Representation Theorems and Theorem Proving in Non-Classical Logics

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Abstract

In this paper we present a method for automated theorem proving in non-classical logics having as algebraic models bounded distributive lattices with certain types of operators. The idea is to use a Priestley-style representation for distributive lattices with operators in order to define a class of Kripke-style models with respect to which the logic is sound and complete. If this class of Kripke-style models is elementary, it can then be used for a translation to clause form; satisfiability of the resulting clauses can be checked by resolution. We illustrate the ideas by several examples.

1 Introduction

Efficient reasoning on incomplete, vague and imprecise knowledge requires the development of efficient many-valued theorem provers. Since many non-classical logics that occur in a natural way in practical applications can be proved to be sound and complete with respect to certain classes of distributive lattices with operators, in this paper we will focus on this kind of logics. One of the main advantages of distributive lattices (with well-behaved operators) is that they usually have good and economical representation theorems. The method for automated theorem proving presented here uses the Priestley representation for distributive lattices with operators. We illustrate the main ideas by several examples, one of which is presented in detail.

The main contributions of this paper are the following:

- A link between algebraic and Kripke-style models for logics based on distributive lattices with operators, established by using Priestley-style representation theorems. This extends the results in [21], and makes more precise some results from [24].
- A method for translation to clause form in logics based on distributive lattices with operators.
- Decidability results for some of the logics considered.

Our work is inspired by the work of Baaz and Fermüller [1] on many-valued resolution, by the results of Hähnle [7, 8], and by our previous work on finitely-valued logics [21] on the one hand; and by the results of Goldblatt [5] on Priestley-type dualities for distributive algebras with operators on the other hand. The method presented in this paper can be seen as a common framework which encompasses existing results in automated theorem proving in both regular finitely-valued logics, and modal logics.

The paper is structured as follows. In Section 2 we give some background information. In Section 3 we briefly present a Priestley representation for distributive lattices with operators, and its use in establishing links between algebraic and Kripke-style models. In Section 4 we describe a method for translation to clause form and automated theorem proving. In Section 5 we give several examples, one of which is presented in detail. We end by mentioning related approaches and plans for future work.

2 Preliminaries

Algebra. In what follows we assume known standard notions, such as partially-ordered set, lattice, order-filter and order-ideal in partially-ordered sets, meet- and join-irreducible elements, and (prime) filter and (prime) ideal in lattices. For definitions and further information we refer to [3]. Let $(X, \leq)$ be a partially ordered set and let $R \subseteq X^{n+1}$ be a $(n+1)$-ary relation on X. R is an *increasing relation* if it has the property that for all $\overline{x} \in X^n$ and every $y, z \in X$, if $R(\overline{x}, y)$ and $y \leq z$ then $R(\overline{x}, z)$. R is a *decreasing relation* if for all $\overline{x} \in X^n$ and every $y, z \in X$, if $R(\overline{x}, y)$ and $z \leq y$ then $R(\overline{x}, z)$. Given a partially-ordered set $(X, \leq)$, $\mathcal{O}(X)$ will denote the set of order-filters of X. For the basic notions of universal algebra needed in what follows, such as satisfiability of equations in an algebra or a class of algebras (notation: $A \overset{a}{\models} t_1 = t_2$ resp. $\mathcal{K} \overset{a}{\models} t_1 = t_2$) and the definition of a variety, we refer to [2]. The variety generated by a class $\mathcal{K}$ of algebras will be denoted $HSP(\mathcal{K})$.

Logic. Roughly speaking, a logic is described by a pair $\mathcal{L} = (\mathsf{Fma}(\mathcal{L}), \vdash_{\mathcal{L}})$, where $\mathsf{Fma}(\mathcal{L})$ is the set of formulae in the logic and $\vdash_{\mathcal{L}}$ is a relation between sets of formulae and formulae called *provability relation* (for simplicity we only consider here the propositional case). A formula ϕ is a *theorem* of $\mathcal{L}$ (denoted $\vdash_{\mathcal{L}} \phi$) if $\emptyset \vdash_{\mathcal{L}} \phi$. We call a logic $\mathcal{L}$ *sound* w.r.t. a class $\mathcal{K}$ of models if every theorem ϕ of $\mathcal{L}$ is true in all models in $\mathcal{K}$; we call $\mathcal{L}$ *complete* w.r.t. $\mathcal{K}$ if every formula ϕ that is true in all models in $\mathcal{K}$ is a theorem of $\mathcal{L}$.

