On the Universal Theory of Varieties of Distributive Lattices with Operators: Some Decidability and Complexity Results

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Abstract. In this paper we establish a link between satisfiability of universal sentences with respect to varieties of distributive lattices with operators and satisfiability with respect to certain classes of relational structures. We use these results for giving a method for translation to clause form of universal sentences in such varieties, and then use results from automated theorem proving to obtain decidability and complexity results for the universal theory of some such varieties.

1 Introduction

In this paper we give a method for automated theorem proving in the universal theory of certain varieties of distributive lattices with well-behaved operators. For this purpose, we use extensions of Priestley's representation theorem for distributive lattices. The advantage of our method is that we avoid the explicit use of the full algebraic structure of such lattices, instead using sets endowed with a reflexive and transitive relation and with additional functions and relations that correspond to the operators in the lattices in a standard way. Our interest in such algebras is motivated by the fact that many existing non-classical logics are sound and complete with respect to varieties of distributive lattices with additional well-behaved operators. Moreover, uniform word problems in lattices also occur in more general contexts such as database dependency theory [6].

The main contributions of this paper are the following:

- We establish a link between satisfiability of universal sentences with respect to varieties of distributive lattices with operators and satisfiability with respect to classes of relational structures. This extends the results from [19].
- We use these results for giving a method for translation to clause form of universal sentences in such varieties.
- We use existing results from automated theorem proving to obtain decidability and complexity results.

We first studied this type of relationships in the context of finitely-valued logics in [18], and then extended the ideas to certain classes of non-classical logics in [20]. This paper shows that the idea is much more general, and can be used for the whole universal theory of certain varieties of distributive lattices with operators. In particular, the method presented here subsumes in a natural way both existing methods for translating modal logics to classical logic and methods for automated theorem proving in finitely-valued logics based on distributive lattices with operators. The approach has the following advantages:

– It avoids the problems that occur when ACI-operators have to be considered (as is the case in algebraic automated reasoning for lattices).
– Known saturation-based techniques for theories of reflexive and transitive relations, such as ordered chaining with selection, can be used successfully.
– Decidability and complexity results follow in many cases as consequences of existing decision procedures based on ordered resolution or ordered chaining.
– We obtain decidability and complexity results for uniform word problems in certain non locally finite varieties of distributive algebras with operators (as far as we know, no such results were known).
– Considerations concerning the structure of the sets of clauses generated with our method make certain algebraic properties of these varieties visible.

The applicability of our method depends on the possibility of finding the appropriate relational structures that can replace the algebras in the variety in the automated theorem proving process. It is known from modal logic that such structures may not always exist. Another limitation is given by the fact that, in general, resolution is a semi-decision procedure, and it may be hard or impossible to obtain resolution-based decision procedures for the classes of clauses generated by the method we describe. However, we show that in many cases the method is applicable and leads to decision procedures.

The idea of using representation theorems for establishing a link between the algebraic and relational semantics of non-classical logics goes back to Jónsson and Tarski [11], who for this purpose used an extension of Stone's representation theorem for Boolean algebras with operators. Our work is influenced by the results of Goldblatt [9], who showed that the "modal case" is an illustration of more general results from universal algebra. He gives an extension of the Priestley duality to join and meet hemimorphisms, which we extended in [19] to lattices endowed with certain classes of anti(hemi)morphisms. In this paper we use the results in [9] and [19] and show that the use of representation theorems has applications which range far beyond the area of applications in modal logics.

The paper is structured as follows. In Section 2 the main notions and results needed in the paper are presented. Section 3 contains the main results. Section 4 contains some run examples and a comparison to a more standard approach. Section 5 contains some conclusions and plans for future work.

2 Preliminaries

This section contains the main notions and results needed in this paper.

Partially ordered sets and lattices. We assume known standard notions, such as partially-ordered set, order-filter and order-ideal in a partially-ordered set, cf. [7]. Given a partially-ordered set $(X, \leq)$, by $\mathcal{O}(X)$ we denote the set of order-filters of X. A *lattice* is a partially-ordered set $(L, \leq)$ with the property that every two elements $x, y \in L$ have a supremum and an infimum (denoted $x \vee y$ resp. $x \wedge y$) in L. Alternatively, a non-empty set L together with two binary operations $\vee$ and $\wedge$ on L is called *lattice* if $\vee$ and $\wedge$ are associative, commutative and idempotent and satisfy the absorption laws. A *distributive lattice* is a lattice that satisfies either of the distributive laws. A lattice L has a *first element* if there is an element $0 \in L$ such that $0 \leq x$ for every $x \in L$; it has a *last element* if there is an element $1 \in L$ such that $x \leq 1$ for every $x \in L$. A lattice having both a first and a last element is called *bounded*. The *pseudocomplement* of an element $a \in L$ (denoted by $\neg a$) is the largest element of $\{c \in L \mid a \wedge c = 0\}$ (if any). Given $a, b \in L$, the *pseudocomplement of a relative to b* (denoted by $a \Rightarrow b$) is the largest element of $\{c \in L \mid a \wedge c \leq b\}$ (if any). A filter in a lattice L is a non-empty order-filter closed under meets. A filter F is said to be *prime* if $F \neq L$ and for every $x, y \in L$, if $x \vee y \in F$ then $x \in F$ or $y \in F$. Ideals and prime ideals are defined dually.

Priestley representation for bounded distributive lattices. The Priestley representation theorem [16] states that every bounded distributive lattice A is isomorphic to the lattice of clopen (i.e. closed and open) order filters of the ordered topological space having as points the prime filters of A, ordered by inclusion, and the topology generated by the sets of the form $X_a = \{F \mid F \text{ prime filter}, a \in F\}$ and their complements as a subbasis. The partially ordered set of all prime filters of A, ordered by inclusion, and endowed with the topology mentioned above will be denoted $D(A)$ (we will refer to it as the dual of A). If we denote the lattice of clopen order filters of an ordered topological space X by $\mathsf{ClopenOF}(X)$, the Priestley representation theorem states that there exists an isomorphism of bounded lattices, $\eta_A : A \to \mathsf{ClopenOF}(D(A))$.

Universal algebra. For the necessary notions of universal algebra we refer e.g. to [5]. For every signature Σ and every arity function $a : \Sigma \to \mathbb{N}$, a Σ-algebra is a structure $(A, \{\sigma_A\}_{\sigma \in \Sigma})$, where for every $\sigma \in \Sigma$, $\sigma_A : A^{a(\sigma)} \to A$. If the signature Σ is known we may use the notation A for the Σ-algebra $(A, \{\sigma_A\}_{\sigma \in \Sigma})$. A Σ-algebra A has a *bounded distributive lattice reduct* if there exist operations $\vee, \wedge, 0, 1$ in Σ such that $(A, 0, 1, \vee_A, \wedge_A)$ is a bounded distributive lattice. A *distributive p-lattice* (resp. *Heyting algebra*) is an algebra $(A, 0, 1, \vee, \wedge, \neg)$ (resp. $(A, 0, 1, \vee, \wedge, \Rightarrow, \neg)$) with a bounded distributive lattice reduct such that for every $a, b \in A$, $\neg a$ is the pseudocomplement of a, and $a \Rightarrow b$ is the relative pseudocomplement of a with respect to b.

