Duality and Canonical Extensions of Bounded Distributive Lattices with Operators, and Applications to the Semantics of Non-Classical Logics II

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Abstract

The main goal of this paper is to explain the link between the algebraic models and the Kripke-style models for certain classes of propositional non-classical logics. We consider logics that are sound and complete with respect to varieties of distributive lattices with certain classes of well-behaved operators for which a Priestley-style duality holds, and present a way of constructing topological and non-topological Kripke-style models for these types of logics. Moreover, we show that, under certain additional assumptions on the variety of the algebraic models of the given logics, soundness and completeness with respect to these classes of Kripke-style models follows by using entirely algebraical arguments from the soundness and completeness of the logic with respect to its algebraic models.

1 Introduction

The goal of this paper is to give a better understanding of the link between the algebraic models and the Kripke-style models for certain classes of propositional logics. In what follows we will focus on logics which are sound and complete with respect to distributive lattices with operators.

The idea of using representation theorems for establishing a link between the algebraic and relational semantics of non-classical logics goes back to Jónsson and Tarski [JT51, JT52], who for this purpose used an extension of Stone's representation theorem for Boolean algebras with operators. Representation theorems were also used in the study of modal systems by Lemmon [Lem66a, Lem66b]. In [Gol89], Goldblatt shows that the case of modal logics is an illustration of more general theoretical results; among other things, he studies the general structure of the so-called *complex varieties* of Boolean algebras with operators.

In this paper we consider a more general class of logics, having as algebraic models distributive lattices with well-behaved operators rather than boolean algebras with operators. We assume given a class of algebraic models for a certain propositional logic $\mathcal{L}$ and present a natural way of constructing Kripke-style models for $\mathcal{L}$. Assume that the logic $\mathcal{L}$ has as class of algebraic models a variety $\mathcal{V}_{\mathcal{L}}$ of bounded distributive lattices with operators, and there exists a dual equivalence (e.g. induced by the Priestley duality) between the variety $\mathcal{V}_{\mathcal{L}}$ (seen as a category) and a corresponding category $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$ of Priestley spaces. Then the objects of the category $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$ can be regarded as (topological or non-topological) Kripke-style frames for the logic $\mathcal{L}$. Moreover, the properties of the elements in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$ may suggest ways of defining more general classes of Kripke-models for $\mathcal{L}$. Among the questions that arise we mention the following:

1. Under which assumptions does the fact that $\mathcal{L}$ is sound and complete with respect to $\mathcal{V}_{\mathcal{L}}$ imply that $\mathcal{L}$ is sound and complete with respect to the Kripke frames corresponding to the elements of $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$?

2. Under which assumptions do the properties of $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$ suggest a way of defining more general classes of Kripke-style models for $\mathcal{L}$, with respect to which $\mathcal{L}$ is sound and complete?

3. What can we infer about even larger classes of Kripke-style frames? (The Kripke-style frames corresponding to elements in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$ are relational spaces endowed with a partial order. However, on the one hand the anti-symmetry of this relation may not be essential, and on the other hand it may

not be preserved under certain classical constructions on Kripke-frames. When defining Kripke-style frames, logicians prefer to consider spaces endowed with relations which are only reflexive and transitive.)

In this paper we focus on answering the first two questions. We briefly mention topological Kripke models, and then consider non-topological Kripke models. In the topological Kripke models we define, every meaning function is required to have as values *clopen* order-filters, whereas in the non-topological Kripke models considered in Section 3.2.2 the meaning functions are required to have as values order-filters. Topological Kripke models may be useful for instance in the study of the semantics of program languages, where the topology provides additional information about properties of states (e.g. verifiability, refutability, observability, cf. [RB98]). However, the use of topological models is not fully satisfactory in all situations, especially when one of the ultimate goals is automated theorem proving. Non-topological Kripke models are the type of models usually used by logicians for the study of non-classical logics. We present several possibilities of choosing classes of Kripke-style models with respect to which the logic is sound and complete, and prove that in order to prove the soundness and completeness of $\mathcal{L}$ with respect to this class of models it is sufficient to show that the variety $\mathcal{V}_{\mathcal{L}}$ is closed under canonical embedding algebras (this kind of properties have been studied in the first part of this paper, [SS98a]).

The paper is structured as follows: In Section 2 we briefly introduce the notations and notions that will be used later; we also present a brief overview of the results in the first part of the paper [SS98a]. We start Section 3 with some generalities on propositional non-classical logics, and their algebraic and Kripke-style semantics. We then present various ways of defining classes of Kripke-style models for certain logics for which an algebraic semantics is known. The results are schematically summarized in Section 4. In Section 5 we present some conclusions and remarks concerning further possibilities of defining Kripke-style models for non-classical logics.

2 Preliminaries

This section contains a brief presentation of the main notions needed in this paper. In order to make the paper self-contained we also include a brief presentation of the main theoretical results of the first part of this paper [SS98a].

2.1 Partially ordered sets, lattices, universal algebra

In what follows we assume known standard notions, such as partially-ordered set, order-filter (or upwards-closed set) and order-ideal (or downwards-closed set) in a partially-ordered set. For the definitions and further informations we refer to [DP90] or to the first part of our paper [SS98a]. Let $(X, \leq)$ be a partially ordered set and let $R \subseteq X^{n+1}$ be a $(n+1)$-ary relation on X. R is an *increasing relation* if it has the property that for all $x \in X^n$ and every $y, z \in X$, if $R(x, y)$ and $y \leq z$ then $R(x, z)$. R is a *decreasing relation* if it has the property that for all $x \in X^n$ and every $y, z \in X$, if $R(x, y)$ and $z \leq y$ then $R(x, z)$. For every $y \in X$, the *inverse image* of y with respect to R is the set $R^{-1}(y) = \{(x_1, \dots, x_n) \mid R(x_1, \dots, x_n, y)\}$. Given a partially-ordered set $(X, \leq)$, by $\mathcal{O}(X)$ we will denote the set of order-filters of X.

A *lattice* is a partially-ordered set $(L, \leq)$ with the property that every two elements $x, y \in L$ have a supremum (denoted $x \vee y$) and an infimum (denoted by $x \wedge y$) in L. Alternatively, a non-empty set L together with two binary operations $\vee$ and $\wedge$ on L is called *lattice* if $\vee$ and $\wedge$ are associative, commutative and idempotent and satisfy the absorbtion laws. A *distributive lattice* is a lattice that satisfies either of the distributive laws. We say that a lattice L has a *first element* if there is an element $0 \in L$ such that $0 \leq x$ for every $x \in L$. A lattice L has a *last element* if there is an element $1 \in L$ such that $x \leq 1$ for every $x \in L$. A lattice having both a first and a last element is called *bounded*. In what follows we will denote the first element in a lattice (if any) by 0 and the last element (if any) by 1. Let L be a lattice. A filter of L is a non-empty upwards-closed set closed under meet, and an ideal is a non-empty downwards-closed set closed under join. A filter F is said to be *prime* if $F \neq L$ and for every $x, y \in L$, if $x \vee y \in F$ then $x \in F$ or $y \in F$. Prime ideals are defined dually.

For the basic notions of universal algebra needed in what follows, we refer for instance to [BS81]. For every signature Σ and every arity function $a : \Sigma \to N$, a Σ-algebra is a structure $(A, \{\sigma_A\}_{\sigma \in \Sigma})$, where for every $\sigma \in \Sigma$, $\sigma_A : A^{a(\sigma)} \to A$. If the signature Σ is assumed to be known we will sometimes use the notation A for the Σ-algebra $(A, \{\sigma_A\}_{\sigma \in \Sigma})$. We say that a Σ-algebra has a *distributive lattice reduct* if there exist operations $\vee, \wedge$ in Σ such that $(A, \vee_A, \wedge_A)$ is a distributive lattice. Algebras with a *bounded distributive lattice reduct* are defined in a similar way. A variety is a class of algebras which is closed under homomorphic images, subalgebras and direct products. For a given class $\mathcal{K}$ of algebras, the variety generated by $\mathcal{K}$ is the class of all algebras that are homomorphic images of a subalgebra of a product of elements in $\mathcal{K}$. The variety generated by $\mathcal{K}$ will be denoted $HSP(\mathcal{K})$.

