, F_{n+1} \in D(A)^{n+1}$, $D(f_A)(F_1, \dots, F_n, F_{n+1})$ if and only if $\forall a_1, \dots, a_n, ((a_1 \notin F_1 \wedge \dots \wedge a_n \notin F_n) \Rightarrow f_A(a_1, \dots, a_n) \in F_{n+1})$.

It is easy to see that $D(f_A)$ is increasing. Let $F \subseteq G$ and assume that $D(f_A)(F_1, \ldots, F_n, F)$. Let $a_1, \ldots, a_n$ be such that $a_1 \notin F_1, \ldots, a_n \notin F_n$. From the fact that $D(f_A)(F_1, \ldots, F_n, F)$ it follows that $f_A(a_1, \ldots, a_n) \in F \subseteq G$. Thus we proved that $D(f_A)(F_1, \ldots, F_n, G)$.

We now show that for every $F \in D(A)$, $D(f_A)^{-1}(F)$ is closed in the product topology on $D(A)^n$ of the lower topology. Let $(G_1, \ldots, G_n) \in D(A)^n$ be such that $(G_1, \ldots, G_n) \notin D(f_A)^{-1}(F)$. Then there exist $a_1, \ldots, a_n$ such that $a_1 \notin G_1, \ldots, a_n \notin G_n$ and $f(a_1, \ldots, a_n) \notin F$. Hence, there exist $a_1, \ldots, a_n$ such that $G_1 \notin X_{a_i}, \ldots, G_n \notin X_{a_n}$ and $f(a_1, \ldots, a_n) \notin F$. Let $N = D(A) \backslash X_{a_1} \times \ldots \times D(A) \backslash X_{a_n}$. For every $(F_1, \ldots, F_n) \in N$ we have $a_i \notin F_i$ for every $1 \leq i \leq n$. We know that $f(a_1, \ldots, a_n) \notin F$. It follows therefore that $(F_1, \ldots, F_n) \notin D(f_A)^{-1}(F)$. This shows that there exists an open neighborhood of $(G_1, \ldots, G_n)$ (in the product topology on $D(A)^n$ of the lower topology) which is contained in the complementary of $D(f_A)^{-1}(F)$. Thus, $D(f_A)^{-1}(F)$ is closed in the product topology on $D(A)^n$ of the lower topology.

In order to show that for every $U_1, \ldots, U_n \in \mathsf{ClopenOF}(D(A))$ the set $\{F \mid \exists F_1 \notin U_1, \ldots, F_n \notin U_n : D(f_A)(F_1, \ldots, F_n, F)\}$ is clopen, taking into account the form of the clopen order-filters of $D(A)$, it suffices to show that the necessary and sufficient condition for $F \in X_{f_A(a_1, \ldots, a_n)}$ is ($\exists G_1, \ldots, G_n \in D(A)$ such that $a_1 \notin G_1, \ldots, a_n \notin G_n$ and $D(f_A)(G_1, \ldots, G_n, F)$).

The sufficiency of the condition follows easily: assume that there exist $G_1, \ldots, G_n \in D(A)$ such that ($a_1 \notin G_1, \ldots, a_n \notin G_n$ and $D(f_A)(G_1, \ldots, G_n, F)$). Since $D(f_A)(G_1, \ldots, G_n, F)$, it follows that for all $b_1, \ldots, b_n$, if $b_1 \notin G_1, \ldots, b_n \notin G_n$ then $f_A(b_1, \ldots, b_n) \in F$. In particular taking $b_1 = a_1, \ldots, b_n = a_n$ it follows that $f_A(a_1, \ldots, a_n) \in F$, i.e. $F \in X_{f_A(a_1, \ldots, a_n)}$. In order to prove the necessity, we use an argument similar to the one used by Goldblatt in Theorem 2.2.1 in [Gol89]. We construct inductively a sequence of prime filters $G_1, \ldots, G_n$ such that for every $i \leq n$ the following hold:

(i) $a_i \notin G_i$,

(ii) If $y_k \notin G_k$ for all $k \leq i$, then $f_A(y_1, \ldots, y_i, a_{i+1}, \ldots, a_n) \in F$.

Let $j \leq n$ be fixed and assume that for every $i < j$, G_i has been defined such that the conditions (i) and (ii) are satisfied. Then G_j will be defined such that (i) and (ii) are satisfied for j.

Let $H_j = \{z \mid \exists y_1 \notin G_1, \ldots, \exists y_{j-1} \notin G_{j-1} : f_A(y_1, \ldots, y_{j-1}, z, a_{j+1}, \ldots, a_n) \notin F\}$.

Claim 1. H_j is closed under all finite meets.

Proof: Assume that $z_1, z_2 \in H_j$. Then for all $i < j$ there exist $y_i^1, y_i^2 \in G_i$ such that $f_A(y_1^1, \ldots, y_{j-1}^1, z_1, a_{j+1}, \ldots, a_n) \notin F$ and $f_A(y_1^2, \ldots, y_{j-1}^2, z_2, a_{j+1}, \ldots, a_n) \notin F$. Let $y_i = y_i^1 \vee y_i^2$ for $i = 1, \ldots, j-1$. Since f_A is order-reversing in every argument, it follows that $f_A(y_0, \ldots, y_{j-1}, z_k, a_{j+1}, \ldots, a_n) \notin F$ for $k = 1, 2$. Therefore, $f_A(y_0, \ldots, y_{j-1}, z_1 \wedge z_2, a_{j+1}, \ldots, a_n) = f_A(y_0, \ldots, y_{j-1}, z_1, a_{j+1}, \ldots, a_n) \vee f_A(y_0, \ldots, y_{j-1}, z_2, a_{j+1}, \ldots, a_n) \notin F$ and thus, $z_1 \wedge z_2 \in H_j$.

Note also that $1 \in H_j$. It follows that $\bigwedge I \in H_j$ for every finite (possible empty) subset $I \subseteq H_j$.

Claim 2. H_j is separated from a_j, i.e. for every finite subset $I \subseteq H_j$, $\bigwedge I \not\leq a_j$.

Proof: Assume that there exists a finite subset $I \subseteq H_j$, such that $\bigwedge I \leq a_j$. Since H_j is closed under all finite meets, $z = \bigwedge I \in H_j$, and, therefore, there exist $y_1 \notin G_1, \ldots, y_{j-1} \notin G_{j-1}$ such that $f_A(y_1, \ldots, y_{j-1}, z, a_{j+1}, \ldots, a_n) \notin F$.

On the other hand, since f_A is order-reversing in every argument and $\bigwedge I \leq a_j$, it follows that $f_A(y_1, \ldots, y_{j-1}, \bigwedge I, a_{j+1}, \ldots, a_n) \geq f_A(y_1, \ldots, y_{j-1}, a_j, a_{j+1}, \ldots, a_n)$, and since by the induction hypothesis $f_A(y_1, \ldots, y_{j-1}, a_j, a_{j+1}, \ldots, a_n) \in F$ it follows that $f_A(y_1, \ldots, y_{j-1}, \bigwedge I, a_{j+1}, \ldots, a_n) \in F$, which is a contradiction.

Since H_j is separated from a_j, there exists a prime filter G_j such that $H_j \subseteq G_j$ and $a_j \notin G_j$. Thus, (i) is fulfilled by G_j. In order to prove that (ii) also holds, let $y_1, \ldots, y_j$ such that $y_k \notin G_k$ for every $k \leq j$. Thus, $y_j \notin G_j$ and therefore $y_j \notin H_j$. It follows that for all $\overline{y}_1 \notin G_1, \ldots, \overline{y}_{j-1} \notin G_{j-1}$, $f_A(\overline{y}_1, \ldots, \overline{y}_{j-1}, y_j, a_{j+1}, \ldots, a_n) \in F$, hence, in particular, taking $\overline{y}_1 = y_1, \ldots, \overline{y}_{j-1} = y_{j-1}$, $f_A(y_1, \ldots, y_j, a_{j+1}, \ldots, a_n) \in F$.

(6) The proof is analogous to that of (5), taking into account the definition of $D(g_A)$ for $g \in Ma$. $\square$

Lemma 2 shows that D maps objects of $\mathsf{DLO}_{\mathbf{\Sigma}}$ to objects of $\mathsf{RPS}_{\mathbf{\Sigma}}$.

2.2.2 Definition of D on morphisms

For every morphism $h : A_1 \to A_2$ in DLO_Σ, let $D(h) : D(A_2) \to D(A_1)$ be defined for every $F \in D(A_2)$ by $D(h)(F) = h^{-1}(F)$.

Lemma 3 *Let $h : A_1 \to A_2$ be a morphism of DLO_Σ. Then*

(1) $D(h) : D(A_2) \to D(A_1)$ defined by $D(h)(F) = h^{-1}(F)$ for every $F \in D(A_2)$ is continuous and preserves the order $\leq$, the operations Lh, La and the relations in Jh, Mh, Ja and Ma in passing from $D(A_2)$ to $D(A_1)$. Moreover, it satisfies the following conditions:

(B1) for every $f \in Jh \cup Ma$ and every $y_1, \ldots, y_n \in D(A_1)$, if $D(f_{A_1})(y_1, \ldots, y_n, D(h)(z))$ then $\exists x_1, \ldots, x_n \in D(A_2)$ such that $D(f_{A_2})(x_1, \ldots, x_n, z)$ and $y_i \leq D(h)(x_i)$, for every $1 \leq i \leq n$,

(B2) for every $f \in Mh \cup Ja$ and every $y_1, \ldots, y_n \in D(A_1)$, if $D(f_{A_1})(y_1, \ldots, y_n, D(h)(z))$ then $\exists x_1, \ldots, x_n \in D(A_2)$ such that $D(f_{A_2})(x_1, \ldots, x_n, z)$ and $D(h)(x_i) \leq y_i$, for every $1 \leq i \leq n$.

(2) If h is injective, then $D(h)$ is surjective.

(3) If h is surjective, then $D(h)$ is an order-embedding.

Proof: In order to prove (1) we only have to show that the conditions (B1) and (B2) are satisfied for $f \in Ja$ resp. $f \in Ma$ (for the rest we refer to [Gol89], Theorem 2.3.2).

Let $f \in Ja$ be a join-hemiantimorphism. Since the proof is similar to the one for join-hemimorphisms given in [Gol89] up to replacing some of the prime filters (resp. order filters) with their complements, we just present a sketch of the proof.

Note first that if f has arity 0 then we are exactly in the situation of 0-ary meet-hemimorphisms, considered in [Gol89]. Assume that $D(f_{A_1})(G_1, \ldots, G_n, D(h)(F))$, where $G_1, \ldots, G_n \in D(A_1)$ and $F \in D(A_2)$. We will show that there exist $F_1, \ldots, F_n \in D(A_2)$ such that $D(f_{A_2})(F_1, \ldots, F_n, F)$ and for every $i = 1, \ldots, n$, $D(h)(F_i) \leq G_i$. The construction is carried on inductively such that for every $i \leq n$,

(i) $F_i \cap h(A_1 \backslash G_i) = \emptyset$,

(ii) if $y_k \notin F_k$ for $k \leq i$, and $x_p \notin G_p$ for $i < p \leq n$, then $f_{A_2}(y_1, \ldots, y_i, h(x_{i+1}), \ldots, h(x_n)) \in F$.

Let $j \leq n$ and suppose that $F_1, \ldots, F_{j-1}$ have been inductively defined to satisfy (i) and (ii). Define $H_j \subseteq A_2$ as follows:

$$H_j = \{z \mid \left\{ \begin{array}{l} \exists y_1 \notin F_1, \ldots, y_{j-1} \notin F_{j-1}; \\ \exists x_{j+1} \notin G_{j+1}, \ldots, x_n \notin G_n, \end{array} \right. \quad \text{s.t. } f_{A_2}(y_1, \ldots, y_{j-1}, z, h(x_{j+1}), \ldots, h(x_n)) \notin F\}.$$

Claim 1. H_j is closed under finite meets.