Given a set X, the term algebra over Σ in the variables X will be denoted $\mathsf{Term}_\Sigma(X)$. An *equation* is an expression of the form $t_1 = t_2$ where $t_1, t_2 \in \mathsf{Term}_\Sigma(X)$; an *implication* is an expression of the form $\beta_1 \wedge \cdots \wedge \beta_m \to \alpha$, where $\beta_1, \ldots, \beta_m, \alpha$ are equations. A *conditional equation* (or *quasi-equation*) is an expression which is either an equation or an implication. A Σ-algebra A *satisfies* a quasi-equation γ (notation: $A \models \gamma$) if the quasi-equation is true for every substitution of elements in A for the variables. A class $\mathcal{K}$ of algebras satisfies γ (notation: $\mathcal{K} \models \gamma$) iff all algebras in $\mathcal{K}$ satisfy γ. Truth of conditional equations is preserved under isomorphic images, subalgebras, and products. Truth of equations is additionally preserved under homomorphic images. A variety is the class of all algebras that satisfy a set of identities, or, alternatively, a class of algebras which is closed under homomorphic images, subalgebras and direct products.

Logic. Let $\mathcal{K}$ be a class of algebras. The *elementary theory* of $\mathcal{K}$ is the collection of all closed formulae in first-order predicate logic with equality that are valid in $\mathcal{K}$. The *universal theory* of $\mathcal{K}$ is the collection of those closed formulae valid in $\mathcal{K}$ which are of the form $\forall x_1 \dots \forall x_k (\bigwedge_{i=1}^{m} ((\neg) t_{i1} = s_{i1} \vee \dots \vee (\neg) t_{in_i} = s_{in_i}))$. The *universal Horn theory* of $\mathcal{K}$ is the collection of those closed formulae valid in $\mathcal{K}$ which are of the form $\forall x_1 \dots \forall x_k (t_{11} = t_{12} \wedge \dots \wedge t_{n1} = t_{n2} \rightarrow s_1 = s_2)$. The *equational theory* of $\mathcal{K}$ is the set of all closed formulae valid in $\mathcal{K}$ which are of the form $\forall x_1 \dots x_k (t = s)$. Given a recursively enumerable set E of conditional Σ-equations we say that the *word problem* for E is decidable if we can decide for every $t, s \in \mathsf{Term}_\Sigma(X)$ whether $s \equiv_E t$, where $\equiv_E$ denotes the congruence on $\mathsf{Term}_\Sigma(X)$ generated by E. We say that the *uniform word problem* for E is decidable if the universal Horn theory of the class of all models of E is decidable. McKinsey [13] showed that for every class $\mathcal{K}$ of Σ-algebras which is closed under direct products, if a sentence of the form

$$\forall x_1 \dots \forall x_k (s_{11} = s_{12} \wedge \dots \wedge s_{n1} = s_{n2} \rightarrow t_{11} = t_{12} \vee \dots \vee t_{m1} = t_{m2})$$

is true in $\mathcal{K}$, then there exists $j \in \{1, \dots, m\}$ such that

$$\forall x_1 \dots \forall x_k (s_{11} = s_{12} \wedge \dots \wedge s_{n1} = s_{n2} \rightarrow t_{j1} = t_{j2})$$

is true in $\mathcal{K}$. In particular it follows that for every class $\mathcal{K}$ of algebras which is closed under direct products, if its universal Horn theory is decidable, then its universal theory is decidable.

Decidability results for distributive lattices. Decidability of the theories related to various classes of algebras has been studied extensively. In what follows we will present existing decidability and complexity results for the variety of distributive lattices. It is known (cf. e.g. [4], p.16) that the elementary theory of every non-trivial variety of lattices is undecidable. Thus, the elementary theory of the variety DLat of distributive lattices is undecidable. The uniform word problem for distributive lattices is decidable (since $\mathsf{DLat} = ISP(2)$, where 2 is the 2-element lattice), and has been proved to be co-NP-hard by Bloniarz et al. [10]. By the result of McKinsey [13] mentioned above it follows that the universal theory of the variety of distributive lattices is decidable. (In 1920, Skolem [17] gave a polynomial time decision procedure for the uniform word problem for general lattices, which cannot be used for the variety of distributive lattices.)

Struth [21] gives a calculus based on non-symmetric rewriting (modulo ACI) for the elementary theory of finite distributive lattices. Besides the possibility of extending this calculus to families of well-behaved operators on lattices, and the complexity results established for (boolean) Tarskian set constraints by McAllester et al. [12], and Mielniczuk and Pacholski [14], we are not aware of any systematic study on automated theorem proving or decidability and complexity results for varieties of distributive lattices with additional operators.

Resolution as a decision procedure. We assume known the usual notions and notations in first-order logic and resolution. For details we refer to any text on automated theorem proving. Unrefined resolution is only a semi-decision procedure for first-order logic. However, for some classes of formulae known to be decidable, the resolution principle can be adapted in order to obtain decision procedures. The main idea is to find a complete resolution refinement (usually an ordering refinement, possibly combined with the use of a selection function) which is terminating on the specified class of clauses. Termination may be proved for instance by finding a depth and a length limit for the resolvents.

In this paper reflexive and transitive relations will play an important rôle. In the presence of this kind of relations, superposition and ordered chaining have successfully been used for obtaining decidability results. The *superposition calculus* is a refutationally complete inference system for arbitrary first-order clauses with equality. Its inference rules are restricted versions of paramodulation, resolution, and factoring, parametrized by a total reduction ordering $\succ$ on ground expressions and a selection function S. The *ordered chaining calculus* is an extension of the superposition calculus to more general reflexive and transitive relations. Its inference rules are restricted versions of (positive and negative) chaining, resolution, and factoring, parametrized by a total reduction ordering $\succ$ on ground expressions and a selection function S. In both cases, S assigns to each clause a (possibly empty) multiset of negative literals. For details cf. [1, 2]. Superposition with selection and simplification has been proved to be a decision procedure for the monadic class with equality [3]. Ordered chaining with selection was used to obtain decision procedures for the relational translation of propositional modal logics with modal operators satisfying the axiom $\mathsf{4}$ [8].

3 On the universal theory of subvarieties of DLO_Σ

We start by presenting some results on a Priestley representation for distributive lattices with operators. We show that this helps to establish a link between satisfiability of universal sentences with respect to varieties of distributive lattices with operators and satisfiability with respect to certain classes of relational structures. These results are used for giving a method for translation to clause form of universal sentences in such varieties.