We continue by presenting some results on Priestley representation for distributive lattices with operators. For more details we refer to [DP90, Gol89, SS97, SS98a].

2.2 Priestley duality for distributive lattices with operators

The Priestley representation theorem [Pri70, Pri72] states that every distributive lattice A is isomorphic to the lattice of clopen (i.e. closed and open) order filters of the ordered topological space having as points the prime filters of A, ordered by inclusion, and the topology generated by the sets of the form $X_a = \{F \mid F \text{ prime filter}, a \in F\}$ and their complements as a subbasis (if A is finite, this topology is discrete). In what follows, the partially ordered set of all prime filters of A, ordered by inclusion, and endowed with the topology mentioned above will be denoted $D(A)$ (we will refer to it as the dual of A). Given an ordered topological space $X = (X, \leq, \tau)$, its lattice of clopen order filters will be denoted by $E(X)$ or, sometimes, by $\mathsf{ClopenOF}(X)$. Thus, the Priestley representation theorem states that every distributive lattice A is isomorphic to $E(D(A))$. A dual equivalence exists between the category $\mathsf{D_{01}}$ of bounded distributive lattices and a category P of Priestley spaces (compact, totally order-disconnected ordered topological spaces). This duality was extended to bounded distributive lattices with operators in the following classes: lattice homomorphisms and antimorphisms, join- and meet-hemimorphisms (maps of arbitrary arity, preserving finite joins resp. meets in every argument), and join- resp. meet-antihemimorphisms (maps of arbitrary arity, mapping finite meets in joins (resp. joins in meets) in every argument) [Gol89, SS98a].

Let Σ be a signature containing function symbols in the classes mentioned above. In order to distinguish these classes, we will write $\Sigma = Lh \cup La \cup Jh \cup Mh \cup Ja \cup Ma$, where Lh, La, Jh, Mh, Ja, and Ma can possibly be empty. Let DLO_Σ be the category having as objects bounded distributive lattices with operators in Σ (and such that the operators in the classes Lh, La, Jh, Mh, Ja, and Ma are respectively lattice homomorphisms and antimorphisms, join- and meet-hemimorphisms, and join- resp. meet-hemiantimorphisms), and as morphisms homomorphisms of such lattices. Let RPS_Σ be a corresponding category having as objects relational Priestley spaces (with additional properties cf. [SS98a]), and as morphisms continuous bounded morphisms, i.e. continuous maps $\phi : X_1 \to X_2$ which preserve the order $\leq$, the operations in Lh, La and the relations in Jh, Mh, Ja and Ma in passing from X_1 to X_2, and, additionally, for all $z \in X_1$, satisfy the following boundedness condition:

(i) $R_{X_2}(y_1, \ldots, y_n, \phi(z))$ implies $\exists x_1, \ldots, x_n \in X_1(R_{X_1}(x_1, \ldots, x_n, z)$ and $y_i \leq \phi(x_i)$, for every $1 \leq i \leq n)$, for every $y_1, \ldots, y_n \in X_2$, and every $R \in Jh \cup Ma$,

(ii) $R_{X_2}(y_1, \ldots, y_n, \phi(z))$ implies $\exists x_1, \ldots, x_n \in X_1(R_{X_1}(x_1, \ldots, x_n, z)$ and $\phi(x_i) \leq y_i$, for every $1 \leq i \leq n)$, for every $y_1, \ldots, y_n \in X_2$, and every $Q \in Mh \cup Ja$.

In [SS98a] we showed that for every algebra $A \in \mathsf{DLO}_\Sigma$, the operators in Σ induce in a canonical way functions and relations on its Priestley dual $D(A)$, and, conversely, for every relational Priestley space X, its set of clopen order-filters can be endowed with a bounded lattice structure with additional operators. Moreover, we showed that the Priestley duality for distributive lattices induces a dual equivalence between DLO_Σ and RPS_Σ, defined by means of two functors, $D : \mathsf{DLO}_\Sigma \to \mathsf{RPS}_\Sigma$ and $E : \mathsf{RPS}_\Sigma \to \mathsf{DLO}_\Sigma$. For details we refer to the first part of the paper [SS98a]. In what follows, for every $A \in |\mathsf{DLO}_\Sigma|$, and $X \in |\mathsf{RPS}_\Sigma|$ we will denote the corresponding isomorphisms by $\eta_A : A \to E(D(A))$ resp. $\epsilon_X : X \to D(E(X))$.

2.3 Ordered Σ-relational structures

In [SS98a], inspired by the definition of relational Priestley spaces, we introduced the notion of *ordered relational structures*, and a corresponding notion of *morphism* of ordered relational structures.

An ordered Σ-relational structure is a partially ordered set endowed with additional maps and relations

$$(X, \leq, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma})$$

where for every $H \in Lh$, $H_X : X \to X$ is an order-preserving map, for every $K \in La$, $K_X : X \to X$ is an order-reversing map, for every $R \in Jh \cup Ja$ with arity n, $R_X \subseteq X^{n+1}$ is an increasing relation, and for every $Q \in Mh \cup Ma$ with arity n, $Q_X \subseteq X^{n+1}$ is a decreasing relation. We showed that for every ordered Σ-relational structure X, its set $\mathcal{O}(X)$ of order-filters can be endowed with a bounded lattice structure (where join is union, meet is intersection, $0 = \emptyset$ and $1 = X$), with additional operators which can be defined in a canonical way starting from the maps and relations on X, as showed below:

(1) Every order preserving operation on X, $H \in Lh$ induces a lattice morphism $h_H : \mathcal{O}(X) \to \mathcal{O}(X)$, defined for every order-filter U of X by $h_H(U) = H_X^{-1}(U)$.

(2) Every order reversing operation on X, $K \in La$ induces a lattice antimorphism $k_K : \mathcal{O}(X) \to \mathcal{O}(X)$ defined for every order-filter U of X by $k_K(U) = X \backslash K_X^{-1}(U)$.

(3) Every increasing relation $R_X \subseteq X^{n+1}$, $R \in Jh$, induces a join hemimorphism $f_R : \mathcal{O}(X)^n \to \mathcal{O}(X)$, defined for every $U_1, \ldots, U_n \in \mathcal{O}(X)$ by

$$f_R(U_1, \ldots, U_n) = \{x \in X \mid \exists x_1 \in U_1, \ldots, x_n \in U_n : R_X(x_1, \ldots, x_n, x)\}.$$

(4) Every decreasing relation $Q_X \subseteq X^{n+1}$, $Q \in Mh$, induces a meet hemimorphism $g_Q : \mathcal{O}(X)^n \to \mathcal{O}(X)$, defined for every $U_1, \ldots, U_n \in \mathcal{O}(X)$ by

$$g_Q(U_1, \ldots, U_n) = \{x \in X \mid \forall x_1, \ldots, x_n\, (Q_X(x_1, \ldots, x_n, x) \Rightarrow \exists i, x_i \in U_i)\}.$$

(5) Every increasing relation $R'_X \subseteq X^{n+1}$, $R' \in Ja$, induces a join hemimorphism $f'_{R'} : \mathcal{O}(X)^n \to \mathcal{O}(X)$, defined for every $U_1, \ldots, U_n \in \mathcal{O}(X)$ by

$$f'_{R'}(U_1, \ldots, U_n) = \{x \in X \mid \exists x_1 \notin U_1, \ldots, x_n \notin U_n : R'_X(x_1, \ldots, x_n, x)\}.$$

(6) Every decreasing relation $Q' \subseteq X^{n+1}$, $Q' \in Ma$, induces a meet hemimorphism $g'_{Q'} : \mathcal{O}(X)^n \to \mathcal{O}(X)$, defined for every $U_1, \ldots, U_n \in \mathcal{O}(X)$ by

$$g'_{Q'}(U_1, \ldots, U_n) = \{x \in X \mid \forall x_1, \ldots, x_n\, (Q'_X(x_1, \ldots, x_n, x) \Rightarrow \exists i, x_i \notin U_i)\}.$$

The class of ordered Σ-relational structures will be denoted by ORS_Σ. Bounded morphisms of ordered Σ-relational structures are maps that preserve the order $\leq$, the operations in Lh, La and the relations in Jh, Mh, Ja, and satisfy the conditions (i) and (ii) in the definition of morphisms in RPS_Σ. Moreover, if $f : X_1 \to X_2$ is a bounded morphism of ordered Σ-relational spaces, then $\mathcal{O}(f) : \mathcal{O}(X_2) \to \mathcal{O}(X_1)$ defined by $\mathcal{O}(f)(U) = f^{-1}(U)$ for every $U \in \mathcal{O}(X_2)$ is a homomorphism in DLO_Σ; if f is order-embedding, then $\mathcal{O}(f)$ is surjective, and if f is surjective, then $\mathcal{O}(f)$ is injective.