Proof: Let $z_1, z_2 \in H_j$. For z_i ($i \in \{1,2\}$), there exist $y_1^i \notin F_1, \ldots, y_{j-1}^i \notin F_{j-1}$, and there exist $x_{j+1}^i \notin G_{j+1}, \ldots, x_n^i \notin G_n$, such that $f_{A_2}(y_1^i, \ldots, y_{j-1}^i, z_i, h(x_{j+1}^i), \ldots, h(x_n^i)) \notin F$. Let $y_k = y_k^1 \vee y_k^2$ for every $k \in \{1, \ldots, j-1\}$ and $x_p = x_p^1 \vee x_p^2$ for every $p \in \{j+1, \ldots, n\}$. Then it is easy to see that $f_{A_2}(y_1, \ldots, y_{j-1}, z_i, h(x_{j+1}), \ldots, h(x_n)) \notin F$ for $i = 1, 2$, hence $f_{A_2}(y_1, \ldots, y_{j-1}, z_1 \wedge z_2, h(x_{j+1}), \ldots, h(x_n)) \notin F$. This shows that for z_1, z_2 there exist $y_1 \notin F_1, \ldots, y_{j-1} \notin F_{j-1}, x_{j+1} \notin G_{j+1}, \ldots, x_n \notin G_n$ such that $f_{A_2}(y_1, \ldots, y_{j-1}, z_1 \wedge z_2, h(x_{j+1}), \ldots, h(x_n)) \notin F$, i.e. that $z_1 \wedge z_2 \in H_j$. It is also easy to see that $1 \in H_j$. It follows therefore that H_j is closed under all finite (possibly empty) meets.

Claim 2. H_j is separated from $h(A_1 \backslash G_j)$.

Proof: Assume that $\bigwedge I \leq \bigvee J$ for some finite $I \subseteq H_j$ and $J \subseteq h(A_1 \backslash G_j)$. Then $\bigwedge I \in H_j$, hence there exist $y_1 \notin F_1, \ldots, y_{j-1} \notin F_{j-1}; x_{j+1} \notin G_{j+1}, \ldots, x_n \notin G_n$ such that

$$f_{A_2}(y_1, \ldots, y_{j-1}, \bigwedge I, h(x_{j+1}), \ldots, h(x_n)) \notin F.$$

On the other hand, since $f \in Ja$, it is order reversing, hence

$$f_{A_2}(y_1, \ldots, y_{j-1}, \bigwedge I, h(x_{j+1}), \ldots, h(x_n)) \geq f_{A_2}(y_1, \ldots, y_{j-1}, \bigvee J, h(x_{j+1}), \ldots, h(x_n)),$$

and thus $f_{A_2}(y_1, \ldots, y_{j-1}, \bigvee J, h(x_{j+1}), \ldots, h(x_n)) \notin F$.

Note now that if $J \subseteq h(A_1 \backslash G_j)$ is a finite subset, then $\bigvee J = h(b)$ for some $b \in A_1 \backslash G_j$. Thus it follows that $f_{A_2}(y_1, \ldots, y_{j-1}, h(b), h(x_{j+1}), \ldots, h(x_n)) \notin F$. If $j > 1$ this contradicts the induction hypothesis (ii) for $j-1$. If $j = 1$ then we have $h(f_{A_1}(b, x_2, \ldots, x_n)) = f_{A_2}(h(b), h(x_2) \ldots, h(x_n)) \notin F$, hence $f_{A_1}(b, x_2, \ldots, x_n) \notin h^{-1}(F) = D(h)(F)$. Since $b \notin G_1$ and $x_i \notin G_i$ for every $i \geq 2$, this contradicts $D(f_{A_1})(G_1, \ldots, G_n, D(h)(F))$.

Hence, by the prime filter theorem, there exists a prime filter $F_j \in D(A_2)$ such that $H_j \subseteq F_j$ and $F_j \cap h(A_1 \backslash G_j) = \emptyset$. Thus, (i) is fulfilled by F_j; from the fact that $H_j \subseteq F_j$ it follows that if $z \notin F_j$ then $z \notin H_j$, hence for every $y_i \notin F_i$, $i < j$, every $y_j \notin F_j$, and every $x_p \notin G_p$, $j < p \leq n$, $f_{A_2}(y_1, \ldots, y_{j-1}, y_j, h(x_{j+1}), \ldots, h(x_n)) \in F$, i.e. (ii) holds too.

The case $f \in Ma$ can be solved similarly combining arguments from the cases Mh and Ja.

For (2) we refer to [Gol89].

(3) Assume that $h : A_1 \to A_2$ is onto. Then $D(h)$ is obviously order-preserving. We show that it is an order-embedding. Let $F_1, F_2 \in D(A_2)$. Assume that $D(h)(F_1) \leq D(h)(F_2)$. We want to show that $F_1 \subseteq F_2$. Let $b \in F_1$ be arbitrary. From the surjectivity of h, $b = h(a)$ for some $a \in A_1$. Thus, $b = h(a)$ for some $a \in h^{-1}(F_1) = D(h)(F_1) \subseteq D(h)(F_2)$. Hence, $b = h(a) \in F_2$. This proves that $F_1 \subseteq F_2$. □

Lemma 3 shows that D maps morphisms of DLO_Σ to morphisms of RPS_Σ.

2.3 The functor E

In this section we define a contravariant functor $E : \mathsf{RPS}_\Sigma \to \mathsf{DLO}_\Sigma$. We present separately the definition of E on objects and on morphisms, as well as its main properties which are needed later.

2.3.1 Definition of E on objects

For every relational Priestley space

$$(X, \leq, \tau, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma})$$

let $E(X)$ be the bounded lattice of clopen order-filters of X (where $0 = \emptyset, 1 = X$, the join is defined by taking the union, and the meet by taking the intersection), endowed with additional operators defined as shown in what follows.

For every $H \in Lh$ (i.e. such that $H_X : X \to X$ is continuous and order-preserving), $E(H_X) : E(X) \to E(X)$ is defined by

$$E(H_X)(U) = H_X^{-1}(U) = \{x \in X \mid H_X(x) \in U\}.$$

For every $K \in La$ (i.e. such that $K_X : X \to X$ is continuous and order-reversing), $E(K_X) : E(X) \to E(X)$ is defined by

$$E(K_X)(U) = X \backslash K_X^{-1}(U) = \{x \in X \mid K_X(x) \notin U\}.$$

If $R \in Jh$ has arity n (i.e. $R_X \subseteq X^{n+1}$ is an $(n+1)$-ary increasing relation satisfying conditions (4a) and (4b) in the definition of relational Priestley spaces), then $E(R_X) : E(X)^n \to E(X)$ is defined by

$$E(R_X)(U_1, \ldots, U_n) = \{x \in X \mid \exists x_1 \in U_1, \ldots, x_n \in U_n : R_X(x_1, \ldots, x_n, x)\}.$$

If $Q \in Mh$ has arity n (i.e. $Q_X \subseteq X^{n+1}$ is a $(n+1)$-ary decreasing relation satisfying conditions (5a) and (5b)), then $E(Q_X) : E(X)^n \to E(X)$ is defined by

$$E(Q_X)(U_1, \ldots, U_n) = \{x \in X \mid \forall x_1, \ldots, x_n (Q_X(x_1, \ldots, x_n, x) \Rightarrow \exists i, x_i \in U_i)\}.$$

If $R' \in Ja$ has arity n (i.e. $R'_X \subseteq X^{n+1}$ is an $(n+1)$-ary increasing relation satisfying conditions (6a) and (6b)), then $E(R'_X) : E(X)^n \to E(X)$ is defined by

$$E(R'_X)(U_1, \ldots, U_n) = \{x \in X \mid \exists x_1 \notin U_1, \ldots, x_n \notin U_n : R'_X(x_1, \ldots, x_n, x)\}.$$

If $Q' \in Ma$ has arity n (i.e. $Q'_X \subseteq X^{n+1}$ is a $(n+1)$-ary decreasing relation satisfying conditions (7a) and (7b)), then $E(Q'_X) : E(X)^n \to E(X)$ is defined by

$$E(Q'_X)(U_1, \ldots, U_n) = \{x \in X \mid \forall x_1, \ldots, x_n (Q'_X(x_1, \ldots, x_n, x) \Rightarrow \exists i, x_i \notin U_i)\}.$$

From the definition of the objects of RPS_Σ it follows immediately that the maps above are well-defined.

2.3.2 Definition of E on morphisms

For every continuous bounded morphism $\phi : X_1 \to X_2$ let $E(\phi) : E(X_2) \to E(X_1)$ be defined for every $U \in E(X_2)$ by $E(\phi)(U) = \phi^{-1}(U)$.

Lemma 4 *Let $\phi : X_1 \to X_2$ be a bounded morphism. The following hold.*

(1) For every clopen order-filter U of X_2, $\phi^{-1}(U)$ is a clopen order-filter of X_1.

(2) $E(\phi) : E(X_2) \to E(X_1)$ is a homomorphism in $\mathsf{DLO}_\mathbf{\Sigma}$.

Proof: (1) follows from the fact that ϕ is continuous and preserves $\leq$. (2) follows easily from the fact that ϕ is a bounded morphism. □

2.4 Duality

We now show that the functors D and E define a dual equivalence between the categories $\mathsf{DLO}_\mathbf{\Sigma}$ and $\mathsf{RPS}_\mathbf{\Sigma}$.

Lemma 5 *For every $A \in |\mathsf{DLO}_\mathbf{\Sigma}|$, there exists an isomorphism $\eta_A : A \to E(D(A))$.*

Proof: Let $\eta_A : A \to E(D(A))$ be defined by $\eta_A(a) = X_a$. By the Priestley duality for distributive lattices we know that η_A is an isomorphism of bounded distributive lattices, and using the results from Lemma 2 it is not difficult to see that it preserves the operations, namely for every $h \in Lh, k \in La, f \in Jh, g \in Mh, f' \in Ja, g' \in Ma$:

$$\eta_A(h_A(a)) = X_{h_A(a)} = D(h_A)^{-1}(X_a) = E(D(h_A))(\eta_A(a)),$$
$$\eta_A(k_A(a)) = X_{k_A(a)} = X \setminus D(k_A)^{-1}(X_a) = E(D(k_A))(\eta_A(a)),$$
$$\begin{aligned}
\eta_A(f_A(a_1, \ldots, a_n)) &= X_{f_A(a_1,\ldots,a_n)} = \{F \mid \exists F_1 \in X_{a_1}, \ldots, F_n \in X_{a_n} : D(f_A)(F_1, \ldots, F_n, F)\} = \\
&= E(D(f_A))(X_{a_1}, \ldots, X_{a_n}) = E(D(f_A))(\eta_A(a_1), \ldots, \eta_A(a_n)),\\
\eta_A(g_A(a_1, \ldots, a_n)) &= X_{g_A(a_1,\ldots,a_n)} = \{F \mid \forall F_1, \ldots, F_n, D(g_A)(F_1, \ldots, F_n) \Rightarrow \exists i : F_i \in X_{a_i}\} = \\
&= E(D(g_A))(X_{a_1}, \ldots, X_{a_n}) = E(D(g_A))(\eta_A(a_1), \ldots, \eta_A(a_n)),\\
\eta_A(f'_A(a_1, \ldots, a_n)) &= X_{f'_A(a_1,\ldots,a_n)} = \{F \mid \exists F_1 \notin X_{a_1}, \ldots, F_n \notin X_{a_n} : D(f'_A)(F_1, \ldots, F_n, F)\} = \\
&= E(D(f'_A))(X_{a_1}, \ldots, X_{a_n}) = E(D(f'_A))(\eta_A(a_1), \ldots, \eta_A(a_n)),\\
\eta_A(g'_A(a_1, \ldots, a_n)) &= X_{g'_A(a_1,\ldots,a_n)} = \{F \mid \forall F_1, \ldots, F_n, D(g'_A)(F_1, \ldots, F_n) \Rightarrow \exists i : F_i \notin X_{a_i}\} = \\
&= E(D(g'_A))(X_{a_1}, \ldots, X_{a_n}) = E(D(g'_A))(\eta_A(a_1), \ldots, \eta_A(a_n)).
\end{aligned}$$
□

Lemma 6 *Let $X \in |\mathsf{RPS}_\mathbf{\Sigma}|$. Then there exists a bounded homeomorphism $\epsilon_X : X \to D(E(X))$.*

Proof: Let $\epsilon_X : X \to D(E(X))$ be defined by $\epsilon_X(x) = F_x = \{Y \in E(X) \mid x \in Y\}$. From the Priestley duality, ϵ_X is a isomorphism of ordered spaces and a homeomorphism (see e.g. [DP90, Gol89]). We only have to prove that ϵ_X preserves the operators.

Claim 1. Let $H \in Lh$ and $x \in X$. Then $F_{H_X(x)} = D(E(H_X))(F_x)$.
Proof: It follows from the fact that $Y \in F_{H_X(x)}$ if and only if $H_X(x) \in Y$ if and only if $x \in H_X^{-1}(Y)$ if and only if $x \in E(H_X)(Y)$ if and only if $E(H_X)(Y) \in F_x$ if and only if $Y \in E(H_X)^{-1}(F_x) = D(E(H_X))(F_x)$.