Definition 1. *Let A be an algebra with a bounded lattice reduct. A* lattice antimorphism *on A is a function $k : A \to A$ which maps* 0 *to* 1, 1 *to* 0, *joins to meets and meets to joins. A* join hemimorphism *on A is a function $f : A^n \to A$ that preserves* 0 *and all finite joins in every argument. A* meet hemimorphism *on A is a function $g : A^n \to A$ that preserves* 1 *and all finite meets in every argument. A* join hemiantimorphism *on A is a function $f' : A^n \to A$ that maps* 1 *to* 0 *and meets to joins in every argument. A* meet hemiantimorphism *on A is a function $g' : A^n \to A$ that maps* 0 *to* 1 *and joins to meets in every argument.*

Let Σ be a signature containing function symbols in several classes; in order to distinguish these classes, we will write $\Sigma = Lh \cup La \cup Jh \cup Mh \cup Ja \cup Ma$, where

Lh, La, Jh, Mh, Ja, and Ma may be empty. Let DLO_Σ be the class of all bounded distributive lattices with operators in Σ, $(A, \vee, \wedge, 0, 1, \{\sigma_A\}_{\sigma\in\Sigma})$, such that if σ is an operation symbol in Lh, La, Jh, Mh, Ja, or Ma, then σ_A is, respectively, a lattice homomorphism, lattice antimorphism, join or meet hemimorphism, or join or meet hemiantimorphism. DLO_Σ is a variety.

3.1 Priestley representation for DLO_Σ and Σ-relational structures

In [19] we showed that, given an algebra $A \in \mathsf{DLO}_\Sigma$, the operators in Σ induce in a canonical way functions and relations on its Priestley dual $D(A)$ which, in their turn, induce operators on $\mathsf{ClopenOF}(D(A))$. Taking into account these correspondences, we showed that the canonical isomorphism $\eta_A : A \to \mathsf{ClopenOF}(D(A))$ from the Priestley duality is an isomorphism of algebras in DLO_Σ. For details, including a categorical duality theorem, we refer to [7, 9, 18, 19]. The Priestley duality has been extended to distributive p-lattices and Heyting algebras (cf. e.g. [15], [9]). The dual spaces $(X, \leq, \tau)$ satisfy in this case the additional condition that for every clopen order-filter U, $X \setminus \downarrow U$ is clopen.

Definition 2. *Let $(X, \leq)$ be such that $\leq$ is a reflexive and transitive relation on X, and let $R \subseteq X^{n+1}$. R is called* increasing *if for every $\overline{x} \in X^n$ and every $y, z \in X$, if $R(\overline{x}, y)$ and $y \leq z$ then $R(\overline{x}, z)$; R is called* decreasing *if for every $\overline{x} \in X^n$ and every $y, z \in X$, if $R(\overline{x}, y)$ and $z \leq y$ then $R(\overline{x}, z)$.*

For every set X endowed with a reflexive and transitive relation $\leq$, its set $\mathcal{H}(X)$ of hereditary (i.e. upwards-closed with respect to $\leq$) subsets can be endowed with a bounded lattice structure (where join is union, meet is intersection, $0 = \emptyset$ and $1 = X$). We can canonically define additional operators on $\mathcal{H}(X)$ as showed below.

Theorem 1. *Let $(X, \leq)$ be a set endowed with a reflexive and transitive relation.*

(1) Every $\leq$-preserving map $H_X : X \to X$ induces a lattice morphism $h_H : \mathcal{H}(X) \to \mathcal{H}(X)$, defined for every $U \in \mathcal{H}(X)$ by $h_H(U) = H_X^{-1}(U)$.
(2) Every $\leq$-reversing map $K_X : X \to X$ induces a lattice antimorphism $k_K : \mathcal{H}(X) \to \mathcal{H}(X)$, defined for every $U \in \mathcal{H}(X)$ by $k_K(U) = X \setminus K_X^{-1}(U)$.
(3) Every increasing relation $R_X \subseteq X^{n+1}$ induces a join hemimorphism $f_R : \mathcal{H}(X)^n \to \mathcal{H}(X)$, and a join hemiantimorphism $f'_R : \mathcal{H}(X)^n \to \mathcal{H}(X)$, defined for every $U_1, \dots, U_n \in \mathcal{H}(X)$ by:
$f_R(U_1, \dots, U_n) = \{x \in X \mid \exists x_1, \dots, x_n (x_i \in U_i \text{ for all } i, \text{ and } R_X(x_1, \dots, x_n, x))\}$,
$f'_R(U_1, \dots, U_n) = \{x \in X \mid \exists x_1, \dots, x_n (x_i \notin U_i \text{ for all } i, \text{ and } R_X(x_1, \dots, x_n, x))\}$.
(4) Every decreasing relation $Q_X \subseteq X^{n+1}$ induces a meet hemimorphism $g_Q : \mathcal{H}(X)^n \to \mathcal{H}(X)$, and a meet hemiantimorphism $g'_Q : \mathcal{H}(X)^n \to \mathcal{H}(X)$, defined for every $U_1, \dots, U_n \in \mathcal{H}(X)$ by:
$g_Q(U_1, \dots, U_n) = \{x \in X \mid \forall x_1, \dots, x_n \,(Q_X(x_1, \dots, x_n, x) \to \exists i, x_i \in U_i)\}$,
$g'_Q(U_1, \dots, U_n) = \{x \in X \mid \forall x_1, \dots, x_n \,(Q_X(x_1, \dots, x_n, x) \to \exists i, x_i \notin U_i)\}$.
(5) Moreover, a pseudocomplementation $\neg$ and a relative pseudocomplementation $\Rightarrow$ can be defined on $\mathcal{H}(X)$ by $\neg U = \{x \mid \forall y (x \leq y \to y \notin V)\}$ and $U \Rightarrow V = \{x \mid \forall y ((x \leq y \wedge y \in U) \to y \in V)\}$.

Proof: *(Sketch)* The proof closely follows the proof of the similar results established in [9, 19] for relational structures endowed with partial orders. It can be seen that the antisymmetry of $\leq$ is not needed anywhere in the proof. □

Let $\Sigma = Lh \cup La \cup Jh \cup Mh \cup Ja \cup Ma$ be a signature as discussed above.

Definition 3. *An RT Σ-relational structure is a set endowed with a reflexive and transitive relation $\leq$ and with additional maps and relations indexed by Σ, $(X, \leq, \{\sigma_X\}_{\sigma\in\Sigma})$, where if $\sigma \in Lh$, $\sigma_X : X \to X$ is a $\leq$-preserving map, if $\sigma \in La$, $\sigma_X : X \to X$ is a $\leq$-reversing map, if $\sigma \in Jh \cup Ja$ with arity n, $\sigma_X \subseteq X^{n+1}$ is an increasing relation, and if $\sigma \in Mh \cup Ma$ with arity n, $\sigma_X \subseteq X^{n+1}$ is a decreasing relation.*

The class of RT Σ-relational structures will be denoted by RTS_Σ. For every $X \in RTS_\Sigma$ and every $\sigma \in \Sigma$ let $\sigma_{\mathcal{H}(X)}$ be the operation on $\mathcal{H}(X)$ associated with σ_X as explained in Theorem 1. The corresponding algebra is again denoted by $\mathcal{H}(X)$. By Theorem 1, $\mathcal{H}(X) \in \mathsf{DLO}_\Sigma$. Conversely, for every $A \in \mathsf{DLO}_\Sigma$, the ordered space $U(D(A))$, obtained from $D(A)$ by ignoring the topology, is in RTS_Σ. $\mathsf{ClopenOF}(D(A))$ is a subalgebra (in DLO_Σ) of $\mathcal{H}(D(A)) = \mathcal{O}(D(A))$.