Let $\{X_i \mid i \in I\}$ be a family of ordered Σ-relational structures, $X_i = (X_i, \leq_i, \{H_{X_i}\}_{H \in Lh}, \{K_{X_i}\}_{K \in La}, \{R_{X_i}\}_{R \in Jh}, \{Q_{X_i}\}_{Q \in Mh}, \{R'_{X_i}\}_{R' \in Ja}, \{Q'_{X_i}\}_{Q' \in Ma})$. The sum $\sum_{i \in I} X_i$ of this family is the relational structure $(X, \leq, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma})$, where:

- $X = \coprod_{i \in I} X_i$ (the disjoint union),
- for every $x, y \in X$ $x \leq y$ if and only if $x, y \in X_i$ and $x \leq_i y$ for some $i \in I$,
- for every $x \in X$ and every $H \in Lh \cup La$, $H_X(x) = H_{X_i}(x)$ where i is such that $x \in X_i$,
- for every $x_1, \ldots, x_n \in X^n$, and every $R \in Jh \cup Mh \cup Ja \cup Ma$ such that $R_X \subseteq X^n$, $R_X(x_1, \ldots, x_n)$ if and only if for some $i \in I$, $x_1, \ldots, x_n \in X_i$ and $R_{X_i}(x_1, \ldots, x_n)$.

Lemma 1 ([SS98a]) *For every family $\{X_i \mid i \in I\}$ of ordered Σ-relational spaces, there exists an isomorphism between $\mathcal{O}(\sum_{i \in I} X_i)$ and $\prod_{i \in I} \mathcal{O}(X_i)$.*

2.4 Closure under canonical embedding algebras

We recall some results on closure under canonical embedding algebras.

Definition 1 *Let $\mathcal{V}$ be a variety of bounded distributive algebras with operators in the classes mentioned above. We say that the variety $\mathcal{V}$ is closed under embedding algebras if it has the property*

$$(CE) \qquad \text{for every algebra } A \in \mathcal{V},\ \mathcal{O}(D(A)) \in \mathcal{V}.$$

In [JT51, JT52] Jónsson and Tarski showed that classes of Boolean algebras with additive operators axiomatized by certain kinds of (quasi-)equations are closed under canonical embedding algebras. Several generalizations of this result are known.

Theorem 2 ([Gol89]) *Let K is a class of Σ-structures (the order is considered to be discrete) which is closed under ultraproducts, and let $K^+ = \{\mathcal{P}(X) \mid X \in K\}$. Then $HSP(K^+)$ is closed under canonical embedding algebras. In particular, every finitely-generated variety of Boolean algebras with normal and additive operators is closed under canonical embedding algebras.*

Theorem 3 ([GJ94]) *For every distributive algebra A with additive operators, A and its canonical extension $\mathcal{O}(D(A))$ satisfy the same identities.*

Theorem 4 ([SS98a]) *Let K be a class of ordered Σ-relational structures which is closed under ultraproducts, and let $K^+ = \{\mathcal{O}(X) \mid X \in K\}$. Then $HSP(K^+)$ is closed under canonical embedding algebras. In particular, let $A_1, \dots, A_n$ be finite distributive lattices with operators in Σ. Then for every $A \in HSP(A_1, \dots, A_n)$, $\mathcal{O}(D(A)) \in HSP(A_1, \dots, A_n)$.*

The closure under embedding algebras plays an important rôle in our work. In what follows we will show that that under certain circumstances we can thereby establish, by purely algebraic means, soundness and completeness results for certain logics with respect to classes of Kripke-style models obtained in a canonical way from their algebraic models.

3 Propositional logics

In what follows we mention propositional logics which are sound and complete with respect to certain varieties of algebras. Roughly speaking, a logic is described by a pair $\mathcal{L} = (\mathsf{Fma}(\mathcal{L}), \vdash_{\mathcal{L}})$, where $\mathsf{Fma}(\mathcal{L})$ is the set of formulae in the logic and $\vdash_{\mathcal{L}}$ is a relation between sets of formulae and formulae called *provability relation*. Usually, $\mathsf{Fma}(\mathcal{L})$ is constructed starting from a set Var of propositional variables and a set Σ of logical connectives. (If the set of logical connectives is considered fixed, instead of $\mathsf{Fma}(\mathcal{L})$ we will use the notation $\mathsf{Fma}(\mathsf{Var})$.) In this case the logic is described as $\mathcal{L} = (\mathsf{Var}, \Sigma, \vdash_{\mathcal{L}})$. In general the provability relation $\vdash_{\mathcal{L}}$ is introduced by defining a notion of proof, starting from a set of formulae called axioms, and a (finite) set $\{(r_1), \dots, (r_n)\}$ of *rules of inference*. A formula ϕ is said to be a *theorem* of $\mathcal{L}$ (denoted by $\vdash_{\mathcal{L}} \phi$) if $\emptyset \vdash_{\mathcal{L}} \phi$. In this paper we consider a provability relation as given, and we will assume that among the symbols for the logical connectives there is one for conjunction and one for disjunction. Thus, the (propositional) logics considered will be of the form $\mathcal{L} = (\mathsf{Var}, \Sigma \cup \{\vee, \wedge\}, \vdash_{\mathcal{L}})$ where Var is a set of propositional variables, $\vee, \wedge$ are symbols for disjunction resp. conjunction, Σ is a set of additional logical connectives, and $\vdash_{\mathcal{L}}$ is a provability relation.

In the study of non-classical propositional logics (and especially of modal logics) there are two main ways of defining interpretations or models. One possibility is to use *algebras* as models. Propositional variables are interpreted over elements of these algebraic models, and the logical connectives correspond to the algebra operators. The other way is to use Kripke-style semantics. Here the models are known as *frames*, and usually carry a relational structure. Formulae are interpreted as subsets of the frame in a way that depends on the particular structure on these frames. Usually, the algebraic models have the advantage that soundness and completeness results are relatively easy to prove, which is not always the case with the Kripke-style models. On the other hand, Kripke-style models have a much simpler structure than algebraic models, hence are more comfortable to use. In what follows we will briefly explain the above-mentioned notions.

3.1 Algebraic semantics

We start by briefly explaining the notion of an algebraic model for a logic $\mathcal{L} = (\mathsf{Var}, \Sigma \cup \{\vee, \wedge\}, \vdash_{\mathcal{L}})$. Note that the set $\mathsf{Fma}(\mathsf{Var})$ of formulae of $\mathcal{L}$ in the variables Var is the absolutely free $\{0, 1, \wedge, \vee\} \cup \Sigma$-algebra freely generated by Var. A map $f : \mathsf{Var} \to A$, where A is a Σ-algebra is called a *valuation* in A. From the universality property of $\mathsf{Fma}(\mathcal{L})$ it follows that for every $\Sigma \cup \{0, 1, \vee, \wedge\}$-algebra A and every valuation $f : \mathsf{Var} \to A$ there is a unique homomorphism of algebras $\overline{f} : \mathsf{Fma}(\mathcal{L}) \to A$ that extends f.

Definition 2 *Let A be a $\{0, 1, \wedge, \vee\} \cup \Sigma$-algebra and let ϕ be a formula of the logic $\mathcal{L}$ in the variables* Var. *Let $f : \mathsf{Var} \to A$ be an assignment of values in A to the variables in* Var. *We say that (A, f) satisfies ϕ (denoted by $A \overset{a}{\models}_f \phi = 1$) if $\overline{f}(\phi) = 1_A$, where $\overline{f}$ is the unique extension of f to formulae. We say that ϕ is satisfied by A (denoted by $A \overset{a}{\models} \phi = 1$) if for every assignment $f : \mathsf{Var} \to A$, $\overline{f}(\phi) = 1$.*

A class $\mathcal{K}$ of algebras satisfies ϕ (denoted by $\mathcal{K} \overset{a}{\models} \phi = 1$) if and only if for every $A \in \mathcal{K}$, $A \overset{a}{\models} \phi = 1$.