Claim 2. Let $K \in La$ and $x \in X$. Then $F_{K_X(x)} = D(E(K_X))(F_x)$.
Proof: It follows from the fact that $Y \in F_{K_X(x)}$ if and only if $K_X(x) \in Y$ if and only if $x \notin K_X^{-1}(Y)$ if and only if $x \notin E(K_X)(Y)$ if and only if $E(K_X)(Y) \notin F_x$ if and only if $Y \in E(K_X)^{-1}(F_x) = D(E(K_X))(F_x)$.

Claim 3. Let $R \in Jh$ with arity n and $x_1, \ldots, x_{n+1} \in X$. Then $R_X(x_1, \ldots, x_{n+1})$ if and only if $D(E(R_X))(F_{x_1}, \ldots, F_{x_{n+1}})$.
Proof: see e.g. [Gol89].

Claim 4. Let $Q \in Mh$ with arity n and $x_1, \ldots, x_{n+1} \in X$. Then $Q_X(x_1, \ldots, x_{n+1})$ if and only if $D(E(Q_X))(F_{x_1}, \ldots, F_{x_{n+1}})$.
Proof: see e.g. [Gol89].

Claim 5. Let $R' \in Ja$ with arity n and $x_1, \ldots, x_{n+1} \in X$. Then $R'_X(x_1, \ldots, x_{n+1})$ if and only if $D(E(R'_X))(F_{x_1}, \ldots, F_{x_{n+1}})$.
Proof: First assume that $R'_X(x_1, \ldots, x_{n+1})$. Let $Y_1, \ldots, Y_n \in E(X)$ be such that $Y_1 \notin F_{x_1}, \ldots, Y_n \notin F_{x_n}$ (i.e. $x_i \notin Y_i$ for $i = 1, \ldots, n$). By using the assumption that $R'_X(x_1, \ldots, x_{n+1})$, it follows that $x_{n+1} \in E(R'_X)(Y_1, \ldots, Y_n)$, i.e. $E(R'_X)(Y_1, \ldots, Y_n) \in F_{x_{n+1}}$. Thus, $D(E(R'_X))(F_{x_1}, \ldots, F_{x_{n+1}})$.

For the converse, suppose that not $R'_X(x_1, \ldots, x_n, x_{n+1})$. Then $(x_1, \ldots, x_n) \notin R'^{-1}_X(x_{n+1})$. Since $R'^{-1}_X(x_{n+1})$ is closed in the product of the lower topology, there exists a basic open neighborhood N of $(x_1, \ldots, x_n)$ in this topology, which is disjoint from $R'^{-1}_X(x_{n+1})$. By the definition of the lower topology, $N = X \backslash Y_1 \times \ldots \times X \backslash Y_n$ for some clopen sets $Y_1, \ldots, Y_n$. Since $X \backslash Y_1 \times \ldots \times X \backslash Y_n \cap R'^{-1}_X(x_{n+1}) = \emptyset$ it follows that $\forall z_1 \notin Y_1, \ldots, z_n \notin Y_n, \text{not}\, R'_X(z_1, \ldots, z_n, x_{n+1})$, hence $x_{n+1} \notin E(R'_X)(Y_1, \ldots, Y_n)$, i.e. $E(R'_X)(Y_1, \ldots, Y_n) \notin F_{x_{n+1}}$. Since $(x_1, \ldots, x_n) \in N = X \backslash Y_1 \times \ldots \times X \backslash Y_n$, we have on the one hand $Y_i \notin F_{x_i}$ for every $i = 1, \ldots, n$, and on the other hand, by the remarks above, $E(R'_X)(Y_1, \ldots, Y_n) \notin F_{x_{n+1}}$. Thus there exist $Y_1, \ldots, Y_n$ with $Y_i \notin F_{x_i}$ and $E(R'_X)(Y_1, \ldots, Y_n) \notin F_{x_{n+1}}$. This shows that not $D(E(R'_X))(F_{x_1}, \ldots, F_{x_n}, F_{x_{n+1}})$.

Claim 6. Let $Q' \in Ma$ with arity n and $x_1, \ldots, x_{n+1} \in X$. Then $Q'_X(x_1, \ldots, x_{n+1})$ if and only if $D(E(Q_X))(F_{x_1}, \ldots, F_{x_{n+1}})$.
Proof: The proof combines ideas from the proofs of *Claim 4* and *Claim 5*.

□

Theorem 7 *The composites $D \circ E$ and $E \circ D$ are naturally isomorphic to the identity functors on* DLO_Σ *and* RPS_Σ *respectively, and so the two categories are dually equivalent.*

Proof: It is sufficient to show that the isomorphisms $\eta_A : A \to E(D(A))$ and $\epsilon_X : X \to D(E(X))$ are the components of two natural transformations. For details we refer e.g. to [Gol89]. □

3 A category of ordered relational structures

In what follows, inspired by the definition of relational Priestley spaces, we introduce the notion of *ordered relational structures*, and a corresponding notion of *morphism* of ordered relational structures.

3.1 Ordered relational structures

Definition 5 *An ordered Σ-relational structure is a partially ordered set endowed with additional functions and relations*

$$\mathbf{X} = (X, \leq, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma})$$

where for every $H \in Lh$, $H_X : X \to X$ is an order-preserving map, for every $K \in La$, $K_X : X \to X$ is an order-reversing map, for every $R \in Jh \cup Ja$ with arity n, $R_X \subseteq X^{n+1}$ is an increasing relation, and for every $Q \in Mh \cup Ma$ with arity n, $Q_X \subseteq X^{n+1}$ is a decreasing relation.

Remark: Let $\mathbf{X} = (X, \leq, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma})$ be an ordered relational structure. Consider the discrete topology τ on X. The ordered topological space $(X, \leq, \tau)$ is totally order-disconnected, and it is compact if and only if X is finite.

We now show that for every ordered Σ-relational structure X, its set of order-filters can be endowed with a bounded distributive lattice structure, with additional operators which can be defined in a canonical way starting from the maps and relations on X. Note first that, for every partially ordered set $(X, \leq)$, we can define two binary operators on $\mathcal{O}(X)$:

$\vee : \mathcal{O}(X) \to \mathcal{O}(X)$ defined for every $U, V \in \mathcal{O}(X)$ by $U \vee V = U \cup V$,

$\wedge : \mathcal{O}(X) \to \mathcal{O}(X)$ defined for every $U, V \in \mathcal{O}(X)$ by $U \wedge V = U \cap V$.

Thus, $(\mathcal{O}(X), \vee, \wedge, \emptyset, X)$ is a bounded distributive lattice with 0 $(= \emptyset)$ and 1 $(= X)$. The next lemma shows that $\mathcal{O}(X)$ can be endowed with a $\mathsf{DLO}_{\mathbf{\Sigma}}$-structure.

Lemma 8 *Let* $\mathbf{X} = (X, \leq, \{H_X\}_{H\in Lh}, \{K_X\}_{K\in La}, \{R_X\}_{R\in Jh}, \{Q_X\}_{Q\in Mh}, \{R'_X\}_{R'\in Ja}, \{Q'_X\}_{Q'\in Ma})$ *be an ordered* Σ*-relational structure with operations and relations in the classes* Lh, La, Jh, Mh, Ja, Ma. *The following hold:*

(1) Every order preserving operation on X, $H \in Lh$ *induces a lattice morphism* $h_H : \mathcal{O}(X) \to \mathcal{O}(X)$, *defined for every order-filter* U *of* X *by* $h_H(U) = H_X^{-1}(U)$.

(2) Every order reversing operation on X, $K \in La$ *induces a lattice antimorphism* $k_K : \mathcal{O}(X) \to \mathcal{O}(X)$, *defined for every order-filter* U *of* X *by* $k_K(U) = X \backslash K_X^{-1}(U)$.

(3) Every increasing relation $R_X \subseteq X^{n+1}$, $R \in Jh$, *induces a join-hemimorphism* $f_R : \mathcal{O}(X)^n \to \mathcal{O}(X)$, *defined for every* $U_1, \dots, U_n \in \mathcal{O}(X)$ *by*

$$f_R(U_1, \dots, U_n) = \{x \in X \mid \exists x_1 \in U_1, \dots, x_n \in U_n : R_X(x_1, \dots, x_n, x)\}.$$

(4) Every decreasing relation $Q_X \subseteq X^{n+1}$, $Q \in Mh$, *induces a meet hemimorphism* $g_Q : \mathcal{O}(X)^n \to \mathcal{O}(X)$, *defined for every* $U_1, \dots, U_n \in \mathcal{O}(X)$ *by*

$$g_Q(U_1, \dots, U_n) = \{x \in X \mid \forall x_1, \dots, x_n\, (Q_X(x_1, \dots, x_n, x) \Rightarrow \exists i, x_i \in U_i)\}.$$

(5) Every increasing relation $R'_X \subseteq X^{n+1}$, $R' \in Ja$, *induces a join-hemiantimorphism* $f'_{R'} : \mathcal{O}(X)^n \to \mathcal{O}(X)$, *defined for every* $U_1, \dots, U_n \in \mathcal{O}(X)$ *by*

$$f'_{R'}(U_1, \dots, U_n) = \{x \in X \mid \exists x_1 \notin U_1, \dots, x_n \notin U_n : R'_X(x_1, \dots, x_n, x)\}.$$

(6) Every decreasing relation $Q' \subseteq X^{n+1}$, $Q' \in Ma$, *induces a meet-hemiantimorphism* $g'_{Q'} : \mathcal{O}(X)^n \to \mathcal{O}(X)$, *defined for every* $U_1, \dots, U_n \in \mathcal{O}(X)$ *by*

$$g'_{Q'}(U_1, \dots, U_n) = \{x \in X \mid \forall x_1, \dots, x_n\, (Q'_X(x_1, \dots, x_n, x) \Rightarrow \exists i, x_i \notin U_i)\}.$$

Proof: (1) Let $H_X : X \to X$ be order-preserving, and let h_H be defined for every $U \in \mathcal{O}(X)$ by $h_H(U) = H_X^{-1}(U)$. We show that for every $U \in \mathcal{O}(X)$, $h_H(U) \in \mathcal{O}(X)$. Let $x \leq y$. Assume that $x \in h_H(U) = H_X^{-1}(U)$. Then $H_X(x) \in U$. Since H_X is order-preserving and U an order-filter it follows that $H_X(y) \in U$, i.e. that $y \in h_H(U)$. This proves that $h_H(U) \in \mathcal{O}(X)$. The fact that h_H is a lattice homomorphism follows immediately.

(2) Let $K_X : X \to X$ be order-reversing, and let k_K be defined for every $U \in \mathcal{O}(X)$ by $k_K(U) = X \backslash K_X^{-1}(U)$. We show that for every $U \in \mathcal{O}(X)$, $k_K(U) \in \mathcal{O}(X)$. Let $x \leq y$. Assume that $x \in k_K(U)$. Then $x \notin K_X^{-1}(U)$, hence $K_X(x) \notin U$. Since K_X is order-reversing and U an order-filter it follows that $K_X(y) \notin U$, i.e. that $y \in k_K(U)$. This proves that $k_K(U) \in \mathcal{O}(X)$. The fact that k_K is a lattice antimorphism follows immediately.