Notation. As a convention, if not explicitly specified otherwise, in what follows h (resp. k) will denote an operation symbol in Lh (resp. La), f one in $Jh \cup Ja$, and g one in $Mh \cup Ma$. Sometimes, in order to distinguish between elements in Jh and Ja, resp. Mh and Ma, the operation symbols in Ja and Ma will be denoted by f' resp. g'. The symbols in $Jh \cup \dots \cup Ma$ are interpreted as maps for elements in DLO_Σ, and as relations in RTS_Σ. For the sake of clarity we will always overline the operation symbol in the latter case. In particular, in Section 3.3 (Theorem 3) and Section 3.4 the function resp. relation symbols h, k, f, g are in the classes corresponding to the labeling in (Ren).

Let $\phi = \forall x_1, \dots, x_k(\bigwedge_{i=1}^n s_{i1} = s_{i2} \to \bigvee_{j=1}^m t_{j1} = t_{j2})$ (where $s_{il}, t_{jp} \in \mathsf{Term}_{\Sigma'}(\{x_1, \dots, x_k\})$, and Σ' is $\Sigma \cup \{\vee, \wedge, 0, 1\}$ to which possibly $\neg$ and $\Rightarrow$ are adjoined). $ST(\phi)$ denotes the set of all subterms of s_{il} and t_{jp}, $1 \leq i \leq n, 1 \leq j \leq m, l, p \in \{1, 2\}$, $ns = |ST(\phi)|$, $nf = |Lh \cup La|$, $np = |Jh \cup Ja \cup Mh \cup Ma|$, and mp is the maximal arity of an operation in $Jh \cup Ja \cup Mh \cup Ma$.

3.2 A link between algebraic and relational models

We study the link between satisfiability of universal sentences with respect to algebraic and relational models. As algebraic models we consider subvarieties $\mathcal{V}$ of DLO_Σ (possibly with an additional p-lattice or Heyting algebra structure), satisfying the condition (K) below:

(K) There exists a class $\mathcal{K}$ of RT Σ-relational structures such that:
- (i) for every $A \in \mathcal{V}$, the RT Σ-relational structure $U(D(A))$ is in $\mathcal{K}$;
- (ii) for every $X \in \mathcal{K}$, the algebra $\mathcal{H}(X)$ is in $\mathcal{V}$.

Theorem 2. *Let $\phi = \forall x_1, \dots, x_k(\bigwedge_{i=1}^n s_{i1} = s_{i2} \to \bigvee_{j=1}^m t_{j1} = t_{j2})$. Assume that $\mathcal{V}$ satisfies condition (K). Then $\mathcal{V} \models \phi$ iff for every $X \in \mathcal{K}$, $\mathcal{H}(X) \models \phi$.*

Proof: *(Sketch)* The direct implication follows from the fact that, by (K)(ii), for every $X \in \mathcal{K}$, $\mathcal{H}(X) \in \mathcal{V}$; the inverse implication follows from the fact that, by (K)(i), for every $A \in \mathcal{V}$, the RT Σ-relational structure corresponding to $D(A)$ is in $\mathcal{K}$, and that, by the Priestley representation theorem, A is isomorphic to $\mathsf{ClopenOF}(D(A))$ which is a subalgebra of $\mathcal{O}(D(A))$. $\square$

3.3 Structure-preserving translation to clause form

If the class $\mathcal{K}$ is first-order definable, Theorem 2 justifies a structure-preserving translation of universal formulae to sets of clauses, inspired by the method of Tseitin [22] for transforming quantifier-free formulae to clausal normal form.

Theorem 3. *Assume that $\mathcal{V}$ satisfies (K), where the class $\mathcal{K}$ is definable by a finite set C of first-order sentences*[1]. *Let $\phi = \forall x_1, \dots, x_k (\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^m t_{j1} = t_{j2})$. Then $\mathcal{V} \models \phi$ iff the following conjunction is unsatisfiable:*

$$
\begin{cases}
(\mathsf{Dom}) & C, \\
(\mathsf{Her}) & \forall x, y (x \leq y \wedge P_e(x) \rightarrow P_e(y)), \\
(\mathsf{Ren}) & \\
\quad (1,0) & \forall x P_1(x), \ \textit{resp.} \ \forall x \neg P_0(x), \\
\quad (\wedge) & \forall x (P_{e_1 \wedge e_2}(x) \leftrightarrow P_{e_1}(x) \wedge P_{e_2}(x)), \\
\quad (\vee) & \forall x (P_{e_1 \vee e_2}(x) \leftrightarrow P_{e_1}(x) \vee P_{e_2}(x)), \\
\quad (Lh) & \forall x (P_{h(e)}(x) \leftrightarrow P_e(\overline{h}(x))), \\
\quad (La) & \forall x (P_{k(e)}(x) \leftrightarrow \neg P_e(\overline{k}(x))), \\
\quad (Jh) & \forall x (P_{f(e_1, \dots, e_p)}(x) \leftrightarrow \exists x_1, \dots, x_p (\bigwedge_{i=1}^p P_{e_i}(x_i) \wedge \overline{f}(x_1, \dots, x_p, x))), \\
\quad (Mh) & \forall x (P_{g(e_1, \dots, e_p)}(x) \leftrightarrow \forall x_1, \dots, x_p (\overline{g}(x_1, \dots, x_p, x) \rightarrow (\bigvee_{i=1}^p P_{e_i}(x_i)))), \\
\quad (Ja) & \forall x (P_{f(e_1, \dots, e_p)}(x) \leftrightarrow \exists x_1, \dots, x_p (\bigwedge_{i=1}^p \neg P_{e_i}(x_i) \wedge \overline{f}(x_1, \dots, x_p, x))), \\
\quad (Ma) & \forall x (P_{g(e_1, \dots, e_p)}(x) \leftrightarrow \forall x_1, \dots, x_p (\overline{g}(x_1, \dots, x_p, x) \rightarrow (\bigvee_{i=1}^p \neg P_{e_i}(x_i)))), \\
\quad (\Rightarrow) & \forall x (P_{e_1 \Rightarrow e_2}(x) \leftrightarrow \forall y (x \leq y \wedge P_{e_1}(y) \rightarrow P_{e_2}(y))), \\
\quad (\neg) & \forall x (P_{\neg e}(x) \leftrightarrow \forall y (x \leq y \rightarrow \neg P_e(y))), \\
(\mathsf{P}) & \forall x (\bigwedge_{i=1}^n P_{s_{i1}}(x) \leftrightarrow P_{s_{i2}}(x)), \\
(\mathsf{N_1}) & \exists x_1 P_{t_{11}}(x_1) \not\leftrightarrow P_{t_{12}}(x_1), \\
\dots & \dots \\
(\mathsf{N_m}) & \exists x_m P_{t_{m1}}(x_m) \not\leftrightarrow P_{t_{m2}}(x_m),
\end{cases}
$$

where the unary predicates P_e are indexed by elements in $ST(\phi)$.