3 Priestley representation

We first present some results on Priestley representation for distributive lattices with operators (abbreviated in what follows DLO). For details we refer to [3, 5, 21, 24].

The Priestley representation theorem [17] states that every bounded distributive lattice A is isomorphic to the lattice of clopen (i.e. closed and open) order filters of the ordered topological space having as points the prime filters of A, ordered by inclusion, and the topology generated by the sets of the form $X_a = \{F \mid F \text{ prime filter}, a \in F\}$ and their complements as a subbasis. We denote the partially ordered set of all prime filters of A, ordered by inclusion, and endowed with the topology mentioned above, by $D(A)$ (we will refer to it as the dual of A). The lattice of clopen order filters of an ordered topological space $X = (X, \leq, \tau)$ will be denoted by $E(X)$. The Priestley representation theorem states that every bounded distributive lattice A is isomorphic to $E(D(A))$.

In this paper we will also consider other classes of operators on bounded distributive lattices, besides $\vee$ and $\wedge$, as explained in what follows. Let Σ be a signature containing function symbols in several classes. In order to distinguish these classes, we will write $\Sigma = Lh \cup La \cup Jh \cup Mh \cup Ja \cup Ma$, where Lh, La, Jh, Mh, Ja, and Ma may be empty. Let $\mathsf{DLO}_{\mathbf{\Sigma}}$ be the class of all bounded distributive lattices with operators in Σ, such that for every $A \in \mathsf{DLO}_{\mathbf{\Sigma}}$ the operators in the classes Lh, La, Jh, Mh, Ja, and Ma are respectively lattice morphisms and antimorphisms, join and meet hemimorphisms (maps of arbitrary arity, preserving finite joins resp. meets in every argument), and join resp. meet hemiantimorphisms (maps of arbitrary arity, mapping finite meets to joins (resp. joins to meets) in every argument). The main fact proved in [24] is that the additional operators in Σ on an algebra $A \in \mathsf{DLO}_{\Sigma}$ induce functions resp. relations on $D(A)$, which, again, define corresponding operators on both the set $E(D(A))$ of clopen order-filters of $D(A)$ and on the set $\mathcal{O}(D(A))$ of order-filters of $D(A)$. For details on a Priestley duality theorem for DLO we refer to [24]. The main result we will use here is that for every $A \in \mathsf{DLO}_{\Sigma}$, there exists an isomorphism of $\{0, 1, \vee, \wedge\} \cup \Sigma$-algebras $\eta_A : A \simeq E(D(A))$. Note that if A is finite then the Priestley topology on $D(A)$ is discrete, hence $E(D(A)) = \mathcal{O}(D(A))$.

In [21, 23] we used this Priestley representation for DLO to give an automated theorem proving procedure for certain finitely-valued logics. In this paper we show that the ideas are much more general, and can also be applied, with minor modifications, to wider classes of non-classical logics.

3.1 Kripke-style models

In [24], inspired by the definition of the relational Priestley spaces, we introduced *ordered relational structures*.

Definition 1 *An ordered Σ-relational structure is a partially ordered set endowed with maps and relations*

$$(X, \leq, \{H_X\}_{H \in Lh \cup La}, \{R_X\}_{R \in Jh \cup Ja}, \{Q_X\}_{Q \in Mh \cup Ma})$$

where for every $H \in Lh$, $H_X : X \to X$ is order-preserving, for every $K \in La$, $K_X : X \to X$ is order-reversing, for every $R \in Jh \cup Ja$ with arity n, $R_X \subseteq X^{n+1}$ is an increasing relation, and for every $Q \in Mh \cup Ma$ with arity n, $Q_X \subseteq X^{n+1}$ is a decreasing relation.