Definition 3 *The logic $\mathcal{L}$ is said to be* sound *with respect to a class $\mathcal{K}$ of algebras if for every formula ϕ of $\mathcal{L}$, whenever ϕ is a theorem of $\mathcal{L}$, ϕ is satisfied by every algebra of $\mathcal{K}$. $\mathcal{L}$ is said to be* complete *with respect to a class $\mathcal{K}$ of algebras if for every formula ϕ of $\mathcal{L}$, if ϕ is satisfied by every algebra of $\mathcal{K}$ then it is a theorem of $\mathcal{L}$.*

3.2 Kripke-style models for non-classical logics

Let Σ be a signature containing function symbols in the classes considered before, i.e. $\Sigma = Lh \cup La \cup Jh \cup Ja \cup Mh \cup Ma$ be a signature[1]. Let $\mathcal{L} = (\mathsf{Var}, \Sigma \cup \{\vee, \wedge\}, \vdash_{\mathcal{L}})$ be a logic which satisfies the following condition:

(L) $\mathcal{L}$ is sound and complete with respect to a subvariety $\mathcal{V}_{\mathcal{L}}$ of $\mathsf{DLO}_{\mathbf{\Sigma}}$, such that the Priestley duality induces a full duality between the variety $\mathcal{V}_{\mathcal{L}}$ and a subcategory $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$ of the category $\mathsf{RPS}_{\mathbf{\Sigma}}$ of relational Priestley spaces.

The signature of the algebras in $\mathcal{V}_{\mathcal{L}}$ is $\{\wedge, \vee, 0, 1\} \cup \Sigma$; the logical operations $\vee, \wedge$ are interpreted as join respectively meet in the lattice, and the elements in Σ are interpreted as operators in the lattice. The additional operations on the algebras in $\mathcal{V}_{\mathcal{L}}$ belong to the classes (Lh), (La), (Jh), (Ja), (Mh), (Ma) (and may also additionally include e.g. Heyting implication and negation). The duals of the algebras in $\mathcal{V}_{\mathcal{L}}$ are Priestley spaces endowed with additional operation and relation symbols, corresponding to the operators in Σ. We will represent the corresponding operations, indexed by the same families. An element in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$ will be therefore represented by:

$$(X, \leq, \tau, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma}).$$

Kripke-style models for a logic $\mathcal{L}$ with a set of truth values Var consist of a *frame* (usually a relational structure) and a map that assigns to every propositional variables a subset of the frame (the set of all elements in the frame at which the variable "is true"; at all other points the variable is supposed to be "false"). We will refer to such maps as *meaning functions*. The wide use of Kripke-style models in non-classical logics is due, on the one hand, to the fact that they have a simpler structure than the algebraic models, and on the other hand to the fact that, in a certain sense, they provide a "decomposition" of a logic in terms of classical (2-valued) logics (since at every possible world the propositional variables can be interpreted as "true" or "false").

In what follows we discuss the following ways of defining Kripke-style models:

- *topological Kripke-style models* (the spaces in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$).
- *non-topological Kripke-style models.*

  We will show that under certain conditions $\mathcal{L}$ is sound and complete with respect to the following classes:

  (i) $\mathcal{K}_{\mathcal{V}_{\mathcal{L}}\mathsf{Sp}}$, the class of all elements in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$ where the topology is forgotten,

  (ii) $\mathcal{K}_{\mathcal{V}_{\mathcal{L}}}$ the class of all non-topological frames $K = (X, \leq, \{H_X\}_{H \in Lh \cup La}, \{R_X\}_{R \in Jh \cup Ja \cup Mh \cup Ma})$ with $\mathcal{O}(X) \in \mathcal{V}_{\mathcal{L}}$.

[1] We can possibly consider $\Sigma \subseteq \{\Rightarrow, \neg\} \cup Lh \cup La \cup Jh \cup Ja \cup Mh \cup Ma$ if the logic also contains the operation symbols for Heyting implication and negation; the Priestley duality in [SS98a] can be extended with small modifications to also encompass Heyting algebras, cf. also [Gol89].

3.2.1 Topological Kripke-style models

The satisfiability relation $\overset{a}{\models}$ for the algebras of $\mathcal{V}_{\mathcal{L}}$ induces a satisfiability relation $\overset{rt}{\models}$ for the elements in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$ as follows. In order to define $\overset{rt}{\models}$ we require that the meaning functions have as values clopen order-filters[2]. Let K be an object of $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$, and let $\mathsf{ClopenOF}(K)$ denote the lattice of clopen order-filters of the partially ordered topological space K. Let $m : \mathsf{Var} \to \mathsf{ClopenOF}(K)$ be a meaning function. By condition (L), there exists a dual equivalence between the categories $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$ and $\mathcal{V}_{\mathcal{L}}$ (induced by the Priestley duality). It follows therefore that $K = D(A)$ for some $A \in \mathcal{V}_{\mathcal{L}}$.

$$\begin{tikzcd}
\mathsf{Var} \arrow[r, "m"] \arrow[dr] & \mathsf{ClopenOF}(D(A)) \arrow[r, shift left, "\eta_A^{-1}"] & A \arrow[l, shift left, "\eta_A"] \\
& \mathsf{Fma}(\mathsf{Var}) \arrow[u, "\overline{m}"] \arrow[ur, "\overline{\eta_A^{-1} \circ m}"'] &
\end{tikzcd} \tag{1}$$

Definition 4 *Let* $\overline{m} : \mathsf{Fma}(\mathsf{Var}) \to \mathsf{ClopenOF}(K) = \mathsf{ClopenOF}(D(A))$ *be defined by* $\overline{m} = \eta_A \circ \overline{\eta_A^{-1} \circ m}$. *We define:*

(1) $D(A) \overset{rt}{\models}_{m,x} \phi$ *if and only if* $x \in \overline{m}(\phi)$,

(2) $D(A) \overset{rt}{\models}_m \phi$ *if and only if* $\overline{m}(\phi) = D(A)$,

(3) $D(A) \overset{rt}{\models} \phi$ *if and only if* $D(A) \overset{r}{\models}_m \phi$ *for every* $m : \mathsf{Var} \to \mathsf{ClopenOF}(D(A))$.

Lemma 5 *Let* A *be an algebra in* $\mathcal{V}_{\mathcal{L}}$. *Then* $A \overset{a}{\models} \phi$ *if and only if* $D(A) \overset{rt}{\models} \phi$.

Proof: Assume that $A \overset{a}{\models} \phi = 1$. Let $m : \mathsf{Var} \to \mathsf{ClopenOF}(D(A))$ be a meaning function such that for every $p \in \mathsf{Var}$, $m(p)$ is a clopen order-filter. Let $f_m : \mathsf{Var} \to A$ be defined by $f_m = \eta_A^{-1} \circ m$. Since $A \overset{a}{\models} \phi = 1$ it follows that $\overline{f_m}(\phi) = 1$, hence $\overline{m}(\phi) = \eta_A \circ \overline{\eta_A^{-1} \circ m}(\phi) = \eta_A(1) = D(A)$, i.e. $D(A) \overset{rt}{\models}_m \phi$.