Proof: *(Sketch)* By Theorem 2, $\mathcal{V} \models \phi$ iff for every $X \in \mathcal{K}$ and every $m : \{x_1, \dots, x_k\} \rightarrow \mathcal{H}(X)$, $\mathcal{H}(X) \models_m \phi$. The conclusion now follows from the fact the set of formulae $(\mathsf{Dom}) \cup (\mathsf{Her}) \cup (\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N_1}) \cup \dots \cup (\mathsf{N_m})$ is satisfiable iff there exists $X \in \mathcal{K}$ and $m : \{x_1, \dots, x_k\} \rightarrow \mathcal{H}(X)$ such that $\mathcal{H}(X) \not\models_m \phi$. $\square$

The problem of deciding whether a universal formula is true in a variety $\mathcal{V}$ can be reduced to deciding whether a set of clauses corresponding to the conjunction in Theorem 3 is unsatisfiable. In what follows we show that ordered chaining with selection gives a decision procedure in the case when $\mathcal{V}$ is the variety DLO_Σ, the variety of distributive p-lattices or that of Heyting algebras.

[1] The set C contains formulae expressing the properties of $\leq$ (such as reflexivity and transitivity), monotonicity properties of the functions and relations in Σ, as well as the possible interdependence between the functions and relations in $\Sigma \cup \{\leq\}$

3.4 DLO_Σ: Decidability and complexity results

Let now $\mathcal{V} = \mathsf{DLO}_\Sigma$. From the results on Priestley duality for DLO_Σ and by Theorem 1 it follows that DLO_Σ satisfies condition (K) where $\mathcal{K} = RTS_\Sigma$. This class is defined by a set RT of formulae expressing the reflexivity and transitivity of $\leq$, together with the set C_Σ of formulae, corresponding to the fact that in every structure in RTS_Σ the functions in Lh preserve $\leq$, those in La reverse $\leq$, the relations in $Jh \cup Ja$ are increasing and those in $Mh \cup Ma$ are decreasing:

C_{Lh}	$\forall x, y (x \leq y \rightarrow \overline{h}(x) \leq \overline{h}(y))$	$h \in Lh$,
C_{La}	$\forall x, y (x \leq y \rightarrow \overline{k}(y) \leq \overline{k}(x))$	$k \in La$,
$C_{Jh,Ja}$	$\forall x_1, \dots, x_p, x, y (x \leq y \wedge \overline{f}(x_1, \dots, x_p, x) \rightarrow \overline{f}(x_1, \dots, x_p, y))$	$f \in Jh \cup Ja$,
$C_{Mh,Ma}$	$\forall x_1, \dots, x_p, x, y (y \leq x \wedge \overline{g}(x_1, \dots, x_p, x) \rightarrow \overline{g}(x_1, \dots, x_p, y))$	$g \in Mh \cup Ma$.

The set $\mathcal{C}_\Sigma(\phi)$ of clauses generated by translating the conjunction in Theorem 3 to clause form is indicated below. (Note that $|\mathcal{C}_\Sigma(\phi)| = \mathcal{O}(\mathsf{length}(\phi))$.)

(Dom) clause form of the formulae in C_Σ,
(RT) clause form of the reflexivity and transitivity axioms,
(Her) $\{\neg x \leq y, \neg P_e(x), P_e(y)\}$,
(Ren)
(1,0) $\{P_1(x)\}, \{\neg P_0(x)\}$,
($\wedge$) $\{\neg P_{e_1 \wedge e_2}(x), P_{e_1}(x)\}, \{\neg P_{e_1 \wedge e_2}(x), P_{e_2}(x)\}, \{\neg P_{e_1}(x), \neg P_{e_2}(x), P_{e_1 \wedge e_2}(x)\}$,
($\vee$) $\{\neg P_{e_1 \vee e_2}(x), P_{e_1}(x), P_{e_2}(x)\}, \{\neg P_{e_1}(x), P_{e_1 \vee e_2}(x)\}, \{\neg P_{e_2}(x), P_{e_1 \vee e_2}(x)\}$,
(Lh) $\{\neg P_{h(e)}(x), P_e(\overline{h}(x))\}, \{P_{h(e)}(x), \neg P_e(\overline{h}(x))\}$,
(La) $\{P_{k(e)}(x), P_e(\overline{k}(x))\}, \{\neg P_{k(e)}(x), \neg P_e(\overline{k}(x))\}$,
(Jh_1) $\{\neg P_{f(e_1,\dots,e_p)}(x), P_{e_i}(c_i^{f(e_1,\dots,e_p)}(x))\}, i = 1, \dots, p$,
(Jh_2) $\{\neg P_{f(e_1,\dots,e_p)}(x), \overline{f}(c_1^{f(e_1,\dots,e_p)}(x), \dots, c_p^{f(e_1,\dots,e_p)}(x), x)\}$,
(Jh_3) $\{P_{f(e_1,\dots,e_p)}(x), \neg P_{e_1}(y_1), \dots, \neg P_{e_p}(y_p), \neg \overline{f}(y_1, \dots, y_p, x)\}$,
(Mh_1) $\{P_{g(e_1,\dots,e_p)}(x), \neg P_{e_i}(c_i^{g(e_1,\dots,e_p)}(x))\}, i = 1, \dots, p$,
(Mh_2) $\{P_{g(e_1,\dots,e_p)}(x), \overline{g}(c_1^{g(e_1,\dots,e_p)}(x), \dots, c_p^{g(e_1,\dots,e_p)}(x), x)\}$,
(Mh_3) $\{\neg P_{g(e_1,\dots,e_p)}(x), P_{e_1}(y_1), \dots, P_{e_p}(y_p), \neg \overline{g}(y_1, \dots, y_p, x)\}$,
(Ja_1) $\{\neg P_{f(e_1,\dots,e_p)}(x), \neg P_{e_i}(c_i^{f(e_1,\dots,e_p)}(x))\}, i = 1, \dots, p$,
(Ja_2) $\{\neg P_{f(e_1,\dots,e_p)}(x), \overline{f}(c_1^{f(e_1,\dots,e_p)}(x), \dots, c_p^{f(e_1,\dots,e_p)}(x), x)\}$,
(Ja_3) $\{P_{f(e_1,\dots,e_p)}(x), P_{e_1}(y_1), \dots, P_{e_p}(y_p), \neg \overline{f}(y_1, \dots, y_p, x)\}$,
(Ma_1) $\{P_{g(e_1,\dots,e_p)}(x), P_{e_i}(c_i^{g(e_1,\dots,e_p)}(x))\}, i = 1, \dots, p$,
(Ma_2) $\{P_{g(e_1,\dots,e_p)}(x), \overline{g}(c_1^{g(e_1,\dots,e_p)}(x), \dots, c_p^{g(e_1,\dots,e_p)}(x), x)\}$,
(Ma_3) $\{\neg P_{g(e_1,\dots,e_p)}(x), \neg P_{e_1}(y_1), \dots, \neg P_{e_p}(y_p), \neg \overline{g}(y_1, \dots, y_p, x)\}$,
(P) $\{\neg P_{s_{i1}}(x), P_{s_{i2}}(x)\}, \{P_{s_{i1}}(x), \neg P_{s_{i2}}(x)\}, i = 1, \dots, n$,
(N) $\{P_{t_{j1}}(c_j), P_{t_{j2}}(c_j)\}, \{\neg P_{t_{j1}}(c_j), \neg P_{t_{j2}}(c_j)\}, j = 1, \dots, m$,