(3) Let $R_X \subseteq X^n$ be an increasing relation. We show that for every $U_1, \dots, U_n \in \mathcal{O}(X)$, $f_R(U_1, \dots, U_n) \in \mathcal{O}(X)$. Let $x \leq y$. Assume that $x \in f_R(U_1, \dots, U_n)$. Then for some $x_1 \in U_1, \dots, x_n \in U_n$, $R_X(x_1, \dots, x_n, x)$. Since R_X is an increasing relation it follows that $R_X(x_1, \dots, x_n, y)$. Hence we proved that there exist $x_1 \in U_1, \dots, x_n \in U_n$ such that $R_X(x_1, \dots, x_n, y)$, i.e. that $y \in f_R(U_1, \dots, U_n)$. For every $i \in \{1, \dots, n\}$, $f_R(U_1, \dots, U_{i-1}, \emptyset, U_{i+1}, \dots, U_n) = \emptyset$. Also,

$f_R(U_1, \dots, U_{i-1}, U_i \cup V_i, U_{i+1}, \dots, U_n) =$

$$\begin{aligned}
&= \{x \in X \mid \exists x_1 \in U_1, \dots, x_i \in U_i \cup V_i, \dots, x_n \in U_n : R_X(x_1, \dots, x_n, x)\} = \\
&= \{x \in X \mid \exists x_1 \in U_1, \dots, x_i \in U_i, \dots, x_n \in U_n : R_X(x_1, \dots, x_n, x)\} \cup \\
&\quad \cup \{x \in X \mid \exists x_1 \in U_1, \dots, x_i \in V_i, \dots, x_n \in U_n : R_X(x_1, \dots, x_n, x)\} = \\
&= f_R(U_1, \dots, U_{i-1}, U_i, U_{i+1}, \dots, U_n) \cup f_R(U_1, \dots, U_{i-1}, V_i, U_{i+1}, \dots, U_n).
\end{aligned}$$

(4) Let $Q_X \subseteq X^n$ be a decreasing relation. We show that for every $U_1, \ldots, U_n \in \mathcal{O}(X)$, $g_Q(U_1, \ldots, U_n) \in \mathcal{O}(X)$. Let $x \leq y$. Assume that $x \in g_Q(U_1, \ldots, U_n)$, i.e. for every $x_1, \ldots, x_n$, if $Q_X(x_1, \ldots, x_n, x)$, then there exists an $i \in \{1, \ldots, n\}$ with $x_i \in U_i$. Let $x_1, \ldots, x_n$ be such that $Q_X(x_1, \ldots, x_n, y)$. Then, by the fact that Q_X is decreasing, $Q_X(x_1, \ldots, x_n, x)$, hence $x_i \in U_i$ for some $i \in \{1, \ldots, n\}$. This shows that $y \in g_Q(U_1, \ldots, U_n)$. It is easy to see that for every $1 \leq i \leq n$, $g_Q(U_1, \ldots, U_{i-1}, X, U_{i+1}, \ldots, U_n) = X$. Moreover,
$g_Q(U_1, \ldots, U_{i-1}, U_i \cap V_i, U_{i+1}, \ldots, U_n) =$

$$\begin{aligned}
&= \{x \in X \mid \forall x_1, \ldots, x_n \text{ if } Q_X(x_1, \ldots, x_n, x) \text{ then either } x_j \in U_j \text{ for some } j \neq i, \text{ or } x_i \in U_i \cap V_i\} = \\
&= \{x \in X \mid \forall x_1, \ldots, x_n \text{ if } Q_X(x_1, \ldots, x_n, x) \text{ then } x_j \in U_j \text{ for some } j\} \cap \\
&\quad \cap \{x \in X \mid \forall x_1, \ldots, x_n \text{ if } Q_X(x_1, \ldots, x_n, x) \text{ then either } x_j \in U_j \text{ for some } j \neq i, \text{ or } x_i \in V_i\} = \\
&= g_Q(U_1, \ldots, U_{i-1}, U_i, U_{i+1}, \ldots, U_n) \cap g_Q(U_1, \ldots, U_{i-1}, V_i, U_{i+1}, \ldots, U_n).
\end{aligned}$$

The proofs of (5) and (6) are similar. □

This shows that for every ordered Σ-relational structure

$$(X, \leq, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma}),$$

$$(\mathcal{O}(X), \vee, \wedge, \emptyset, X, \{h_H\}_{H \in Lh}, \{k_K\}_{K \in La}, \{f_R\}_{R \in Jh}, \{g_Q\}_{Q \in Mh}, \{f'_{R'}\}_{R' \in Ja}, \{g'_{Q'}\}_{Q' \in Ma})$$

is a bounded distributive lattice with operators in Σ which will be denoted $\mathcal{O}(X)$.

Remark 9 Note that every relational Priestley space

$$(X, \leq, \tau, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma})$$

can be in particular regarded as an ordered Σ-relational space, if we ignore its topology. It is easy to see that there the inclusion $\mathsf{ClopenOF}(X) \subseteq \mathcal{O}(X)$ is an injective homomorphism of bounded distributive lattices with operators in Σ, i.e. $\mathsf{ClopenOF}(X)$ is a subalgebra of $\mathcal{O}(X)$.

3.2 Morphisms of ordered Σ-relational structures

Bounded morphisms of ordered Σ-relational structures are bounded morphisms as defined in Definition 4.

Lemma 10 *Let $\phi : X_1 \to X_2$ be a bounded morphism of ordered Σ-relational spaces. Then*

(1) $\mathcal{O}(\phi) : \mathcal{O}(X_2) \to \mathcal{O}(X_1)$ defined by $\mathcal{O}(\phi)(U) = \phi^{-1}(U)$ for every $U \in \mathcal{O}(X_2)$ is a homomorphism in DLO_Σ.

(2) If ϕ is order-embedding, then $\mathcal{O}(\phi)$ is surjective.

(3) If ϕ is surjective, then $\mathcal{O}(\phi)$ is injective.

Proof: (1) follows immediately from the definition of a bounded morphism.

(2) We assume that ϕ is an order-embedding and show that $\mathcal{O}(\phi)$ is surjective.

Let $U \in \mathcal{O}(X_1)$. Let $\uparrow \phi(U) \in \mathcal{O}(X_2)$ be the order-filter generated by $\phi(U)$. We show that $\mathcal{O}(\phi)(\uparrow\phi(U)) = U$. By definition, $\mathcal{O}(\phi)(\uparrow\phi(U)) = \phi^{-1}(\uparrow\phi(U)) = \{x \mid \phi(x) \in \uparrow\phi(U)\} = \{x \mid \phi(x) \geq \phi(x_1) \text{ for some } x_1 \in U\}$. It is easy to see that for every $x \in U, \phi(x) \geq \phi(x_1)$ with $x_1 = x \in U$. Thus, $U \subseteq \mathcal{O}(\phi)(\uparrow\phi(U))$. Conversely, let $x \in \mathcal{O}(\phi)(\uparrow\phi(U))$. Then $\phi(x) \geq \phi(x_1)$ for some $x_1 \in U$. From the fact that ϕ is an order-embedding it follows that $x \geq x_1$, hence, since U is an order-filter, $x \in U$. Thus, $\mathcal{O}(\phi)(\uparrow\phi(U)) = U$.

(3) Assume that ϕ is surjective. We show that $\mathcal{O}(\phi)$ is injective. Let $U_1, U_2 \in \mathcal{O}(X_2)$ be such that $\mathcal{O}(\phi)(U_1) = \mathcal{O}(\phi)(U_2)$. Then $\phi^{-1}(U_1) = \phi^{-1}(U_2)$. Let $x \in U_1$. Since ϕ is onto, $x = \phi(y)$ for some $y \in X_1$. Thus, $x = \phi(y)$ for some $y \in \phi^{-1}(U_1) = \phi^{-1}(U_2)$. It follows therefore that $x = \phi(y) \in U_2$. The converse inclusion follows similarly. Thus $U_1 = U_2$. □

4 Canonical embedding algebras

In the introduction we mentioned that in the study of modal logics a link can be established between algebraic and Kripke-style models of these logics in the case when the class of their algebraic models are varieties closed under canonical embedding algebras.

Definition 6 *Let $\mathcal{V}$ be a variety of bounded distributive lattices with operators in the classes considered above. We say that the variety $\mathcal{V}$ is closed under embedding algebras if it has the property*

(CE) *for every algebra $A \in \mathcal{V}$, $\mathcal{O}(D(A)) \in \mathcal{V}$.*

In this section we present some of the cases when a variety is closed under canonical embedding algebras. We first point out that, due to a result of Gehrke and Jónsson, every variety of distributive lattices with additive operators is closed under canonical embedding algebras. Then, we show that finitely-generated varieties of distributive lattices with more general classes of operators (which need not be order-preserving) are closed under canonical embedding algebras.

4.1 Distributive lattices with additive operators

We briefly present here the results contained in [GJ94] for distributive lattices with additive operators.

For every bounded distributive lattice A with additive operators let $D(A)$ be its Priestley dual. The *canonical extension* of A is defined in [GJ94] as being the extension A^σ of A (unique up to equivalence) such that the natural isomorphism from A to $E(D(A))$ can be extended to an isomorphism from A^σ onto the lattice of all isotone maps from $D(A)$ into the two-element ordered set $\{0, 1\}$, with $0 \leq 1$. It follows therefore that A^σ is isomorphic to $\mathcal{O}(D(A))$. The following theorem is proved in [GJ94].

Theorem 11 ([GJ94]) *For every distributive algebra A with additive operators, A and its canonical extension A^σ satisfy the same identities.*

The case of maps which are anti-isotone, or isotone in some of their arguments and anti-isotone in others is not considered in [GJ94]. In what follows we also take into account antimorphisms, join-hemiantimorphisms and meet-hemiantimorphisms as operators.

4.2 Distributive lattices with non-additive operators

In this section we will provide a partial answer to the question whether, given a variety $\mathcal{V}$ of bounded distributive lattices with operators (not necessarily additive), $\mathcal{V}$ is closed under canonical embedding algebras (i.e. for every $A \in \mathcal{V}$, $\mathcal{O}(D(A)) \in \mathcal{V}$).

From Lemma 8 it follows that for every algebra A in $\mathsf{DLO}_{\mathbf{\Sigma}}$, $\mathcal{O}(D(A))$ is again in $\mathsf{DLO}_{\mathbf{\Sigma}}$. In what follows we will show that if $A \in HSP(A_1, \dots, A_n)$, where A_i is finite for every $i = 1, \dots, n$, then $\mathcal{O}(D(A)) \in HSP(A_1, \dots, A_n)$. The proof will proceed as follows:

1. We show that if for every $C \in ISP(A_1, \dots, A_n)$, $\mathcal{O}(D(C)) \in ISP(A_1, \dots, A_n)$, then $HSP(A_1, \dots, A_n)$ is closed under canonical embedding algebras.
2. We show that, in order to prove that for every $C \in ISP(A_1, \dots, A_n)$, $\mathcal{O}(D(C)) \in ISP(A_1, \dots, A_n)$, it is sufficient to show that if $X = \sum_{j \in J} D(B_j)$ for a family $\{B_j \mid j \in J\}$ such that for every $j \in J$, $B_j \in \{A_1, \dots, A_n\}$ then:
   (a) there is an ultrapower X^I/U of X having a bounded morphism which is *onto*,
   $$X^I/U \stackrel{\phi}{\to} D(\mathcal{O}(X)),$$
   (b) if Y is an ultrapower of X then $\mathcal{O}(Y) \in ISP(A_1, \dots, A_n)$.

   (The sums and ultrapowers are defined in Definition 7 and resp. Definition 8.)
3. We show that (a) and (b) hold.

The next sections will be dedicated to proving these facts.

Our goal is to prove that if $A_1, \ldots, A_n$ are finite distributive lattices with operators and $A \in HSP(A_1, \ldots, A_n)$ then $\mathcal{O}(D(A)) \in HSP(A_1, \ldots, A_n)$. In what follows, unless explicitly specified otherwise, $A_1, \ldots, A_n$ denote finite distributive lattices with operators.

Let $A \in HSP(A_1, \ldots, A_n)$. Then there exists $C \in ISP(A_1, \ldots, A_n)$ (i.e. there exists an injective homomorphism $i : C \to \prod_{j \in J} B_j$ where for every $j \in J, B_j \in \{A_1, \ldots, A_n\}$), and a surjective morphism $f : C \to A$.

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[d, "f"'] & \prod_{j\in J} B_j \\ A & \end{tikzcd}$$

If we apply the functor D to the diagram above, by Lemma 3 the following holds:

$$\begin{tikzcd} D(C) & D(\prod_{j\in J} B_j) \arrow[l, "D(i)"'] \\ D(A) \arrow[u, "D(f)"] & \end{tikzcd}$$

where $D(i)$ is a surjective bounded morphism and $D(f)$ a bounded morphism which is an order-embedding. By Lemma 10 we have:

$$\begin{tikzcd} \mathcal{O}(D(C)) \arrow[r, "\mathcal{O}(D(i))"] \arrow[d, "\mathcal{O}(D(f))"'] & \mathcal{O}(D(\prod_{j\in J} B_j)) \\ \mathcal{O}(D(A)) & \end{tikzcd}$$

where $\mathcal{O}(D(i))$ is an injective homomorphism and $\mathcal{O}(D(f))$ is a surjective homomorphism.

Consequence 12 *Assume that for every $C \in ISP(\{A_1, \ldots, A_n\})$, $\mathcal{O}(D(C)) \in ISP(\{A_1, \ldots, A_n\})$. Then for every $A \in HSP(A_1, \ldots, A_n)$, $\mathcal{O}(D(A)) \in HSP(A_1, \ldots, A_n)$.*

This proves 1. on page 15.

Our goal is now to show that for every $C \in ISP(A_1, \ldots, A_n)$, $\mathcal{O}(D(C)) \in ISP(\{A_1, \ldots, A_n\})$.

Let $C \in ISP(A_1, \ldots, A_n)$. Then there exists an injective homomorphism $i : C \to \prod_{j \in J} B_j$, where for every $j \in J$, $B_j \in \{A_1, \ldots, A_n\}$. Since $A_1, \ldots, A_n$ are all finite, we know that for every $j \in J$, $B_j \simeq \mathcal{O}(D(B_j))$. Thus, there exists an injective homomorphism $\bar{i} : C \to \prod_{j \in J} \mathcal{O}(D(B_j))$.