Conversely, assume that $D(A) \overset{rt}{\models} \phi$. Let $f : \mathsf{Var} \to A$ be an arbitrary assignment. Let $m_f : \mathsf{Var} \to \mathsf{ClopenOF}(D(A))$ be defined by $m_f = \eta_A \circ f$. Since $D(A) \overset{rt}{\models} \phi$ it follows that $\overline{m}_f(\phi) = D(A)$, where $\overline{m}_f = \eta_A \circ \overline{\eta_A^{-1} \circ \eta_A \circ f} = \eta_A \circ \overline{f}$. Hence, $\overline{f}(\phi) = 1$. □

Topological Kripke-style models are not very usual in the study of non-classical logics. However, in recent work by Brink and Rewitzky [BR98, RB98], topology and Priestley duality were used in the study of program semantics. More precisely, [BR98, RB98] use Priestley duality for establishing a link between the predicate transformers semantics [Dij75, Dij76] and a version of topological relational semantics for programs. The idea of the predicate transformer semantics is to describe the meaning of a non-deterministic program by viewing it as a mapping from predicates to predicates (a 'predicate' can be seen as a certain property of states). In the algebraic sense, a predicate transformer is just a unary (often well behaved) operator on a lattice [RB98]. In the relational model of program semantics, a program is thought of as a binary input-output relation over states. A state is thought of as a snapshot of the current values of the program variables (some of them possibly undefined). Brink and Rewitzky additionally assume that the set of all states is endowed with an order $\leq$ and a topology. Intuitively, given two states s_1, s_2, $s_1 \leq s_2$ if s_2 contains at least as much information as s_1, in the sense that all program variables defined in the state s_1 are defined in the state s_2 and they have the same value; and there may be some program variables which are undefined in s_1 but defined in s_2. Brink and Rewitzky use the topology for distinguishing between properties of states (identified with subsets of states): a *verifiable* (*refutable*, resp. *observable*) property is one for which the corresponding subset of states is *open* (*closed*, resp. *clopen*). The results above can be applied to the kind of logics used for reasoning about properties of programs. Thus, Lemma 5 shows, given a logic $\mathcal{L}$ for reasoning about a class of programs with the

[2] The fact that every meaning function m has the property that $m(p)$ is a clopen order-filter for every propositional variable can alternatively be expressed by saying that for every propositional variable p, its meaning $m(p)$ is a continuous and order-preserving map from X to $\{0,1\}$, i.e. that the meaning of p varies "continuously and preserving the order" in the space X.

property that $\mathcal{L}$ is sound and complete with respect to the predicate transformer semantics, that $\mathcal{L}$ is also sound and complete with respect to the corresponding topological relational semantics.

3.2.2 Non-topological Kripke-style models

In what follows we will consider non-topological Kripke-style models. The previous considerations suggest a possible definition of such Kripke-style models for logics that satisfy the condition (L). Our goal in what follows is to explain in an intuitive way: (i) how to define a class $\mathcal{K}_{\mathcal{L}}$ of non-topological Kripke frames such that $\mathcal{L}$ is sound and complete with respect to $\mathcal{K}_{\mathcal{L}}$; (ii) why, for certain choices of $\mathcal{K}_{\mathcal{L}}$, the satisfiability relation $\overset{r}{\models}$ on $\mathcal{K}_{\mathcal{L}}$ can be defined in a canonical way using the algebraic satisfiability relation $\overset{a}{\models}$ for algebraic models; and (iii) why, for some choices of such classes $\mathcal{K}_{\mathcal{L}}$, it is possible to prove the soundness and completeness of $\mathcal{L}$ with respect to $\mathcal{K}_{\mathcal{L}}$ by only using the soundness and completeness of $\mathcal{L}$ with respect to $\mathcal{V}_{\mathcal{L}}$, and the additional properties of the variety $\mathcal{V}_{\mathcal{L}}$ (e.g. the existence of the dual equivalence with the category $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$).

In what follows we will define a semantics for the logic $\mathcal{L}$ in terms of ordered Σ-relational spaces.

Definition 5 (Kripke-style model) *A* Kripke-style model *based on an ordered Σ-relational space*

$$X = (X, \leq, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma})$$

is a system $M = (X, m)$, where $m : \mathsf{Var} \to \mathcal{P}(X)$ is a meaning function that assigns to every variable $p \in \mathsf{Var}$ an order-filter of X.

In order to show that the meaning function can be extended to formulae in a canonical way (that generalizes e.g. the way meaning functions are extended to formulae in modal logic) we will use the fact that the set of order-filters of every ordered Σ-relational space can be endowed with a $\mathsf{DLO}_{\mathbf{\Sigma}}$-structure. The extension of the meaning function $m : \mathsf{Var} \to \mathcal{O}(X)$ is the unique morphism of $\{0, 1, \vee, \wedge\} \cup \Sigma$-algebras, $\overline{m} : \mathsf{Fma}(\mathsf{Var}) \to \mathcal{O}(X)$ that extends m.

Definition 6 *We say that a Kripke-style model $M = (K, m)$* satisfies *a formula ϕ at the state x (denoted by $K \overset{r}{\models}_{x,m} \phi$) if and only if $x \in \overline{m}(\phi)$. A formula ϕ is* true in a Kripke-style model *$M = (K, m)$ (denoted by $K \overset{r}{\models}_m \phi$) if and only if $\overline{m}(\phi) = X$. The formula ϕ is* true in a Kripke-style frame *K (denoted by $K \overset{r}{\models} \phi$) if and only if it is true in every Kripke-style model based on K.*

We illustrate this definition by means of three examples, in order to show how usual notions of satisfiability can be obtained in a natural way:

Example 1 (E_{fde}) *The fragment E_{fde} of relevance logic has been proved sound and complete with respect to the variety of De Morgan algebras [AB75]. A De Morgan algebra is a distributive lattice endowed with a lattice antimorphism $\sim$, which satisfies the De Morgan laws and the law of double negation. The Priestley duals of the De Morgan algebras are partially-ordered topological spaces $(X, \leq, \tau, g)$ endowed with a continuous order-reversing map $g : X \to X$. Every order-reversing map $g : X \to X$ on a partially ordered set $(X, \leq)$ induces a lattice antimorphism $\sim$ on $\mathcal{O}(X)$ defined for every $U \in \mathcal{O}(X)$ by $\sim U = \{x \in X \mid g(x) \notin U\}$. By Definition 6, for every Kripke-style model $M = (K, m)$, where $K = (X, \leq, g)$, and $m : \mathsf{Var} \to \mathcal{O}(X)$,*

$$K \overset{r}{\models}_{x,m} \sim \phi \quad \textit{iff} \quad x \in \overline{m}(\sim \phi) = \sim \overline{m}(\phi) \quad \textit{iff} \quad g(x) \notin \overline{m}(\phi) \quad \textit{iff} \quad K \overset{r}{\not\models}_{g(x),m} \phi.$$

Example 2 *Let $\mathcal{L}$ be a logic which is sound and complete with respect to a variety of bounded distributive lattices endowed with one unary join-hemimorphism f. The Priestley duals of bounded distributive lattices endowed with one unary join-hemimorphism are partially-ordered topological spaces $(X, \leq, R_f)$ endowed with one binary increasing relation R_f. Conversely, every binary increasing relation R on a partially-ordered set $(X, \leq)$ induces a join-hemimorphism f_R on $\mathcal{O}(X)$ defined for every $U \in \mathcal{O}(X)$ by $f_R(U) = \{x \in X \mid \exists y \in U : R(y, x)\}$. By Definition 6, for every Kripke-style model $M = (K, m)$, where $K = (X, \leq, R)$ with $R \subseteq X \times X$ increasing, and $m : \mathsf{Var} \to \mathcal{O}(X)$,*

$$
\begin{aligned}
K \overset{r}{\models}_{x,m} f(\phi) \quad & \textit{iff} \quad x \in \overline{m}(f(\phi)) = f_R(\overline{m}(\phi)) = \{z \mid \exists y \in \overline{m}(\phi) : R(y,z)\} \\
& \textit{iff} \quad \exists y (R(y,x) \textit{ and } K \overset{r}{\models}_{y,m} \phi).
\end{aligned}
$$

Note the similarity of this satisfiability notion with the definition of satisfiability in modal logic.

Example 3 (Heyting Algebras) *It is known (e.g. [Gol89]) that for every partially-ordered set $(X, \leq)$, $\mathcal{O}(X)$ can be endowed with a Heyting algebra structure, by defining*

$$U \Rightarrow V = \{x \mid \forall y, \textit{ if } y \geq x \textit{ and } y \in U \textit{ then } y \in V\}.$$

By Definition 6, for every Kripke-style model $M = (K, m)$, where $K = (X, \leq)$, and $m : \mathsf{Var} \to \mathcal{O}(X)$,

$$
\begin{aligned}
K \overset{r}{\models}_{x,m} \phi_1 \Rightarrow \phi_2 \quad & \textit{iff} \quad x \in \overline{m}(\phi_1 \Rightarrow \phi_2) = \overline{m}(\phi_1) \Rightarrow \overline{m}(\phi_2) \\
& \textit{iff} \quad x \in \{z \mid \forall y, \textit{ if } y \geq z \textit{ and } y \in \overline{m}(\phi_1) \textit{ then } y \in \overline{m}(\phi_2)\} \\
& \textit{iff} \quad \forall y \geq x, \textit{ if } K \overset{r}{\models}_{y,m} \phi_1 \textit{ then } K \overset{r}{\models}_{y,m} \phi_2.
\end{aligned}
$$

This is the usual definition of satisfiability for the Heyting implication $\Rightarrow$ in intuitionistic logic.