where the predicate symbols P_e are indexed by subterms in $ST(\phi)$, $c_i^{f(e_1,\dots,e_p)}$ are Skolem functions obtained from the existential quantifiers in the transformation of terms of the form $f(e_1, \dots, e_p)$, where $p = a(f)$; $c_1, \dots, c_m$ are Skolem constants introduced by the existential quantifiers in $(\mathsf{N_1}), \dots, (\mathsf{N_m})$ in Theorem 3; and $\overline{f}, \overline{g}$ for $f \in Jh \cup Ja, g \in Mh \cup Ma$ are also considered predicate symbols.

The following result is a direct consequence of Theorem 3.

Corollary 1. *$\mathsf{DLO}_\Sigma \models \phi$ iff $\mathcal{C}_\Sigma(\phi)$ is unsatisfiable.*

We now show that ordered chaining with selection is a decision procedure for $\mathcal{C}_\Sigma(\phi)$. We assume given a reduction ordering $\succ$ which is total on ground terms. Based on $\succ$, an ordering on literals (also denoted by $\succ$) will be defined. Let c be the complexity measure defined for every ground literal L by $c_L = (\mathsf{max}_L, p_L, s_L)$ where max_L is the maximal term occurring in L, p_L is 1 if L is negative and 0 if L is positive, and s_L is 1 if L is of the form $(\neg)s \leq t$ with $s \succ t$, and 0 otherwise. (The choice of c_L was inspired by [8].) c induces a well-founded ordering $\succ_c$ on ground literals, defined by $L \succ_c L'$ iff $c_L > c_{L'}$ (in the lexicographic combination of $\succ$ and $>$, where $1 > 0$). Let $\succ$ be a total and well-founded extension of $\succ_c$. (Such an ordering is left-to-right admissible in the sense used in [2].) Let S be the selection function that selects (i) all negative occurrences of literals containing $\leq$, and (ii) all occurrences of negative literals containing a predicate symbol in $Jh \cup \dots \cup Ma$ in clauses which do not contain $\leq$. The chaining calculus based on the literal ordering $\succ$ and the selection function S will be denoted $\mathsf{C}_S^{\succ}$.

Theorem 4. $\mathsf{C}_S^{\succ}$ *decides the unsatisfiability of* $\mathcal{C}_\Sigma(\phi)$ *in exponential time.*

Proof: *(Sketch)* It can be shown that, due to ordering constraints and the choice of S, no $\mathsf{C}_S^{\succ}$ inferences between clauses in $(\mathsf{RT}) \cup (\mathsf{Her})$ and clauses in $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ are possible, and all clauses obtained by $\mathsf{C}_S^{\succ}$ inferences from $(\mathsf{RT}) \cup (\mathsf{Her})$ are redundant. Using the definition of $\succ$ on literals, it can be shown that all clauses obtained by ordered resolution with selection from $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ have term depth 1 and either (i) are ground (and contain only one constant), or (ii) contain only one variable (occurring in every literal) and no constant or, (iii) are factors of $(Jh_3), (Ja_3), (Mh_3)$ or (Ma_3). Moreover, all negative occurrences of a predicate symbol in $Jh \cup Ja \cup Mh \cup Ma$ must occur in clauses of type (iii). Due to the definition of $\succ$, neither the term depth of clauses nor the number of variables in the clause increase by ordered resolution. For every constant c_i (resp. every variable x) the number of all possible atoms for the clauses containing c_i (resp. x) and of term depth at most 1 is $ns \cdot (mp \cdot ns + nf + 1) + np \cdot (mp \cdot ns + nf + 1)^{mp+1}$ (ns resp. np is the number of all unary, resp. at most mp-ary predicate symbols; among the function symbols one also has to count the (unary) Skolem functions associated to the subterms in $ST(\phi)$, of which there are at most $mp \cdot ns$). This shows that, assuming np, nf, and mp are constant, the number clauses that can be generated by ordered resolution with selection from $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ is of the order $3^{\mathcal{O}(ns^{m+1})}$. $\square$

Remark. The above proof shows that the clauses containing the predicate symbol $\leq$ are not needed in order to prove unsatisfiability of $\mathcal{C}_\Sigma(\phi)$. The reason is that every algebra in DLO_Σ is a sublattice of a lattice whose Priestley dual has the discrete ordering, i.e. $\mathsf{DLO}_\Sigma = IS(\{L \in \mathsf{DLO}_\Sigma \mid D(L) \text{ discretely ordered}\})$, and, hence, a universal formulae is valid in DLO_Σ iff it is valid in every algebra in DLO_Σ whose dual is discretely ordered. All varieties in this subsection have this property; in Section 3.5 we discuss two varieties which do not have this property, i.e. for which $\leq$ has to be explicitly taken into account.

Example 1: The variety D_{01} of bounded distributive lattices. Let $\Sigma = \emptyset$. In this case $\mathsf{DLO}_\Sigma = \mathsf{D}_{01}$. The considerations above show that D_{01} fulfills condition (K), $\mathcal{K}$ being the class RTS of all sets endowed with a reflexive and transitive relation. In the translation to clause form only the set $\mathcal{C}(\phi) = (\mathsf{RT}) \cup (\mathsf{Her}) \cup (\mathsf{Ren})(0,1) \cup (\mathsf{Ren})(\wedge) \cup (\mathsf{Ren})(\vee) \cup (\mathsf{P}) \cup (\mathsf{N})$ of clauses needs to be taken into account. (In this case $(\mathsf{Ren}) = (\mathsf{Ren})(0,1) \cup (\mathsf{Ren})(\wedge) \cup (\mathsf{Ren})(\vee)$.)