Definition 7 *Let $\{X_i \mid i \in I\}$ be a family of ordered Σ-relational structures, $X_i = (X_i, \leq_i, \{H_{X_i}\}_{H \in Lh}, \{K_{X_i}\}_{K \in La}, \{R_{X_i}\}_{R \in Jh}, \{Q_{X_i}\}_{Q \in Mh}, \{R'_{X_i}\}_{R' \in Ja}, \{Q'_{X_i}\}_{Q' \in Ma})$. The sum $\sum_{i \in I} X_i$ of this family is the relational structure $(X, \leq, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma})$, where:*

- $X = \coprod_{i \in I} X_i$ *(the disjoint union),*
- *for every $x, y \in X$ $x \leq y$ if and only if $x, y \in X_i$ and $x \leq_i y$ for some $i \in I$,*
- *for every $x \in X$ and every $H \in Lh \cup La$, $H_X(x) = H_{X_i}(x)$ where i is such that $x \in X_i$,*
- *for every $x_1, \ldots, x_n \in X^n$, and every $R \in Jh \cup Mh \cup Ja \cup Ma$, $R_X(x_1, \ldots, x_n)$ if and only if for some $i \in I$, $x_1, \ldots, x_n \in X_i$ and $R_{X_i}(x_1, \ldots, x_n)$.*

It is easy to see that if for every $i \in I$, H_{X_i} is order-preserving, then also H_X is order-preserving; if for every $i \in I$, K_{X_i} is order-reversing, then also K_X is order-reversing; if for every $i \in I$, R_{X_i} is increasing, then also R_X is increasing; if for every $i \in I$, R_{X_i} is decreasing, then also R_X is decreasing.

Also, for every $i \in I$, the *canonical embedding* $j_i : X_i \to \coprod_{i \in I} X_i$ defined by $j_i(x) = (x, i)$ is a bounded morphism (which is an order-embedding).

Lemma 13 *For every family $\{X_i \mid i \in I\}$ of relational spaces, there exists an isomorphism between $\mathcal{O}(\sum_{i \in I} X_i)$ and $\prod_{i \in I} \mathcal{O}(X_i)$.*

Proof: Let $j_i : X_i \to \sum_{i \in I} X_i$ be the canonical embedding and let $\phi_i = \mathcal{O}(j_i) : \mathcal{O}(\sum_{i \in I} X_i) \to \mathcal{O}(X_i)$ be defined by $\phi_i(U) = j_i^{-1}(U)$. By Lemma 10, $\mathcal{O}(j_i)$ is onto.

Let $k : \mathcal{O}(\sum_{i \in I} X_i) \to \prod_{i \in I} \mathcal{O}(X_i)$ be defined by $k(U) = (\phi_i(U))_{i \in I} = (j_i^{-1}(U))_{i \in I}$. Obviously, k is a homomorphism. It is onto because for every $(U_i)_{i \in I}$, where $U_i \in \mathcal{O}(X_i)$, $(U_i)_{i \in I} = k\left(\bigcup_{i \in I} U_i\right)$. Moreover, k is injective: assume that $k(U) = k(V)$ for some $U, V \in \mathcal{O}(\sum_{i \in I} X_i)$. It follows that for every $i \in I$ and every $x \in X_i$, $(x, i) \in U$ if and only if $x \in j_i^{-1}(U) = \phi_i(U) = \phi_i(V) = j_i^{-1}(V)$ if and only if $(x, i) \in V$. □

As a consequence, it immediately follows that, for the family $\{B_j \mid j \in J\}$ considered before, there exists an isomorphism
$$k : \mathcal{O}(\sum_{j \in J} D(B_j)) \simeq \prod_{j \in J} \mathcal{O}(D(B_j)).$$

Lemma 14 *Let $C \in ISP(A_1, \dots, A_n)$ (i.e. such that there exists an injective homomorphism $i : C \to \prod_{j \in J} B_j$, where $\{B_j \mid j \in J\} \subseteq \{A_1, \dots, A_n\}$). Then the algebra $\mathcal{O}(D(C))$ is isomorphic to a subalgebra of $\mathcal{O}(D(\mathcal{O}(\sum_{j \in J} D(B_j))))$.*

Proof: From Lemma 13 and the fact that there exists an injective homomorphism $\overline{i} : C \to \prod_{j \in J} \mathcal{O}(D(B_j))$ it follows that $k^{-1} \circ \overline{i} : C \xrightarrow{\overline{i}} \prod_{j \in J} \mathcal{O}(D(B_j)) \xrightarrow{k^{-1}} \mathcal{O}(\sum_{j \in J} D(B_j))$ is an injective homomorphism. Therefore, by Lemma 3 and Lemma 10, $\mathcal{O}(D(k^{-1} \circ \overline{i})) : \mathcal{O}(D(C)) \longrightarrow \mathcal{O}(D(\mathcal{O}(\sum_{j \in J} D(B_j))))$ is an injective homomorphism. □

The following considerations will help us to prove that $\mathcal{O}(D(\mathcal{O}(\sum_{j \in J} D(B_j)))) \in ISP(A_1, \dots, A_n)$. Let $X = \sum_{j \in J} D(B_j)$.

(a) We first show that there is an ultrapower X^I/U of X having a bounded morphism ϕ which is *onto*, $X^I/U \xrightarrow{\phi} D(\mathcal{O}(X))$. Therefore there exists an injective homomorphism
$$\mathcal{O}(D(\mathcal{O}(X))) \xrightarrow{\mathcal{O}(\phi)} \mathcal{O}(X^I/U).$$

(b) We then show that if Y is an ultrapower of $X = \sum_{j \in J} D(B_j)$, where $B_j \in \{A_1, \dots, A_n\}$ for every $j \in J$, then $\mathcal{O}(Y) \in ISP(A_1, \dots, A_n)$.

Lemma 15 *Assume that the conditions (a) and (b) above hold. Then for every $C \in ISP(A_1, \dots, A_n)$, $\mathcal{O}(D(C)) \in ISP(A_1, \dots, A_n)$.*

Proof: Follows from the the remarks above, from the fact that, by Lemma 14 and the assumption (a), there exists an injective homomorphism
$$\mathcal{O}(\phi) \circ k^{-1} \circ \overline{i} : \mathcal{O}(D(C)) \xrightarrow{k^{-1} \circ \overline{i}} \mathcal{O}(D(\mathcal{O}(\textstyle\sum_{j \in J} D(B_j)))) \xrightarrow{\mathcal{O}(\phi)} \mathcal{O}((\textstyle\sum_{j \in J} D(B_j))^I/U),$$
and from the fact that, by (b), $\mathcal{O}((\sum_{j \in J} D(B_j))^I/U) \in ISP(A_1, \dots, A_n)$. □

This proves 2. on page 15. The fact that (a) and (b) hold will be proved in Section 4.2.3. In order to prove these assertions we start by defining ultraproducts of ordered relational structures.

4.2.1 Ultraproducts of relational spaces

Let J be an index set and let $U \subseteq \mathcal{P}(J)$ be an ultrafilter (i.e. a maximal filter). Let $\{X_j \mid j \in J\}$ be a family of relational structures of the same type $\mathcal{R}$ (where for every j, $X_j = (X_j, \{R_j\}_{R\in\mathcal{R}})$). Let $\sim_U$ be the relation defined for every $x, y \in \prod_{j\in J} X_j$ by:

$$x \sim_U y \text{ if and only if } \{j \in J \mid x_j = y_j\} \in U.$$

It is easy to see that the relation $\sim_U$ is an equivalence relation on $\prod_{j\in J} X_j$. For every $x \in \prod_{j\in J} X_j$ let $[x]_U$ denote the equivalence class of x with respect to the equivalence relation $\sim_U$. For every relation R on $\prod_{j\in J} X_j$ let R_U be a relation on the quotient $(\prod_{j\in J} X_j)/\sim_U$ defined by

$$R_U([x^1]_U, \ldots, [x^n]_U) \text{ if and only if } \{j \in J \mid R_j(x^1_j, \ldots, x^n_j)\} \in U.$$

Definition 8 (Ultraproducts) *For every family of relational spaces $\{X_j \mid j \in J\}$ (where for every j, $X_j = (X_j, \{R_j\}_{R\in\mathcal{R}})$), and for every ultrafilter $U \subseteq \mathcal{P}(J)$, the quotient structure*

$$((\prod_{j\in J} X_j)/\sim_U, \{R_U\}_{R\in\mathcal{R}})$$

is called ultraproduct *of the family $\{X_j \mid j \in J\}$ with respect to U.*

In what follows we will denote $\prod_{j\in J} X_j/\sim_U$ by $\prod_{j\in J} X_j/U$. If $X_j = X$ for every $j \in J$, X^J/U is called an *ultrapower* of X.

Ordered Σ-relational structures are in particular relational structures: the functions corresponding to the operation symbols in Lh and La can in particular be seen as relations. Thus, ultraproducts of ordered Σ-relational structures can be defined in the usual way. The next lemma shows that an ultraproduct of ordered Σ-relational structures is again an ordered Σ-relational structure.

Lemma 16 *Let $\{X_j \mid j \in J\}$ be a family of ordered Σ-relational structures of the form $X_j = (X_j, \leq_j, \{H_{X_j}\}_{H\in Lh}, \{K_{X_j}\}_{K\in La}, \{R_{X_j}\}_{R\in Jh}, \{Q_{X_j}\}_{Q\in Mh}, \{R'_{X_j}\}_{R'\in Ja}, \{Q'_{X_j}\}_{Q'\in Ma})$. Let $U \subseteq \mathcal{P}(J)$ be an ultrafilter. Let the following relations be defined on the quotient $\prod_{j\in J} X_j/\sim_U$:*

- *$[x]_U \leq_U [y]_U$ if and only if $\{j \mid x_j \leq_j y_j\} \in U$,*
- *for every $H \in Lh \cup La$, $H_U : \prod_{j\in J} X_j \to \prod_{j\in J} X_j$ is defined by $H_U([x]_U) = [(H_{X_j}(x_j))_{j\in J}]_U$,*
- *for every $R \in Jh \cup Mh \cup Ja \cup Ma$, $R_U([x^1]_U, \ldots, [x^n]_U)$ if and only if $\{j \mid R_{X_j}(x^1_j, \ldots, x^n_j)\} \in U$.*

Then $\prod_{j\in J} X_j/\sim_U$ is again an ordered Σ-relational structure.

Proof: We first show that $\leq_U$ is a partial order. For every x, $[x]_U \leq_U [x]_U$, since $\{j \mid x_j \leq_j x_j\} = J \in U$; assume now that $[x]_U \leq_U [y]_U$ and $[y]_U \leq_U [z]_U$. It follows that $\{j \mid x_j \leq_j y_j\} \in U$ and $\{j \mid y_j \leq_j z_j\} \in U$. We know that $\{j \mid x_j \leq_j y_j\} \cap \{j \mid y_j \leq_j z_j\} \subseteq \{j \mid x_j \leq_j z_j\}$. It follows therefore that $\{j \mid x_j \leq_j z_j\} \in U$. Antisymmetry is proved analogously.

In order to show that for every $H \in Lh$, H_U is order-preserving, let $H \in Lh$ and let x, y be such that $[x]_U \leq_U [y]_U$. It follows that $\{j \mid x_j \leq_j y_j\} \in U$, and since $\{j \mid x_j \leq_j y_j\} \subseteq \{j \mid H_{X_j}(x_j) \leq_j H_{X_j}(y_j)\}$ it follows that $H_U([x]_U) \leq_U H_U([y]_U)$. Similarly it can be shown that for every $K \in La$, K_U is order-reversing.

We now show that for every $R \in Jh \cup Ja$, R_U is increasing. Let $R \in Jh$. Assume that $[x]_U \leq_U [y]_U$ and $R_U([x^1]_U, \ldots, [x^n]_U, [x]_U)$. It follows that $R_U([x^1]_U, \ldots, [x^n]_U, [y]_U)$, since $\{j \mid x_j \leq_j y_j\} \cap \{j \mid R_{X_j}(x^1_j, \ldots, x^n_j, x_j)\} \subseteq \{j \mid R_{X_j}(x^1_j, \ldots, x^n_j, y_j)\}$, hence $\{j \mid R_{X_j}(x^1_j, \ldots, x^n_j, y_j)\} \in U$. The fact that for every $Q \in Mh \cup Ma$, Q_U is decreasing can be proved in a similar way. □

Lemma 17 *Let $\{X_j \mid j \in J\}$ be a finite set of finite ordered Σ-relational structures (such that, as a set, $\{X_j \mid j \in J\} = \{Y_1, \ldots, Y_m\}$, where for every $i = 1, \ldots, m$, Y_i is finite). Let U be an ultrafilter over J. Then $\prod_{j\in J} X_j/U$ is isomorphic to one of the Y_i, namely to that Y_i such that $\{j \in J \mid X_j = Y_i\} \in U$.*

Proof: It follows from the fact that, since J is finite, the ultrafilter U is generated by some $k \in J$, and therefore, by classical properties of ultraproducts (cf. e.g. [BS71], p.124), $\prod_{j\in J} X_j/U \simeq X_k$. □

4.2.2 Elements of model theory

The proof of Theorem 18, inspired by the idea used by Goldblatt in [Gol89], requires some elements of model theory, namely the use of ω-saturated elementary extensions. In this section we briefly summarize the main definitions and results in model theory necessary for this. For further details we refer to [CK73].