Definition 7 *The logic $\mathcal{L}$ is said to be* sound *with respect to a class $\mathcal{K}$ of Kripke-style models if for every formula ϕ of $\mathcal{L}$, whenever ϕ is a theorem of $\mathcal{L}$, ϕ is satisfied by every model in $\mathcal{K}$. $\mathcal{L}$ is said to be* complete *with respect to a class $\mathcal{K}$ of Kripke-style models if for every formula ϕ of $\mathcal{L}$, if ϕ is satisfied by every model of $\mathcal{K}$ then it is a theorem of $\mathcal{L}$.*

3.3 A link between algebraic and Kripke-style models

We now explain the link between satisfiability in the Kripke-style frames and models defined in Section 3.2.2 and the algebraic notion of satisfiability.

Lemma 6 *Let $X = (X, \leq, \{H_X\}_{H \in Lh}, \{K_X\}_{K \in La}, \{R_X\}_{R \in Jh}, \{Q_X\}_{Q \in Mh}, \{R'_X\}_{R' \in Ja}, \{Q'_X\}_{Q' \in Ma})$ be a Kripke-style frame, and let ϕ be a formula in the language of $\mathcal{L}$. Then $X \overset{r}{\models} \phi$ if and only if $\mathcal{O}(X) \overset{a}{\models} \phi = 1$.*

Proof: It follows immediately from the definition of $X \overset{r}{\models} \phi$. □

Lemma 7 *Let A be an algebra in $\mathcal{V}_{\mathcal{L}}$.*

(1) If $D(A) \overset{r}{\models} \phi$ then $A \overset{a}{\models} \phi = 1$.

(2) If A is finite, then $A \overset{a}{\models} \phi = 1$ if and only if $D(A) \overset{r}{\models} \phi$.

Proof: (1) Assume that $D(A) \overset{r}{\models} \phi$. Let $f : \mathsf{Var} \to A$ be an arbitrary assignment of values to the variables. Let $m_f : \mathsf{Var} \to \mathsf{ClopenOF}(D(A)) \subseteq \mathcal{O}(D(A))$ be defined by $m_f(p) = \eta_A \circ f$. The unique extension of m_f to the formulae also has clopen order-filters as values, and it is easy to see that $\overline{m_f} = \eta_A \circ \overline{f}$. Since $D(A) \overset{r}{\models} \phi$, it follows that $\overline{m_f}(\phi) = D(A)$, hence $\overline{f}(\phi) = 1$.

(2) Assume that $A \overset{a}{\models} \phi = 1$. We show that $D(A) \overset{r}{\models} \phi$. Let $m : \mathsf{Var} \to \mathcal{O}(D(A))$ be a meaning function. Since A is finite, the topology of $D(A)$ is discrete, hence the set of clopen order-filters of $D(A)$ coincides with the set $\mathcal{O}(D(A))$ of order-filters of $D(A)$, hence there exist an isomorphism $\eta_A : A \to \mathcal{O}(D(A))$. Let $f : \mathsf{Var} \to A$ be defined by $f = \eta_A^{-1} \circ m$. Since $A \overset{a}{\models} \phi = 1$, it follows that $\overline{f}(\phi) = 1$. It is easy to see that $\overline{f} = \eta_A^{-1} \circ \overline{m}$. Thus, $\eta_A^{-1}(\overline{m}(\phi)) = 1$, hence $\overline{m}(\phi) = D(A)$. The converse follows from (1). □

Corollary 8 *Assume that the logic $\mathcal{L}$ satisfies condition (L), and, moreover, that the variety $\mathcal{V}_{\mathcal{L}}$ is finitely generated, i.e. $\mathcal{V}_{\mathcal{L}} = HSP(A_1, \ldots, A_n)$ for some finite distributive lattices with operators $A_1, \ldots, A_n$. Then*

$$\vdash_{\mathcal{L}} \phi \textit{ if and only if for all } i = 1, \ldots, n, D(A_i) \overset{r}{\models} \phi.$$

In the case when A is not finite, the argument used in Lemma 7 (2) is not valid anymore. We have however the following partial result.

Lemma 9 *Assume that $\mathcal{V}_\mathcal{L} \overset{a}{\models} \phi = 1$. Let $A \in \mathcal{V}_\mathcal{L}$ be such that $\mathcal{O}(D(A)) \in \mathcal{V}_\mathcal{L}$. Then $D(A) \overset{r}{\models} \phi$.*

Proof: Let $m : \mathsf{Var} \to \mathcal{O}(D(A))$ and let $\overline{m}$ be the extension of m to formulae. From the fact that $\mathcal{O}(D(A)) \in \mathcal{V}_\mathcal{L}$ and $\mathcal{V}_\mathcal{L} \overset{a}{\models} \phi = 1$ it follows that $\overline{m}(\phi) = D(A)$, i.e. that $D(A) \overset{r}{\models} \phi$. □

Proposition 10 *Let $\mathcal{L}$ be a logic satisfying condition (L). Assume that $\mathcal{V}_\mathcal{L}$ is closed under canonical embedding algebras. Then the following are equivalent:*

(1) ϕ is a theorem of $\mathcal{L}$,

(2) $A \overset{a}{\models} \phi = 1$ for every $A \in \mathcal{V}_\mathcal{L}$,

(3) $D(A) \overset{r}{\models} \phi$ for every $A \in \mathcal{V}_\mathcal{L}$.

Proof: The equivalence of (1) and (2) is part of our premises. Lemma 7 shows that (3) implies (2) (also without the additional hypothesis that $\mathcal{V}_\mathcal{L}$ is closed under canonical embedding algebras), and Lemma 9 shows that (2) implies (3). □

3.4 Some classes of Kripke-style models induced by $\mathcal{V}_\mathcal{L}$

Proposition 10 tells that, given a logic $\mathcal{L}$ satisfying condition (L), if $\mathcal{V}_\mathcal{L}$ is additionally closed under canonical embedding algebras, then we can choose as a class of Kripke models the class

$$\mathcal{K}_{\mathcal{V}_\mathcal{L}\mathsf{Sp}} = \{U(X) \mid X \in \mathcal{V}_\mathcal{L}\mathsf{Sp}\},$$

where $U(X)$ is the Kripke-style frame obtained from X by "forgetting" the topology.

Proposition 11 *Assume that $\mathcal{L}$ satisfies condition (L). Then $\mathcal{L}$ is complete with respect to $\mathcal{K}_{\mathcal{V}_\mathcal{L}\mathsf{Sp}}$. If additionally $\mathcal{V}_\mathcal{L}$ is closed under canonical embedding algebras then $\mathcal{L}$ is sound and complete with respect to $\mathcal{K}_{\mathcal{V}_\mathcal{L}\mathsf{Sp}}$.*

Proof: By Lemma 7 (1), $\mathcal{L}$ is complete with respect to $\mathcal{K}_{\mathcal{V}_\mathcal{L}\mathsf{Sp}}$. Assume that $\mathcal{V}_\mathcal{L}$ is closed under canonical embedding algebras. In this case, by Proposition 10, it follows that $\mathcal{L}$ is also sound with respect to $\mathcal{K}_{\mathcal{V}_\mathcal{L}\mathsf{Sp}}$. □

Another possible choice for a class of Kripke models for $\mathcal{L}$ is

$$\mathcal{K}_{\mathcal{V}_\mathcal{L}} = \{X \mid X \text{ frame }, \mathcal{O}(X) \in \mathcal{V}_\mathcal{L}\}.$$

Proposition 12 *Assume that $\mathcal{L}$ satisfies condition (L). Let $K_{\mathcal{V}_\mathcal{L}} = \{X \mid \mathcal{O}(X) \in \mathcal{V}_\mathcal{L}\}$. Then $\mathcal{L}$ is sound with respect to $K_{\mathcal{V}_\mathcal{L}}$. If additionally $\mathcal{V}_\mathcal{L}$ is closed under canonical embedding algebras then $\mathcal{L}$ is sound and complete with respect to $K_{\mathcal{V}_\mathcal{L}}$.*

Proof: Assume that ϕ is a theorem of $\mathcal{L}$. Then by (L) we know that $\mathcal{V}_\mathcal{L} \overset{a}{\models} \phi = 1$. The fact that $\mathcal{K}_{\mathcal{V}_\mathcal{L}} \overset{r}{\models} \phi$ follows immediately by Lemma 6 because for every $X \in \mathcal{K}_{\mathcal{V}_\mathcal{L}}$, $\mathcal{O}(X) \in \mathcal{V}_\mathcal{L}$.