The class of all ordered Σ-relational structures will be denoted $\mathsf{ORS}_{\mathbf{\Sigma}}$. For every ordered Σ-relational structure X, its set $\mathcal{O}(X)$ of order-filters can be endowed with a bounded lattice structure (where join is union, meet is intersection, $0 = \emptyset$ and $1 = X$) and with additional operators which can be defined in a canonical way starting from the maps and relations on X as shown in what follows.

Theorem 1 ([5, 24]) *Let $X \in \mathsf{ORS}_{\mathbf{\Sigma}}$. The following hold.*

1. *Every order preserving (resp. order reversing) operation on X, $H \in Lh$ (resp. $K \in La$) induces a lattice morphism (resp. lattice antimorphism) $h_H : \mathcal{O}(X) \to \mathcal{O}(X)$, (resp. $k_K : \mathcal{O}(X) \to \mathcal{O}(X)$) defined for every $U \in \mathcal{O}(X)$ by $h_H(U) = H_X^{-1}(U)$, (resp. $k_K(U) = X \backslash K_X^{-1}(U)$).*

2. *Every increasing relation $R_X \subseteq X^{n+1}$, $R \in Jh$ (resp. $R'_X \subseteq X^{n+1}$, with $R' \in Ja$), induces a join hemimorphism $f_R : \mathcal{O}(X)^n \to \mathcal{O}(X)$ (resp. join hemiantimorphism $f_{R'} : \mathcal{O}(X)^n \to \mathcal{O}(X)$), defined for every $U_1, \dots, U_n \in \mathcal{O}(X)$ by*

$$(Jh) \quad f_R(U_1, \dots, U_n) = \{x \in X \mid \exists x_1 \in U_1, \dots, x_n \in U_n : R_X(x_1, \dots, x_n, x)\},$$

$$(Ja) \quad f_{R'}(U_1, \dots, U_n) = \{x \in X \mid \exists x_1 \notin U_1, \dots, x_n \notin U_n : R'_X(x_1, \dots, x_n, x)\}.$$

   Every decreasing relation $Q_X \subseteq X^{n+1}$, $Q \in Mh$ (resp. $Q'_X \subseteq X^{n+1}$, with $Q' \in Ma$), induces a meet hemimorphism $g_Q : \mathcal{O}(X)^n \to \mathcal{O}(X)$ (resp. meet hemiantimorphism $g_{Q'} : \mathcal{O}(X)^n \to \mathcal{O}(X)$), defined for every $U_1, \dots, U_n \in \mathcal{O}(X)$ by

$$(Mh) \quad g_Q(U_1, \dots, U_n) = \{x \in X \mid \forall x_1, \dots, x_n (Q_X(x_1, \dots, x_n, x) \to \exists i, x_i \in U_i)\},$$

$$(Ma) \quad g_{Q'}(U_1, \dots, U_n) = \{x \in X \mid \forall x_1, \dots, x_n (Q'_X(x_1, \dots, x_n, x) \to \exists i, x_i \notin U_i)\}.$$

3. *A Heyting algebra structure can be defined on $\mathcal{O}(X)$ by defining for every $U_1, U_2 \in \mathcal{O}(X)$*

$$U_1 \Rightarrow U_2 = \{x \in X \mid \forall y, x \leq y \text{ if } y \in U_1 \text{ then } y \in U_2\}$$

$$\neg U_1 = \{x \in X \mid \forall y, x \leq y \text{ implies } y \notin U_1\}$$

In what follows we consider Kripke-style models for logics $\mathcal{L}$ with logical connectives[1] in $\{\vee, \wedge\} \cup \Sigma$, whose formulae are built from propositional variables taken from a (countable) set Var, and satisfying the following condition:

(L) $\mathcal{L}$ is sound and complete w.r.t. a subvariety $\mathcal{V}_\mathcal{L}$ of DLO_Σ, such that the Priestley duality induces a dual equivalence between $\mathcal{V}_\mathcal{L}$ and a suitable class $\mathcal{V}_\mathcal{L}\mathsf{Sp}$ of relational Priestley spaces.