A first-order language $\mathcal{L}$ consists of a set of a set $\mathcal{R}$ of *relation symbols*, a set $\mathcal{F}$ of *function symbols* and a set $\mathcal{C}$ of constants. (In what follows, following [CK73], we do not allow relation or function symbols with arity 0.) The *power* or *cardinal* of a language $\mathcal{L}$, denoted by $\|\mathcal{L}\|$, is defined by $\|\mathcal{L}\| = \omega \vee card(\mathcal{L})$.

A model for a first-order language $\mathcal{L}$ is a pair $\mathcal{A} = (A, \mathcal{I})$, where A is a non-empty set and $\mathcal{I}$ is an interpretation function, mapping the relation symbols (resp. function symbols, constant symbols) of A to relations (resp. functions, constants) with the same arity on A. The *cardinal* or *power* of a model $\mathcal{A} = (A, \mathcal{I})$ is the cardinal of its domain A. If we start with a model $\mathcal{A}$ for the language $\mathcal{L}$ we can expand it to a model for a language $\mathcal{L}' = \mathcal{L} \cup S$, by giving appropriate interpretations for the symbols in S. Given a model $\mathcal{A}$ and a subset $Y \subseteq A$, the expanded model obtained by adding constants for all elements $a \in Y$ will be denoted by $\mathcal{A}_Y$, and its language by $\mathcal{L}_Y$.

If $\mathcal{L}$ is a first-order language and X is a set (members of X are called *variables*) the *terms* of type $\mathcal{L}$ over X are the terms in X over the signature $\mathcal{F}$, the atomic formulae are expressions of the form $t_1 \simeq t_2$, where t_1, t_2 are terms of type $\mathcal{L}$ over X, and $r(t_1, \dots, t_n)$, where $r \in \mathcal{R}$ of arity n, and $t_1, \dots, t_n$ are terms of type $\mathcal{L}$ over X. The set of *(first-order) formulae* of type $\mathcal{L}$ over X is inductively defined starting from the atomic formulae by using the logical operators $\vee, \wedge, \neg, \Rightarrow$ and the quantifiers $\forall$ and $\exists$.

A first-order theory T of a language $\mathcal{L}$ is a collection of sentences over $\mathcal{L}$. The theory $Th(\mathcal{A})$ of a model $\mathcal{A}$ is the set of all sentences which hold in $\mathcal{A}$. A formula $\phi(x_1, \dots, x_n)$ is said to be *consistent* with respect to a theory T if there exists a model $\mathcal{A}$ of T which realizes ϕ. Given a cardinal α, a model $\mathcal{A}$ is said to be *α-saturated* if and only if for every subset $Y \subseteq A$ with fewer than α elements, the expanded model $\mathcal{A}_Y$ realizes every set of formulae $\Gamma(x)$ of $\mathcal{L}_Y$ which is consistent with the theory $Th(\mathcal{A}_Y)$ of $\mathcal{A}_Y$. The following results are known (cf. e.g. [CK73]):

(1) For any set I of power ω there exists a countably incomplete ultrafilter (i.e. an ultrafilter which is not closed under countable intersections) U over I (by Proposition 4.3.4 and Proposition 4.3.5 in [CK73], p.249).

(2) Let I be a set of power ω. Let $\mathcal{L}$ be a language such that $card(\mathcal{L}) \leq \omega$. Let $\mathcal{A}$ be a model for $\mathcal{L}$, and let U be a countably incomplete ultrafilter of I. Then the ultraproduct $\mathcal{A}^I/U$ is ω^+-saturated (and, hence, ω-saturated) (by Theorem 6.1.1, [CK73], p.384). More precisely the following holds:

> For every set $\Sigma(x)$ of formulae in $\mathcal{L}$, if each finite subset of $\Sigma(x)$ is satisfiable in $\mathcal{A}^I/U$, then $\Sigma(x)$ is satisfiable in $\mathcal{A}^I/U$.

(3) For every model $\mathcal{A}$, non-empty index set I, and ultrafilter U over I, the natural embedding $\iota : \mathcal{A} \to \mathcal{A}^I/U$ that associates with every element $a \in A$ the equivalence class of $(a)_{i \in I}$ with respect to $\sim_U$ is an elementary embedding (i.e. for every formula $\phi(x_1, \dots, x_n)$ of $\mathcal{L}$ and any $a_1, \dots, a_n \in A$, $a_1, \dots, a_n$ satisfy ϕ in $\mathcal{A}$ if and only if $\iota(a_1), \dots, \iota(a_n)$ satisfies ϕ in $\mathcal{A}^I/U$).

It follows therefore that if $\mathcal{L}$ is a language such that $card(\mathcal{L}) \leq \omega$, $\mathcal{A}$ a model for $\mathcal{L}$, and I a set of power ω, then there exists a countably incomplete ultrafilter U over I such that (i) $\mathcal{A}^I/U$ is an elementary extension of $\mathcal{A}$; and (ii) for every set $\Sigma(x)$ of formulae in $\mathcal{L}$, if each finite subset of $\Sigma(x)$ is satisfiable in $\mathcal{A}^I/U$, then $\Sigma(x)$ is satisfiable in $\mathcal{A}^I/U$. In conclusion, if $\mathcal{L}$ is a language such that $card(\mathcal{L}) \leq \omega$, then every model $\mathcal{A}$ of $\mathcal{L}$ has a ω-saturated elementary extension $\mathcal{A}_1$, which is an ultrapower of $\mathcal{A}$. This result is used in the proof of Theorem 18.

4.2.3 Closure under canonical embedding algebras

The following theorem extends the results of Goldblatt [Gol89]: we show that the arguments used in the proof of Theorem 3.6.1 in [Gol89] (for the case of relational structures corresponding to modal logic with one n-ary join-hemimorphic modality) can be extended to the more general relational structures considered here.

Theorem 18 *Let X be an ordered Σ-relational structure. Then there exists an ultrapower X^J/U of X having a bounded morphism onto the canonical structure $D(\mathcal{O}(X))$ of X, $\phi : X^J/U \to D(\mathcal{O}(X))$.*

Proof: The main idea of the proof is to associate with every ordered Σ-relational structure X an extension $X_{\mathcal{L}}$ by unary operators associated to elements of $\mathcal{O}(X)$, and, further, an ω-saturated elementary extension X_1 of $X_{\mathcal{L}}$, which is actually an ultrapower of $X_{\mathcal{L}}$. We show that a map $\phi : X_1 \to D(\mathcal{O}(X))$ can be defined in a canonical way. The ω-saturation of X_1 and the fact that the extension is elementary are then used in order to prove that ϕ is a bounded morphism of relational spaces which is onto.

Let $X = (X, \leq, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma})$ be an ordered Σ-relational structure. Let $\mathcal{L}$ be the first-order language with a binary relation symbol $\leq$, a binary relation symbol $=$, a unary function symbol $\overline{H}$ for every $H \in Lh \cup La$, an $(n+1)$-ary relation symbol $\overline{R}$ for every $R \in Jh \cup Mh \cup Ja \cup Ma$ with arity n, and a unary relation symbol $\overline{Y}$ for every order-filter $Y \in \mathcal{O}(X)$. Let $X_1 = (X_1, \leq_1, =_1, \{H_{X_1}\}_{H \in Lh}, \{K_{X_1}\}_{K \in La}, \{R_{X_1}\}_{R \in Jh}, \{Q_{X_1}\}_{Q \in Mh}, \{R'_{X_1}\}_{R' \in Ja}, \{Q'_{X_1}\}_{Q' \in Ma}, \{Y^* \mid Y \in \mathcal{O}(X)\})$ be an ω-saturated elementary extension of $X_{\mathcal{L}} = (X, \leq, =, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_{X_1}\}_{R' \in Ja}, \{Q'_{X_1}\}_{Q' \in Ma}, \{Y \mid Y \in \mathcal{O}(X)\})$. Such an extension exists (by arguments of model theory [CK73]; see also [Gol89] or the remarks in Section 4.2.2). X_1 may be realized as an ultrapower of $X_{\mathcal{L}}$, such that $(X_1, \leq_1, \{H_{X_1}\}_{H \in Lh}, \{K_{X_1}\}_{K \in La}, \{R_{X_1}\}_{R \in Jh}, \{Q_{X_1}\}_{Q \in Mh}, \{R'_{X_1}\}_{R' \in Ja}, \{Q'_{X_1}\}_{Q' \in Ma})$ is an ultrapower of $X = (X, \leq, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_{X_1}\}_{R' \in Ja}, \{Q'_{X_1}\}_{Q' \in Ma})$. Note that the construction presented here differs from that in [Gol89] only in the fact that the additional unary symbols added to the language are indexed here by the elements of $\mathcal{O}(X)$ rather than $\mathcal{P}(X)$ as in the boolean case considered in [Gol89]. In what follows, the case of $R \in Jh$ is essentially given in [Gol89]; here we show that similar arguments can be used in order to extend the results to the cases when the signature also contains function symbols $H \in Lh$, $K \in La$ and relation symbols $Q \in Mh$, $R' \in Ja$ and $Q' \in Ma$.

For every $x \in X_1$ let $F_x = \{Y \in \mathcal{O}(X) \mid Y^*(x)\}$. The sentences

$$\begin{gathered} \forall v \overline{X}(v), \\ \neg \exists v \overline{\emptyset}(v), \\ \forall v, \overline{Y \cap Z}(v) \Leftrightarrow \overline{Y}(v) \wedge \overline{Z}(v), \\ \forall v, \overline{Y \cup Z}(v) \Rightarrow \overline{Y}(v) \vee \overline{Z}(v) \end{gathered}$$

are all true in $X_{\mathcal{L}}$ for every $Y, Z \in \mathcal{O}(X)$, thus they are true in the elementary extension X_1 of $X_{\mathcal{L}}$. Thus,

$$\begin{gathered} X \in F_x, \ \emptyset \notin F_x, \\ Y \cap Z \in F_x \text{ if and only if } Y \in F_x \text{ and } Z \in F_x, \\ Y \cup Z \in F_x \text{ implies } Y \in F_x \text{ or } Z \in F_x. \end{gathered}$$

It follows that F_x is a prime filter of $\mathcal{O}(X)$. Therefore we obtain a map $\phi : X_1 \to D(\mathcal{O}(X))$ defined for every $x \in X_1$ by $\phi(x) = F_x$. We show now that ϕ is a bounded morphism of ordered Σ-relational structures from X_1 onto $D(\mathcal{O}(X))$. (In what follows we will use the same notational conventions for the operations in $\mathcal{O}(X)$ as used in Section 3.1.)

Claim 1. ϕ is onto.
Proof: Let $G \in D(\mathcal{O}(X))$. Let $\Gamma_G = \{\overline{Y}(v) \mid Y \in G\} \cup \{\neg \overline{Y}(v) \mid Y \notin G\}$.

We first show that Γ_G is finitely satisfiable in $X_{\mathcal{L}}$. Indeed, let I and J be two finite subsets of $\mathcal{O}(X)$ with $I \subseteq G$ and $J \subseteq \mathcal{O}(X) \backslash G$. Since G is a prime filter, $\bigcap I \not\subseteq \bigcup J$. (In order to show this, assume that $\bigcap I \subseteq \bigcup J$. Then, since $I \subseteq G$, it follows that $\bigcap I \in G$, hence $\bigcup J \in G$. It follows that $j \in G$ for some $j \in J$, which contradicts the fact that $J \subseteq \mathcal{O}(X) \backslash G$.) Thus, there exists $y \in X$ such that $y \in \bigcap I$ and $y \notin \bigcup J$, i.e. Γ_G is finitely satisfiable in $X_{\mathcal{L}}$.

This shows that Γ_G is finitely satisfiable in X_1. By ω-saturation it follows that there exists $x \in X_1$ that satisfies Γ_G in X_1, i.e. such that $F_x = G$.

Claim 2. ϕ is a morphism.
Proof: We show that ϕ preserves all the functions and relations.