The results in Theorem 4 can be sharpened in this case. Due to the special form of the clauses in $\mathcal{C}(\phi)$, all possible resolvents are either ground and all literals contain the same constant, or all their literals contain the same variable (and no constant), and, additionally, the term depth of all clauses is 0. Thus, only at most $(m+1) \cdot 3^{ns}$ clauses can be generated in this case.

From the special form of the clauses in $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ it follows that if $\mathcal{C}(\phi)$ is satisfiable, then it is satisfied by a model with m points, namely $\{c_1, \dots, c_m\}$. Moreover, $\mathcal{C}(\phi)$ is satisfiable iff there exists a $j \leq m$ such that $\mathcal{C}(\phi_j)$ (obtained from $\mathcal{C}(\phi)$ by only keeping the clauses containing c_j in (N)) is satisfied by the one point model $\{c_j\}$. This is explained by the fact that $\mathsf{D}_{01} = ISP(2)$ (the quasi-variety generated by the 2-element lattice), hence, every conditional equation is true in D_{01} iff it is true in the 2-element lattice whose Priestley dual has one element. Since D_{01} is closed under direct products, it follows [13] that $\mathsf{D}_{01} \models \phi$ iff there exists a j such that $\mathsf{D}_{01} \models \forall x_1, \dots \forall x_k (\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow t_{j1} = t_{j2})$ iff there exists a j such that $2 \models \forall x_1, \dots \forall x_k (\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow t_{j1} = t_{j2})$ iff $2^m \models \phi$. Thus, a universal formula ϕ is true in D_{01} iff it is true in 2^m, a distributive lattice whose Priestley dual has m elements and is discretely ordered.

Example 2: Bounded distributive lattices with lattice (anti)morphisms. The arguments in Theorem 4 can be adapted to bounded distributive lattices endowed with (anti)morphisms. All clauses in $(\mathsf{Ren})(0,1) \cup (\mathsf{Ren})(\wedge) \cup (\mathsf{Ren})(\vee) \cup (\mathsf{Ren})(Lh) \cup (\mathsf{Ren})(La)$ and all possible resolvents have depth at most 1 and are either ground (and all literals contain the same constant) or have exactly one variable (occurring in all literals). The number of all function symbols is in this case nf (no Skolem functions occur). Therefore, at most $(m+1) \cdot 3^{ns \cdot (nf+1)}$ different clauses can be generated.

The fact that a universal formulae is valid in DLO_Σ iff it is valid in every algebra in DLO_Σ whose dual is discretely ordered, opens the way for further results.

Proposition 1. *The satisfiability problem for $\phi = \forall x_1, \dots, x_k (\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^m t_{j1} = t_{j2})$ can be reduced to the satisfiability problem for the monadic class with equality in polynomial time w.r.t. the length of ϕ.*

Proof: *(Sketch)* The clauses in $(\mathsf{Ren}) \cup (\mathsf{P}) \cup (\mathsf{N})$ can be brought to the form of *flat clauses* considered in [3]. This can be done in the following steps:

1. Replace every occurrence of a literal of the form $\overline{f}(t_1, \dots, t_p)$ or $\neg \overline{f}(t_1, \dots, t_p)$ with $\overline{f}(t_1, \dots, t_p) = \top$, resp. $\overline{f}(t_1, \dots, t_p) = \perp$, $f \in Jh \cup \dots \cup Ma$.
   Thus, the relation symbols in $Jh \cup Ja \cup Mh \cup Ma$ are interpreted as function symbols of a different sort (sorts can be represented by unary predicates).
2. Use variable abstraction for the clauses in $J = (Jh_2) \cup (Ja_2)$ and $M = (Mh_2) \cup (Ma_2)$, to bring them in the following form:

(J') $\{\neg P_{f(e_1,\dots,e_p)}(x), y_1 \neq c_1^{f(e_1,\dots,e_p)}(x), \dots, y_p \neq c_p^{f(e_1,\dots,e_p)}(x), \overline{f}(y_1, \dots, y_p, x) = \top\}$

(M') $\{P_{g(e_1,\dots,e_p)}(x), y_1 \neq c_1^{g(e_1,\dots,e_p)}(x), \dots, y_p \neq c_p^{g(e_1,\dots,e_p)}(x), \overline{g}(y_1, \dots, y_p, x) = \top\}$

The set of clauses obtained this way can be regarded as the result of skolemizing a formula $\overline{\phi}$ (in prenex form) in the monadic class with equality. The translation to clause form, the procedure above, and $\mathsf{length}(\overline{\phi})$ are polynomial w.r.t. $\mathsf{length}(\phi)$. □

Superposition with simplification is a decision procedure for the monadic class with equality [3]. The reduction to the monadic class with equality also offers decidability and complexity results for those subvarieties of DLO_Σ in which (i) the conditions in (Dom) are either (a) in C_Σ or (b) expressible in the monadic class with equality, and (ii) in case (b), only $=$ and the predicate symbols corresponding to relations in $Jh \cup \dots \cup Ma$ may occur. An upper bound for the decision problem for the monadic class with equality is NEXPTIME (cf. e.g. [3]). This gives an upper bound for the complexity of the universal Horn theory of such varieties.

3.5 Distributive p-lattices and Heyting algebras

Let B_ω be the variety of distributive p-lattices, and let H be the variety of Heyting algebras. From the Priestley duality for distributive p-lattices and Heyting algebras and from Theorem 1 it follows that both B_ω and H fulfill condition (K), with $\mathcal{K} = RTS$, i.e. (i) for every $A \in \mathsf{B}_\omega$ or H, $D(A) \in RTS$ (if the topology is ignored); and (ii) for every $(X, \leq) \in RTS$, $(\mathcal{H}(X), \cup, \cap, \neg, \emptyset, X) \in \mathsf{B}_\omega$ and $(\mathcal{H}(X), \cup, \cap, \Rightarrow, \neg, \emptyset, X) \in \mathsf{H}$, where $\neg$ and $\Rightarrow$ are as defined in Theorem 1(5).

Let $\phi = \forall x_1, \dots, x_k (\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow \bigvee_{j=1}^m t_{j1} = t_{j2})$. We reduce the problem of deciding whether $\mathcal{V} \models \phi$ to a problem solved in [8]. By the result of McKinsey mentioned before, $\mathcal{V} \models \phi$ iff $\mathcal{V} \models \phi_j$ for some j, where $\phi_j = \forall x_1 \dots x_k (\bigwedge_{i=1}^n s_{i1} = s_{i2} \rightarrow t_{j1} = t_{j2})$. So the problem of deciding $\mathcal{V} \models \phi$ reduces to deciding $\mathcal{V} \models \phi_j$ for $j = 1, \dots, m$. By Theorem 3, $\mathcal{V} \models \phi_j$ iff the set of clauses $\mathcal{C}(\phi_j)$ is unsatisfiable, where $\mathcal{C}(\phi_j)$ is obtained by adjoining to $(\mathsf{RT}) \cup (\mathsf{Her}) \cup (\mathsf{Ren})(\wedge) \cup (\mathsf{Ren})(\vee) \cup (\mathsf{P}) \cup (\mathsf{N})_\mathsf{j}$ the clauses corresponding to $(\mathsf{Ren})(\neg)$ if $\mathcal{V} = \mathsf{B}_\omega$, respectively to $(\mathsf{Ren})(\neg)$ and $(\mathsf{Ren})(\Rightarrow)$ if $\mathcal{V} = \mathsf{H}$ (where $(\mathsf{Ren})(\neg)$ and $(\mathsf{Ren})(\Rightarrow)$ are as in Theorem 3, and $(\mathsf{N})_\mathsf{j}$ is (N) for c_j only).