Assume now that $\mathcal{V}_\mathcal{L}$ is closed under canonical embedding algebras. We show that under this additional assumption $\mathcal{L}$ is complete with respect to $\mathcal{K}_{\mathcal{V}_\mathcal{L}}$. Let ϕ be a formula such that $X \overset{r}{\models} \phi$ for every $X \in K_{\mathcal{V}_\mathcal{L}}$. Let $A \in \mathcal{V}_\mathcal{L}$ be arbitrary. We know that A is isomorphic to a subalgebra of $\mathcal{O}(D(A))$. Since we assumed that $\mathcal{V}_\mathcal{L}$ is closed under canonical embedding algebras, it follows that $\mathcal{O}(D(A)) \in \mathcal{V}_\mathcal{L}$, hence $D(A) \in K_{\mathcal{V}_\mathcal{L}}$. Hence, $D(A) \overset{r}{\models} \phi$, and by Lemma 7 it follows that $A \overset{a}{\models} \phi = 1$. Thus, for every $A \in \mathcal{V}_\mathcal{L}$, $A \overset{a}{\models} \phi$; hence, by the completeness of $\mathcal{L}$ with respect to $\mathcal{V}_\mathcal{L}$, it follows that ϕ is a theorem of $\mathcal{L}$. □

Corollary 13 *Let $\mathcal{L}$ be a logic which is sound and complete with respect to a variety $\mathcal{V}_{\mathcal{L}}$ of distributive lattices with additive operators. Assume that the Priestley duality induces a dual equivalence between $\mathcal{V}_{\mathcal{L}}$ and a suitable variety of relational Priestley spaces $\mathcal{V}\mathsf{Sp}_{\mathcal{L}}$. Then for every formula ϕ in the language of $\mathcal{L}$ and every $A \in \mathcal{V}_{\mathcal{L}}$, $A \overset{a}{\models} \phi = 1$ if and only if $D(A) \overset{r}{\models} \phi$, and thus*

$$\mathcal{L} \vdash \phi \quad \textit{if and only if} \quad \mathcal{V}_{\mathcal{L}}\mathsf{Sp} \overset{r}{\models} \phi,$$

$$\mathcal{L} \vdash \phi \quad \textit{if and only if} \quad \mathcal{K}_{\mathcal{V}_{\mathcal{L}}} \overset{r}{\models} \phi.$$

In [SS98a] we showed that also finitely generated varieties of distributive lattices, i.e. varieties of the form $\mathcal{V} = HSP(A_1, \dots, A_n)$, where $A_1, \dots, A_n$ are finite distributive lattices with operators, are closed under canonical embedding algebras.

Corollary 14 *Let $\mathcal{L}$ be a logic satisfying condition (L), and assume that $\mathcal{V}_{\mathcal{L}} = HSP(A_1, \dots, A_n)$, where $A_1, \dots, A_n$ are finite algebras. The variety $\mathcal{V}_{\mathcal{L}}$ is closed under canonical embedding algebras, hence $\mathcal{L}$ is sound and complete with respect to the following classes of Kripke models: $\mathcal{K}_{\mathcal{V}_{\mathcal{L}}\mathsf{Sp}}$, $\mathcal{K}_{\mathcal{V}_{\mathcal{L}}}$, and $\{D(A_1), \dots, D(A_n)\}$.*

We used the result in Corollary 14 for giving a method for automatic theorem proving in many-valued logics having a finite distributive lattice with operators as algebra of truth values. Let A be this algebra of truth values. By Corollary 14, for every formula ϕ, $\mathcal{L} \vdash_{\mathcal{L}} \phi$ if and only if $D(A) \overset{r}{\models} \phi$. Thus, ϕ is a theorem of $\mathcal{L}$ if and only if for no meaning function $m : \mathsf{Var} \to \mathcal{O}(D(A))$, $D(A) \overset{r}{\not\models}_x \phi$ for some $x \in D(A)$. The last assertion can be stated, equivalently, by saying that a conjunction of (first-order) formulae is unsatisfiable. This offers a possibility for the translation of formulae to clause form, and, ultimately, for theorem proving. More details are contained in [SS97, SS98b].

3.5 Closure properties for some classes of Kripke models

We will now discuss the closure properties of the Kripke-style models of $\mathcal{L}$ under certain canonical constructions.

Proposition 15 *The category $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$ is a subcategory of $\mathsf{RPS}_{\mathbf{\Sigma}}$ which has the following closure properties:*

(1) for every continuous bounded morphism $h : X_1 \to X_2$, if h is an order-embedding and X_2 is in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$, then X_1 is in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$;

(2) for every continuous bounded morphism $h : X_1 \to X_2$, if h is onto and X_1 is in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$, then X_2 is in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$;

(3) for every family $\{X_i\}_{i \in I}$ of elements in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$, the coproduct in $\mathsf{RPS}_{\mathbf{\Sigma}}$ (which is not necessarily the disjoint union) is again in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$.

Proof: (1) Let $h : X_1 \to X_2$ be a continuous bounded morphism which is an order-embedding. Then $E(h) : E(X_2) \to E(X_1)$ is a surjective homomorphism of $\{0, 1, \vee, \wedge\} \cup \Sigma$-algebras. From the fact that X_2 is in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$, it follows that $E(X_2)$ is an algebra in $\mathcal{V}_{\mathcal{L}}$, and since $\mathcal{V}_{\mathcal{L}}$ is a variety, $E(X_1)$ is again in $\mathcal{V}_{\mathcal{L}}$. But then X_1 is in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$.

(2) follows in a similar way using the fact that $\mathcal{V}_{\mathcal{L}}$ is closed under subalgebras.

(3) follows from the fact that $\mathcal{V}_{\mathcal{L}}$ is closed under direct products and from the fact that the functor E, being part of a dual equivalence, maps coproducts in $\mathsf{RPS}_{\mathbf{\Sigma}}$ into products in $\mathsf{DLO}_{\mathbf{\Sigma}}$. □

Proposition 16 *The class $\mathcal{K}_{\mathcal{V}_{\mathcal{L}}} = \{X \mid X \in ORS_{\Sigma}, \mathcal{O}(X) \in \mathcal{V}_{\mathcal{L}}\}$ has the following closure properties:*

(1) for every bounded morphism $h : X_1 \to X_2$, if h is an order-embedding and X_2 is in $\mathcal{K}_{\mathcal{V}_{\mathcal{L}}}$, then X_1 is in $\mathcal{K}_{\mathcal{V}_{\mathcal{L}}}$;

(2) for every bounded morphism $h : X_1 \to X_2$, if h is onto and X_1 is in $\mathcal{K}_{\mathcal{V}_{\mathcal{L}}}$, then X_2 is in $\mathcal{K}_{\mathcal{V}_{\mathcal{L}}}$;

(3) for every family $\{X_i\}_{i \in I}$ of elements in $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$, their sum in $\mathsf{ORS}_{\mathbf{\Sigma}}$ is again in $\mathcal{K}_{\mathcal{V}_{\mathcal{L}}}$.

Proof: We know that for every bounded morphism $h : X_1 \to X_2$, $\mathcal{O}(h) : \mathcal{O}(X_2) \to \mathcal{O}(X_1)$ is a morphism of distributive lattices with operators. In order to prove (1), assume that h is an order-embedding. Then $\mathcal{O}(h)$ is a surjective morphism of distributive lattices, i.e. $\mathcal{O}(X_1)$ is a homomorphic image of $\mathcal{O}(X_2)$. By the definition of $\mathcal{K}_{\mathcal{V}_\mathcal{L}}$, X_2 is in $\mathcal{K}_{\mathcal{V}_\mathcal{L}}$ if and only if $\mathcal{O}(X_2) \in \mathcal{V}_\mathcal{L}$; the fact that X_1 is in $\mathcal{K}_{\mathcal{V}_\mathcal{L}}$ follows now immediately from the fact that $\mathcal{V}_\mathcal{L}$ is a variety, hence it is closed under homomorphic images. (2) follows in a similar way, using the fact that the variety $\mathcal{V}_\mathcal{L}$ is closed under subalgebras.