(1) $\phi(H_{X_1}(x)) = D(h_H)(\phi(x))$ for $H \in Lh$.
Proof: We show that $Y \in F_{H_{X_1}(x)}$ if and only if $Z = h_H(Y) = H_{X_1}^{-1}(Y) \in F_x$. If $Y \in F_{H_{X_1}(x)}$ and $Z = h_H(Y)$, then the following hold in $X_{\mathcal{L}}$, and are therefore true also in X_1:

$$\forall v_1 \forall v_2 (\overline{Y}(v_1) \wedge \overline{H}(v_2) = v_1 \Rightarrow \overline{Z}(v_2)),$$

$$\forall v_1 \forall v_2 (\overline{Z}(v_2) \wedge \overline{H}(v_2) = v_1 \Rightarrow \overline{Y}(v_1)).$$

Taking in particular $v_1 = H_{X_1}(x)$ and $v_2 = x$, it follows that $Y \in F_{H_{X_1}(x)}$ if and only if $h_H(Y) \in F_x$, i.e. if and only if $Y \in D(h_H)(F_x)$.

(2) $\phi(K_{X_1}(x)) = D(k_K)(\phi(x))$ for $K \in La$.
Proof: We show that $Y \in F_{K_{X_1}(x)}$ if and only if $Z = k_K(Y) \notin F_x$. (Then, $Y \in F_{K_{X_1}(x)}$ if and only if $Y \in D(k_K)(F_x)$.) If $Y \in F_{K_{X_1}(x)}$ and $Z = k_K(Y)$, then the following hold in $X_{\mathcal{L}}$, and are therefore true also in X_1:

$$\forall v_1 \forall v_2 (\overline{Y}(v_1) \wedge \overline{K}(v_2) = v_1 \Rightarrow \neg \overline{Z}(v_2)),$$

$$\forall v_1 \forall v_2 (\neg \overline{Z}(v_2) \wedge \overline{K}(v_2) = v_1 \Rightarrow \overline{Y}(v_1)).$$

(3) Let R be a relation in Jh. We show that if $R_{X_1}(x_1, \dots, x_n, x_{n+1})$ then $D(f_R)(F_{x_1}, \dots, F_{x_{n+1}})$.
Proof: Assume that $R_{X_1}(x_1, \dots, x_n, x_{n+1})$ for $x_1, \dots, x_{n+1} \in X_1$. We want to show that for every $Y_i \in F_{x_i}$, $i = 1, \dots, n$, $f_R(Y_1, \dots, Y_n) \in F_{x_{n+1}}$.
Let $Y_i \in F_{x_i}$ for every $i \in \{1, \dots, n\}$. If $Y = f_R(Y_1, \dots, Y_n)$ then

$$\forall v_1, \dots, \forall v_{n+1} (\overline{Y}_1(v_1) \wedge \dots \wedge \overline{Y}_n(v_n) \wedge \overline{R}(v_1, \dots, v_{n+1}) \Rightarrow \overline{Y}(v_{n+1}))$$

is true in $X_{\mathcal{L}}$, hence it is also true in X_1. Therefore, since $R_{X_1}(x_1, \dots, x_{n+1})$, and $Y_i^*(x_i)$ for all $i = 1, \dots, n$, it follows that $Y^*(x_{n+1})$, where $Y = f_R(Y_1, \dots, Y_n)$. Thus, $f_R(Y_1, \dots, Y_n) \in F_{x_{n+1}}$. This proves that $D(f_R)(F_{x_1}, \dots, F_{x_{n+1}})$.

(4) Let Q be a relation in Mh. We show that if $Q_{X_1}(x_1, \dots, x_n, x_{n+1})$ then $D(g_Q)(F_{x_1}, \dots, F_{x_{n+1}})$.
Proof: Assume that $Q_{X_1}(x_1, \dots, x_n, x_{n+1})$ for $x_1, \dots, x_{n+1} \in X_1$. We want to show that for every $Y_1, \dots, Y_n$, if $g_Q(Y_1, \dots, Y_n) \in F_{x_{n+1}}$ then there exists $i \in \{1, \dots, n\}$ with $Y_i \in F_{x_i}$.
Let $Y_1, \dots, Y_n$ be arbitrary, and assume that $Y = g_Q(Y_1, \dots, Y_n) \in F_{x_{n+1}}$. We know that

$$\forall v_1, \dots, \forall v_{n+1} (\overline{Q}(v_1, \dots, v_{n+1}) \wedge \overline{Y}(v_{n+1}) \Rightarrow (\overline{Y}_1(v_1) \vee \dots \vee \overline{Y}_n(v_n)))$$

is true in $X_{\mathcal{L}}$, hence it is also true in X_1. Therefore, since $Q_{X_1}(x_1, \dots, x_n, x_{n+1})$ and $Y^*(x_{n+1})$ (where $Y = g_Q(Y_1, \dots, Y_n)$) it follows that there exists $i \in \{1, \dots, n\}$ such that $Y_i^*(x_i)$, i.e. $Y_i \in F_{x_i}$. This proves that $D(g_Q)(F_{x_1}, \dots, F_{x_{n+1}})$.

(5) Let R' be a relation in Ja. We show that if $R'_{X_1}(x_1, \dots, x_n, x_{n+1})$ then $D(f'_{R'})(F_{x_1}, \dots, F_{x_{n+1}})$.
Proof: The proof proceeds as in (3) taking into account the fact that if $Y = f'_{R'}(Y_1, \dots, Y_n)$, then the formula

$$\forall v_1, \dots, \forall v_{n+1} (\neg \overline{Y}_1(v_1) \wedge \dots \wedge \neg \overline{Y}_n(v_n) \wedge \overline{R}(v_1, \dots, v_{n+1}) \Rightarrow \overline{Y}(v_{n+1})).$$

holds in $X_{\mathcal{L}}$, hence also in X_1.

(6) Let Q' be a relation in Ja. We show that if $Q'_{X_1}(x_1, \dots, x_n, x_{n+1})$ then $D(g'_{Q'})(F_{x_1}, \dots, F_{x_{n+1}})$.
Proof: The proof proceeds as in (4), taking into account the fact that if $Y = g'_{Q'}(Y_1, \dots, Y_n)$, then the formula

$$\forall v_1, \dots, \forall v_{n+1} (\overline{Q}'(v_1, \dots, v_{n+1}) \wedge \overline{Y}(v_{n+1}) \Rightarrow (\neg \overline{Y}_1(v_1) \vee \dots \vee \neg \overline{Y}_n(v_n)))$$

holds in $X_{\mathcal{L}}$, hence also in X_1.

Claim 3. ϕ is a bounded homomorphism.
Proof: We show that ϕ satisfies the boundedness condition:

(1) Let $R \in Jh$. Assume that $D(f_R)(G_1, \dots, G_n, \phi(z))$. We show that there exist $x_1, \dots, x_n \in X_1$ such that $R_{X_1}(x_1, \dots, x_n, z)$ and $\phi(x_i) \geq G_i$ for every $i \in \{1, \dots, n\}$.
Proof: Let $\Gamma = \{\overline{R}(v_1, \dots, v_n, z)\} \cup \{\overline{Y}(v_1) \mid Y \in G_1\} \cup \dots \cup \{\overline{Y}(v_n) \mid Y \in G_n\}$.
If $(x_1, \dots, x_n)$ satisfies Γ in X_1 (when v_i is interpreted as x_i) then $R_{X_1}(x_1, \dots, x_n, z)$ and for every $Y \in G_i$, $Y^*(x_i)$ for every $i = 1, \dots, n$, i.e. $R_{X_1}(x_1, \dots, x_n, z)$ and $G_i \subseteq F_{x_i} = \phi(x_i)$ for every $i = 1, \dots, n$.

We show that Γ is satisfiable in X_1. Using the ω-saturation of X_1 and the closure of each G_i under finite intersection it is enough to show that if $Y_i \in G_i$ then

$$\Gamma_0 = \{\overline{R}(v_1, \dots, v_n, z), \overline{Y}_1(v_1), \dots, \overline{Y}_n(v_n)\}$$

is satisfiable in X_1.
Since $D(f_R)(G_1, \dots, G_n, F_z)$ it follows that for every $Y_i \in G_i, i = 1, \dots, n$, $Y = f_R(Y_1, \dots, Y_n) \in F_z$. But for $Y = f_R(Y_1, \dots, Y_n)$ the sentence

$$\forall v(\overline{Y}(v) \Rightarrow \exists v_1, \dots, \exists v_n(\overline{Y}_1(v_1) \wedge \dots \wedge \overline{Y}_n(v_n) \wedge \overline{R}(v_1, \dots, v_n, v))).$$

is true in $X_{\mathcal{L}}$, hence it is true in X_1. Hence, there exist $x_1, \dots, x_n \in X_1$ such that $Y_i^*(x_i)$ for every $i = 1, \dots, n$ and $R_{X_1}(x_1, \dots, x_n, z)$. But then $x_1, \dots, x_n$ satisfies Γ_0 in X_1. From ω-saturation of X_1 it follows that there exist $x_1, \dots, x_n$ that satisfy Γ.

(2) Let $Q \in Mh$. Assume that $D(g_Q)(G_1, \dots, G_n, \phi(z))$. We show that there exist $x_1, \dots, x_n \in X_1$ such that $Q_{X_1}(x_1, \dots, x_n, z)$ and $\phi(x_i) \leq G_i$ for every $i \in \{1, \dots, n\}$.
Proof: Let $\Gamma = \{\overline{Q}(v_1, \dots, v_n, z)\} \cup \{\neg\overline{Y}(v_1) \mid Y \notin G_1\} \cup \dots \cup \{\neg\overline{Y}(v_n) \mid Y \notin G_n\}$.
If $(x_1, \dots, x_n)$ satisfies Γ in X_1 (when v_i is interpreted as x_i) then $Q_{X_1}(x_1, \dots, x_n, z)$ and for every $Y \notin G_i$, $\neg Y^*(x_i)$ holds for every $i = 1, \dots, n$, i.e. $Q_{X_1}(x_1, \dots, x_n, z)$ and $\phi(x_i) = F_{x_i} \subseteq G_i$ for every $i = 1, \dots, n$.
We show that Γ is satisfiable in X_1. Using the ω-saturation of X_1 and the closure of $\mathcal{O}(X)\backslash G_i$ under finite union for each G_i, it is enough to show that if $Y_i \notin G_i$, $i = 1, \dots, n$, then

$$\Gamma_0 = \{\overline{Q}(v_1, \dots, v_n, z), \neg\overline{Y}_1(v_1), \dots, \neg\overline{Y}_n(v_n)\}$$

is satisfiable in X_1.
Since $D(g_Q)(G_1, \dots, G_n, F_z)$ it follows that for every $Y_1, \dots, Y_n$, if $Y_i \notin G_i, i = 1, \dots, n$, $Y = g_Q(Y_1, \dots, Y_n) \notin F_z$. But for $Y = g_Q(Y_1, \dots, Y_n)$ the sentence

$$\forall v(\neg\overline{Y}(v) \Rightarrow \exists v_1, \dots, \exists v_n(\neg\overline{Y}_1(v_1) \wedge \dots \wedge \neg\overline{Y}_n(v_n) \wedge \overline{Q}(v_1, \dots, v_n, v)))$$

is true in $X_{\mathcal{L}}$, hence it is true in X_1. Hence, there exist $x_1, \dots, x_n \in X_1$ such that $\neg Y_i^*(x_i)$ for every $i = 1, \dots, n$ and $Q_{X_1}(x_1, \dots, x_n, z)$. But then $x_1, \dots, x_n$ satisfies Γ_0 in X_1, hence Γ is satisfiable. $\square$

(3) Let $R' \in Ja$. Assume that $D(f'_{R'})(G_1, \dots, G_n, \phi(z))$. We prove that there exist $x_1, \dots, x_n$ such that $R'_{X_1}(x_1, \dots, x_n, z)$ and $\phi(x_i) \leq G_i$ for every $i \in \{1, \dots, n\}$.
Proof: Let $\Gamma = \{\overline{R}'(v_1, \dots, v_n, z)\} \cup \{\neg\overline{Y}(v_1) \mid Y \notin G_1\} \cup \dots \cup \{\neg\overline{Y}(v_n) \mid Y \notin G_n\}$.
We show that Γ is satisfiable in X_1. Using the ω-saturation of X_1 and the closure of $\mathcal{O}(X)\backslash G_i$ under finite union for each G_i, it is enough to show that if $Y_i \notin G_i$, $i = 1, \dots, n$, then

$$\Gamma_0 = \{\overline{R}'(v_1, \dots, v_n, z), \neg\overline{Y}_1(v_1), \dots, \neg\overline{Y}_n(v_n)\}$$

is satisfiable in X_1.
Since $D(f'_{R'})(G_1, \dots, G_n, F_z)$ it follows that for every $Y_1, \dots, Y_n$, if $Y_i \notin G_i, i = 1, \dots, n$, then $Y = f'_{R'}(Y_1, \dots, Y_n) \in F_z$ (i.e. $Y^*(z)$ holds in X_1). Let $Y_1, \dots, Y_n$ be such that for every $i = 1, \dots, n$, $Y_i \notin G_i$. Then $Y = f'_{R'}(Y_1, \dots, Y_n) \in F_z$. But for $Y = f'_{R'}(Y_1, \dots, Y_n)$ the sentence

$$\forall v(\overline{Y}(v) \Rightarrow \exists v_1, \dots, \exists v_n(\neg\overline{Y}_1(x_1) \wedge \dots \wedge \neg\overline{Y}_n(x_n) \wedge \overline{R}'(v_1, \dots, v_n, v)))$$

is true in $X_{\mathcal{L}}$, hence also in X_1. Hence, there exist $x_1, \dots, x_n \in X_1$ such that $\neg Y_i^*(x_i)$ for every $i = 1, \dots, n$ and $R'_{X_1}(x_1, \dots, x_n, z)$. Thus, there exist $x_1, \dots, x_n$ that satisfy Γ_0, hence Γ is satisfiable. Moreover, for every $Y_i \notin G_i$, we have $Y_i \notin F_{x_i}$. This shows that $\phi(x_i) = F_{x_i} \subseteq G_i$.