Let $\mathsf{C}_S^\succ$ be the chaining calculus where $\succ$ is a total, well-founded ordering on ground literals compatible with the complexity measure c_L defined in Section 3.4 (hence left-to-right admissible [2]), and, if a clause C contains a literal of the form $\neg s \leq t$ with $s \succeq t$, the selection function S selects one such literal.

Theorem 5. *For every $j = 1, \dots, m$, $\mathsf{C}_S^\succ$ (with eager condensation) decides the unsatisfiability of $\mathcal{C}(\phi_j)$.*

Proof: *(Sketch)* The set $\mathcal{C}(\phi_j)$ is in the class of clauses considered in [8]. There it is proved that $\mathsf{C}_S^\succ$ with eager condensation is a decision procedure for this kind of clauses. (We use the fact that $\mathcal{C}(\phi_j)$ has one constant; if $m > 1$, the existence of m constants may cause problems in adapting Lemma 2 in [8].) The complexity of the method is doubly exponential; a single-exponential space complexity can be obtained by splitting the clauses into their variable-disjoint regions and connecting them with the help of auxiliary monadic predicates as pointed out in [8]. □

4 Experiments

We present some concrete, relatively simple examples which illustrate the type of problems that can be solved with the method described in this paper (**RTS**), and the way this method compares to a more standard approach, (**DLat**), that proves that the conjunction of the negation of the formulae above and the axioms for bounded distributive lattices with operators is unsatisfiable (in first-order logic with equality). We considered the following formulae:

- $\phi_1 = \forall a \forall b \forall c\, (a \leq b \rightarrow a \vee (c \wedge b) = (a \vee c) \wedge b)$,
- $\phi_2 = \forall a \forall b \forall c\, ((a \wedge b = c \wedge b \,\&\, a \vee b = c \vee b) \rightarrow a = c)$,
- $\phi_3 = \forall a \forall b \forall c\, ((k^2(a) \leq a \vee k(a) \,\&\, k^3(b) = a \vee k(a) \,\&\, k^2(a) \leq k(a) \vee k(b) \vee k(c) \,\&\, k^3(b) \leq k(a) \vee k(b) \vee k(c)) \rightarrow k^2(a \vee k(b)) \leq (a \wedge k(b \wedge c)) \vee k(a))$, $k \in La$,
- $\phi_4 = \forall a \forall b\, f(k(a \vee b)) = f(k(a)) \vee f(k(b))$, where $f \in Ja$ and $k \in La$,
- $\phi_5 = \forall a \forall b \forall c \forall d\, ((f(a \vee b, d) = f(c \vee b, d) \,\&\, f(a, d) \wedge f(b, d) = f(c, d) \wedge f(b, d)) \rightarrow f(a, d) = f(c, d))$, where $f \in Jh$.

The translation to clause form in **RTS** used the results in Theorem 2 and Theorem 3. According to the proof of Theorem 4, all clauses containing $\leq$ were ignored. In addition, to reduce the number of clauses generated, an inequality $a \leq b$ was directly replaced by $\forall x(P_a(x) \rightarrow P_b(x))$. In **DLat** we experimented with various axioms for distributivity, namely (j) joins over meets, (m) meets over joins, and (b) both. The unsatisfiability of the resulting sets of clauses was checked by SPASS [23]. In both cases we indicate the number of input and derived clauses, memory and time needed by SPASS V0.92 (on a 200 MHz Pentium Pro).

Formula	**Variety**	**RTS** # Cl (in)	# Cl (der)	Mem (KB)	Time (ms)	**DLat**	# Cl (in)	# Cl (der)	Mem (KB)	Time (ms)
ϕ_1	D_{01}	15	19	436	30	j	13	1	382	20
						m	13	354	590	180
						b	14	465	633	230
ϕ_2	D_{01}	18	31	421	20	j	13	2	383	30
						m	14	3347	2388	4770
						b	15	5533	3609	11990
ϕ_3	DLO_{La}	43	28	448	30	j	20	4042	3532	10860
						m	20			∞
						b	21			∞
ϕ_4	$\mathsf{DLO}_{La,Ja}$	23	44	450	80	j	18	1	386	20
						m	18	1	386	20
						b	19	0	385	10
ϕ_5	DLO_{Jh}	38	72	470	90	j	18	5703	4922	16150
						m	18	5341	4541	15100
						b	19			∞

#Cl (in) resp. (der) represents the number of input, resp. derived clauses, and ∞ indicates the fact that execution did not terminate after more than 3 min.

The results above suggest that, except for very regular and simple formulae, or for purely equational formulae, the first method, based on results presented in this paper, behaves better than the second. In the future we plan to analyze more complex examples. We would be also interested to compare the theoretical complexity of our method with that of other methods.

5 Conclusions and plans for future work

In this paper we presented a resolution-based method for automated theorem proving in the universal theory of certain varieties of distributive lattices with operators. The method is based on extensions of the Priestley representation theorem to distributive lattices with operators. Based on it, we obtained decidability and complexity results (upper bounds) for the universal word problem of $\mathsf{D}_{01}, \mathsf{DLO}_{\Sigma}$, and for the variety of distributive p-algebras and that of Heyting algebras. The complexity results agree with those established for (boolean) Tarskian set constraints without functions in [12], but the methods we use are different. The fact that the same type of structures are used as relational models for distributive lattices, distributive p-lattices and Heyting algebras (the only difference is the signature) shows that the restriction of the universal theory of Heyting algebras (or distributive p-lattices) to the signature $\{0, 1, \vee, \wedge\}$ coincides with the universal theory of distributive lattices. This remark is consistent with the remarks in [21] on the similarity of the cut rules necessary for the calculus for distributive lattices developed there and the cut rules in intuitionistic logic.

By analyzing the possible inferences in a suitably chosen ordered chaining calculus, we obtained a better understanding of the structure of such varieties.

These results seem to open a promising field of research that we would like to explore in future work. We expect to be able to use similar ideas for other varieties of distributive lattices or Heyting algebras with operators. One problem to be solved is to find conditions for such varieties that would give decidability results. It would also be important to find conditions which, given a variety $\mathcal{V}$ of distributive lattices with operators, ensure that a class $\mathcal{K}$ of (first-order definable) relational structures can be found, such that condition (K) is satisfied.

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