In order to prove (3) note that, by Lemma 1, for every family $\{X_i \mid i \in I\}$ of relational spaces, there exists an isomorphism between $\mathcal{O}(\sum_{i\in I} X_i)$ and $\prod_{i\in I} \mathcal{O}(X_i)$. Let $\{X_i \mid i \in I\}$ be a family of elements in $\mathcal{K}_{\mathcal{V}_\mathcal{L}}$. Then for every $i \in I$, $\mathcal{O}(X_i) \in \mathcal{V}_\mathcal{L}$, hence, using the fact that every variety is closed under products, $\prod_{i\in I} \mathcal{O}(X_i) \simeq \mathcal{O}(\sum_{i\in I} X_i) \in \mathcal{V}_\mathcal{L}$. Thus, $\sum_{i\in I} X_i \in \mathcal{K}_{\mathcal{V}_\mathcal{L}}$. □

It follows that the category $\mathcal{K}_{\mathcal{V}_\mathcal{L}}$ is closed under inner substructures (substructures where the inclusion map is an order-embedding bounded morphism), epimorphic bounded morphisms, and sums.

4 Summary

We now give a concise presentation of some of the results proved in the previous sections, concerning the link between algebraic and Kripke-style semantics for these classes of non-classical logics.

First we present the link between valuations, satisfiability and validity in algebraic models, topological Kripke models and non-topological Kripke models for logics that satisfy the condition (L).

Models	**Valuations**	**Truth w.r.t. a valuation**	**Validity**
Algebraic models: $\mathcal{V}_\mathcal{L}$	$f : \mathsf{Var} \to A$	$A \overset{a}{\models}_f \phi = 1$ iff $\overline{f}(\phi) = 1$	$A \overset{a}{\models} \phi = 1$ iff $\forall f : \mathsf{Var} \to A$ $A \overset{a}{\models}_f \phi = 1$
Kripke models based on p.o.: $\mathcal{K}_{\mathcal{V}_\mathcal{L}\mathsf{Sp}}$ or $\mathcal{K}_\mathcal{V}$	$m : \mathsf{Var} \to \mathcal{O}(X)$	$K \overset{r}{\models}_m \phi$ iff $\overline{m}(\phi) = X$	$K \overset{r}{\models} \phi$ iff $\mathcal{O}(X) \overset{a}{\models} \phi = 1$
Topological models: $\mathcal{V}_\mathcal{L}\mathsf{Sp}$	$m : \mathsf{Var} \to \mathsf{ClopenOF}(X)$	$K \overset{rt}{\models}_m \phi$ iff $\overline{m}(\phi) = X$	$K \overset{rt}{\models} \phi$ iff $\mathsf{ClopenOF}(X) \overset{a}{\models} \phi = 1.$

The next table summarizes the main results on soundness and completeness of the logic $\mathcal{L}$, assuming that $\mathcal{L}$ is sound and complete with respect to $\mathcal{V}_\mathcal{L}$.

Kripke models	$\mathcal{V}_\mathcal{L}$ subvariety of DLO_Σ	$\mathcal{V}_\mathcal{L}$ closed under canonical embedding algebras
Non-topological		
$\mathcal{K}_{\mathcal{V}_\mathcal{L}\mathsf{Sp}}$	complete	sound and complete
$\mathcal{K}_{\mathcal{V}_\mathcal{L}}$	sound	sound and complete
Topological		
$\mathcal{V}_\mathcal{L}\mathsf{Sp}$	sound and complete	sound and complete

Finally, we schematically present the link between properties of morphisms in the variety of algebraic models and in the categories of Kripke-style models for the logic $\mathcal{L}$.

$\mathcal{V}_{\mathcal{L}}$	$\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$	Relational Σ-structures
morphisms of DLO_{Σ}	continuous bounded morphisms	bounded morphisms
$f : A_1 \to A_2$	$D(f) : D(A_2) \to D(A_1)$	
injective morphism of DLO_{Σ}	surjective, continuous bounded morphism	
surjective morphism of DLO_{Σ}	order-embedding, continuous bounded morphism	
$E(h) : E(X_2) \to E(X_1)$	$h : X_1 \to X_2$	
injective morphism of DLO_{Σ}	surjective, continuous bounded morphism	
surjective morphism of DLO_{Σ}	order-embedding, continuous bounded morphism	
$\mathcal{O}(h) : \mathcal{O}(X_2) \to \mathcal{O}(X_1)$		$h : X_1 \to X_2$
injective morphism of DLO_{Σ}		surjective bounded morphism
surjective morphism of DLO_{Σ}		order-embedding bounded morphism

5 Conclusions

The main goal of this paper was to show that, for certain classes of propositional logics, known to be sound and complete with respect to certain varieties of distributive lattices with additional operators, there exists a link between their algebraic and their Kripke-style models.

In the first part [SS98a] we presented an extension of the Priestley duality to distributive lattices with operators in the following classes: lattice homomorphisms and antimorphisms; join- and meet-hemimorphisms; and join- and meet-hemiantimorphisms. These results extend the results presented in [Gol89]. We then presented some cases of varieties that are closed under canonical embedding algebras. In particular, we extended some of the results of Goldblatt [Gol89] to more general classes of relational structures. These results were then applied in this part of the paper to the study of non-classical logics. We presented various ways of defining classes of Kripke-style models for certain logics for which an algebraic semantics is known. We showed that, under certain circumstances, the soundness and completeness of a logic $\mathcal{L}$ with respect to a variety $\mathcal{V}_{\mathcal{L}}$ of algebras implies its soundness and completeness with respect to a class of relational Priestley spaces, $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$, regarded as (non-topological) frames, or, alternatively, to the class of all ordered relational structures X of a certain similarity type with the property that $\mathcal{O}(X)$ (endowed with operators corresponding to the relations on X) is an element in $\mathcal{V}_{\mathcal{L}}$. In particular, we showed that these soundness and completeness results can be established by completely algebraic means in the case when the class of algebraic models is closed under canonical embedding algebras.

We would like to point out, however, that in the study of non-classical logics (in particular in the study of modal logics) it is often desirable to have soundness and completeness with respect to a class of Kripke models consisting of all relational structures that satisfy a certain set of additional conditions. It it easy to see that not every such relational structure can be endowed with a Priestley topology that would make it a member of $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$. In many situations it can be seen that the ordered Kripke models of a given logic are exactly those models that satisfy a set E of conditions (which are exactly the non-topological conditions fulfilled by the category of relational Priestley spaces associated to the variety of algebraic models of that logic). Therefore, for defining a larger class of such non-topological Kripke models one might consider the class $\mathcal{K}$ of all relational structures that satisfy the class E of all conditions satisfied by the elements of $\mathcal{V}_{\mathcal{L}}\mathsf{Sp}$, which do not involve the Priestley topology on these spaces. If this class has the property that for every $K \in \mathcal{K}$, $\mathcal{O}(K) \in \mathcal{V}_{\mathcal{L}}$ then soundness and completeness of $\mathcal{L}$ with respect to $\mathcal{K}$ follows immediately. However, we do not have the guarantee that the class $\mathcal{K}$ defined this way is an elementary class. The question is now: under which circumstances is the logic $\mathcal{L}$ sound and complete with respect to an elementary class $\mathcal{K}$ of Kripke-style models? In order to answer this question it may be necessary to investigate, on the one hand the form of the additional conditions in E, and on the other hand the closure properties of such classes of ordered relational structures under certain typical operations, such as bounded epimorphisms, inner substructures, unions, and ultraproducts.

Also more general problems, as for instance how to determine the set of (more general) Kripke-style structures endowed only with a transitive and reflexive relation instead of a partial order, can be considered. For an example of such general Kripke-style models for logics whose models are distributive lattices with operators (not necessarily boolean) we refer to [IO96]. We think, however, that it is should be possible to show that for every such generalized space there exists an ordered space that satisfies the same formulae. This will be subject for future work.

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