(4) Let $Q' \in Ma$. Assume that $D(g'_{Q'})(G_1, \dots, G_n, \phi(z))$. We show that there exist $x_1, \dots, x_n \in X_1$ such that $Q'_{X_1}(x_1, \dots, x_n, z)$ and $\phi(x_i) \geq G_i$ for every $i \in \{1, \dots, n\}$.
Proof: The proof proceeds as the previous ones; we choose the set Γ of formulae as in (1), and then note that in $X_{\mathcal{L}}$ (hence also in X_1) the following sentence holds:

$$\forall v(\neg\overline{Y}(v) \Rightarrow \exists v_1, \dots, \exists v_n(\overline{Y}_1(v_1) \wedge \dots \wedge \overline{Y}_n(v_n) \wedge \overline{Q}'(v_1, \dots, v_n, v))).$$

Theorem 19 *Let K be a class of ordered relational structures closed under ultraproducts. Let $\{X_i \mid i \in I\}$ be a family of elements in K, and let Y be an ultrapower of $\sum_{i\in I} X_i$. Then $\mathcal{O}(Y) \in ISP(K^+)$, where $K^+ = \{\mathcal{O}(X) \mid X \in K\}$.*

Proof: The proof extends the proof of Lemma 3.5 in [Gol88] to the richer structure of the relational spaces considered here. Let $X = \sum_{i\in I} X_i$ and let $Y = X^J/U$ for some index set J and some ultrafilter $U \subseteq \mathcal{P}(J)$. For every $t \in Y$ let $f_t \in X^J$ be such that $t = [f_t]_U$. For every $j \in J$ there exists $i_j \in I$ with $f_t(j) \in X_{i_j}$.

Construct $Y_t = (\prod_{j\in J} X_{i_j})/U$. We know that for every $j \in J$, $X_{i_j} \in K$, hence, since K is closed under ultraproducts, $Y_t \in K$ for every $t \in Y$. Thus, $\mathcal{O}(Y_t) \in K^+$ for every $t \in Y$.

Note that $\prod_{j\in J} X_{i_j} \subseteq X^J$. We now closely analyze Y_t. An element of Y_t is of the form $g^U = \{h \in \prod_{j\in J} X_{i_j} \mid g \sim_U h\} \subseteq [g]_U$, where the last equivalence class is in Y (the two equivalence classes in Y_t and Y are not equal in general).

Let $\theta_t : (\prod_{j\in J} X_{i_j})/U \to X^J/U$ be defined by $\theta_t(g^U) = [g]_U$. It is easy to see that θ_t is well-defined and that $\theta_t(f_t^U) = [f_t]_U = t$. Note that for every $g, h \in \prod_{j\in J} X_{i_j}$ $g^U = h^U$ if and only if $\{j \mid g(j) = h(j) \in Y_{i_j}\} \in U$ if and only if $g \sim_U h$ if and only if $[g]_U = [h]_U$. (In what follows we will denote the functions and relations on $(\prod_{j\in J} X_{i_j})/U$ by indexing them with $(\prod_{j\in J} X_{i_j})/U$, and those on X^J/U by indexing them with U.)

Claim 1. θ_t is an order-embedding.
Proof: For every $g, h \in \prod_{j\in J} X_{i_j} \subseteq X^J$, $g^U \leq h^U$ if and only if $\{j \in J \mid g(j) \leq h(j)\} \in U$ if and only if $[g]_U \leq [h]_U$.

Claim 2. θ_t is a morphism.
Proof: We show that θ_t preserves the operations and relations.

(1) For every $H \in Lh \cup La$, and every $g \in \prod_{j\in J} X_{i_j}$, $\theta_t(H_{\prod_{j\in J} X_{i_j}/U}(g^U)) = \theta_t((H_{\prod_{j\in J} X_{i_j}}(g))^U) = [H_{\prod_{j\in J} X_{i_j}}(g)]_U = [H_{X^J}(g)]_U = H_U([g]_U) = H_U(\theta_t(g^U))$.

(2) Let $R \in Jh \cup Mh \cup Ja \cup Ma$ and $g_1, \ldots, g_n, g_{n+1} \in \prod_{j\in J} X_{i_j}$. If $R_{\prod X_{i_j}/U}(g_1^U, \ldots, g_{n+1}^U)$ then $\{j \mid R_{X_{i_j}}(g_1(j), \ldots, g_{n+1}(j))\} \in U$, and since $g_k(j) \in X_{i_j}$ for every $j \in J$, by the definition of R_X it follows that $\{j \mid R_X(g_1(j), \ldots, g_{n+1}(j))\} \in U$, hence $R_U([g_1]_U, \ldots, [g_{n+1}]_U)$.

Claim 3. θ_t is a bounded morphism.
Proof: Assume that in X^J/U we have $R_U([f_1]_U, \ldots, [f_n]_U, \theta_t(g^U))$ with $g \in \prod_{j\in J} X_{i_j}$ and $f_k \in X^J$ for every k. Then $F = \{j \mid R_X(f_1(j), \ldots, f_n(j), g(j))\} \in U$, where $g(j) \in X_{i_j}$. Since U is an ultrafilter, $F \neq \emptyset$. Let j be an arbitrary element of F. By the definition of R_X, since $g(j) \in X_{i_j}$, it follows that for every $k = 1, \ldots, n$, $f_k(j) \in X_{i_j}$ and $R_{X_{i_j}}(f_1(j), \ldots, f_n(j), g(j))$. For every $k \in \{1, \ldots, n\}$ let h_k be an arbitrary member of $\prod_{j\in J} X_{i_j}$ which agrees with f_k on F. Then $h_k \sim_U f_k$, hence $\theta_t(h_k^U) = [f_k]_U$; and $F \subseteq \{j \mid R_{X_{i_j}}(h_1(j), \ldots, h_n(j), g(j))\}$. Since U is a filter it follows that $\{j \mid R_{X_{i_j}}(h_1(j), \ldots, h_n(j), g(j))\} \in U$. Thus, there exist $h_1^U, \ldots, h_n^U \in (\prod_{j\in J} X_{i_j})/U$ such that $R_{(\prod X_{i_j})/U}(h_1^U, \ldots, h_n^U, g^U)$ and $\theta_t(h_k^U) = [f_k]_U$.

This shows that for every $t \in Y$, θ_t is a bounded morphism from Y_t to Y.

Let $\theta : \sum_{t\in Y} Y_t \to Y$, such that for every $t \in Y$, $\theta_{|Y_t} = \theta_t$. It can be seen that θ is well-defined and it is a bounded morphism. Also, θ is surjective because for every $t \in Y$, $t = \theta_t(f_t^U) = \theta(f_t^U)$. It follows that there exists an injective homomorphism

$$\mathcal{O}(Y) \to \mathcal{O}(\sum_{t\in Y} Y_t) \simeq \prod_{t\in Y} \mathcal{O}(Y_t) \in IP(K^+).$$

Thus, $\mathcal{O}(Y) \in ISP(K^+)$. □

Corollary 20 *Let $\{D(A_j) \mid j \in J\}$ be a finite set, where for every $j \in J$, A_j is finite, such that $\{A_j \mid j \in J\} = \{B_1, \ldots, B_m\}$. Let Y be an ultrapower of $\sum_{j\in J} D(A_j)$. Then $\mathcal{O}(Y) \in ISP(B_1, \ldots, B_m)$.*

Proof: From Lemma 17 we know that $K = \{D(A_j) \mid j \in J\}$ is closed under ultraproducts. Thus, the conclusion follows from Theorem 19. □

Corollary 21 *$A \in HSP(A_1, \ldots, A_n)$, A_i finite for every $i = 1, \ldots, n$. Then $\mathcal{O}(D(A)) \in HSP(A_1, \ldots, A_n)$.*

Proof: The result follows immediately from Lemma 15, Theorem 18, Theorem 19, and Corollary 20. □

An even more general result can be proved, that extends Theorem 3.6 in [Gol88]. The proof proceeds as the one given by Goldblatt.

Corollary 22 *Let K be a class of ordered Σ-relational structures which is closed under ultraproducts, and let $K^+ = \{\mathcal{O}(X) \mid X \in K\}$. Then $HSP(K^+)$ is closed under canonical embedding algebras.*

Proof: We first prove that $ISP(K^+)$ is closed under canonical embedding algebras. Let $B \in ISP(K^+)$. Then there exists a family $\{X_i \mid i \in I\} \subseteq K$ such that B is isomorphic to a subalgebra of $\prod_{i \in I} \mathcal{O}(X_i)$. By Lemma 13, $\prod_{i \in I} \mathcal{O}(X_i)$ is isomorphic to $\mathcal{O}(\sum_{i \in I} X_i)$. It follows that $\mathcal{O}(D(B))$ is isomorphic to a subalgebra of $\mathcal{O}(D(\mathcal{O}(\sum_{i \in I} X_i)))$. By Theorem 18, there exists an ultrapower $(\sum_{i \in I} X_i)^J/U$ of $\sum_{i \in I} X_i$, and a bounded morphism $(\sum_{i \in I} X_i)^J/U \rightarrow D(\mathcal{O}(\sum_{i \in I} X_i))$ which is onto. Therefore, $\mathcal{O}(D(B))$ is a subalgebra of $\mathcal{O}(\sum_{i \in I} X_i)^J/U)$, which, by Theorem 19, is an element in $ISP(K^+)$. Thus, we showed that $ISP(K^+)$ is closed under canonical embedding algebras. Let now $A \in HSP(K^+)$. Then A is a homomorphic image of some $B \in ISP(K^+)$, and hence, $\mathcal{O}(D(A))$ is a homomorphic image of $\mathcal{O}(D(B))$. We proved that $ISP(K^+)$ is closed under canonical embedding algebras, so $\mathcal{O}(D(B)) \in ISP(K^+)$, and, therefore, $\mathcal{O}(D(A)) \in HSP(K^+)$. □

5 Conclusions

In this paper we presented an extension of the Priestley duality to distributive lattices with operators in the following classes: lattice homomorphisms and antimorphisms; join- and meet-hemimorphisms; and join- and meet-hemiantimorphisms. These results can be seen as an extension of the results presented in [Gol89]. We then used these results in order to study the closure under canonical embedding algebras of certain varieties of distributive lattices with operators. Our results extend both the results in [Gol89] (we consider distributive lattices with operators instead of Boolean algebras with operators) and those in [GJ94] (we also consider certain classes of antimorphisms). It turns out that this type of results can be applied to the study of non-classical logics. In the second part of this paper [SS98] we will present a way of defining classes of Kripke-style models for certain logics for which an algebraic semantics is known, and we will show that, under certain circumstances, the soundness and completeness of a logic $\mathcal{L}$ with respect to a variety $\mathcal{V}_\mathcal{L}$ of algebras implies its soundness and completeness with respect to a class of relational Priestley spaces, $\mathcal{V}_\mathcal{L}\mathsf{Sp}$, regarded as (non-topological) frames, or, alternatively, to the class of all ordered relational structures X of a certain similarity type with the property that $\mathcal{O}(X)$ (endowed with operators corresponding to the relations on X) is an element in $\mathcal{V}_\mathcal{L}$.

An extension to more general classes of operators that are hemimorphisms in some of their arguments and hemiantimorphisms in the others is not completely immediate; in future work we would like to also study these classes of operators.

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