Definition 2 *$\mathcal{K}_\mathcal{L}$ is the subclass of ORS_Σ consisting of all relational structures that satisfy all the conditions not involving the topology which characterize the class $\mathcal{V}_\mathcal{L}\mathsf{Sp}$.*

The Kripke models for $\mathcal{L}$ we consider here are relational structures[2] $(X, \leq, \{R_X\}_{R\in\Sigma})$ in $\mathcal{K}_\mathcal{L}$, endowed with a hereditary meaning function $m : \mathsf{Var} \to \mathcal{O}(X)$.

Definition 3 *Let $X \in \mathsf{ORS}_\Sigma$, let $m : \mathsf{Var} \to \mathcal{O}(X)$ be a meaning function, $\overline{m} : \mathsf{Fma}(\mathcal{L}) \to \mathcal{O}(X)$ its unique extension to a morphism of algebras, let $x \in X$, and let $\phi \in \mathsf{Fma}(\mathcal{L})$. We define:*

(1) $X \overset{r}{\models}_{m,x} \phi$ iff $x \in \overline{m}(\phi)$;

(2) $X \overset{r}{\models}_{m} \phi$ iff $\overline{m}(\phi) = X$;

(3) $X \overset{r}{\models} \phi$ iff $X \overset{r}{\models}_{m} \phi$ for every $m : \mathsf{Var} \to \mathcal{O}(X)$.

Lemma 2 *For any $X \in \mathsf{ORS}_\Sigma$, $X \overset{r}{\models} \phi$ iff $\mathcal{O}(X) \overset{a}{\models} \phi = 1$.*

Theorem 3 *Let $\mathcal{L}$ be a logic that satisfies condition (L). If for every $X \in \mathcal{K}_\mathcal{L}$, $\mathcal{O}(X) \in \mathcal{V}_\mathcal{L}$, then $\mathcal{L}$ is sound and complete w.r.t. the class $\mathcal{K}_\mathcal{L}$. Assume that $\mathcal{V}_\mathcal{L} = HSP(\mathcal{A})$ and let $\mathcal{K}_\mathcal{A}$ be the class of all ordered relational structures satisfying all non-topological conditions corresponding to the Priestley duals of the elements of $\mathcal{A}$. If for every $X \in \mathcal{K}_\mathcal{A}$, $\mathcal{O}(X) \in \mathcal{V}_\mathcal{L}$, then $\mathcal{L}$ is sound and complete w.r.t. $\mathcal{K}_\mathcal{A}$.*

Proof: Without loss of generality we only prove the second part of the theorem. Let ϕ be a theorem of $\mathcal{L}$. Then $\mathcal{V}_\mathcal{L} \overset{a}{\models} \phi = 1$. Since, by assumption, for every $X \in \mathcal{K}_\mathcal{A}$, $\mathcal{O}(X) \in \mathcal{V}_\mathcal{L}$, it follows that $X \overset{r}{\models} \phi$ for every $X \in \mathcal{K}_\mathcal{A}$. Conversely, assume that for every $X \in \mathcal{K}_\mathcal{A}$, $X \overset{r}{\models} \phi$. Let $A \in \mathcal{A}$ be an algebra, and $f : \mathsf{Var} \to A$ an arbitrary valuation. By the Priestley duality for $\mathcal{V}_\mathcal{L}$, there exists an isomorphism $\eta_A : A \simeq E(D(A))$. Let $\iota : E(D(A)) \to \mathcal{O}(D(A))$ be the inclusion, and let $m : \mathsf{Var} \to \mathcal{O}(D(A))$ be defined by $m = \iota \circ \eta_A \circ f$. It is easy to see that $\overline{m} = \iota \circ \eta_A \circ \overline{f}$. We know that $D(A) \in \mathcal{K}_\mathcal{A}$. Then $\overline{m}(\phi) = D(A)$, hence $\overline{f}(\phi) = 1$. Since $\mathcal{V}_\mathcal{L} = HSP(\mathcal{A})$ and $\mathcal{L}$ is sound and complete w.r.t. $\mathcal{V}_\mathcal{L}$ it follows that $\vdash_\mathcal{L} \phi$. □

Corollary 4 *If $\mathcal{V}_\mathcal{L}$ is finitely generated by the family $\{A_1, \dots, A_n\}$ of finite algebras in $\mathcal{V}_\mathcal{L}$ then $\mathcal{L}$ is sound and complete w.r.t. $\mathcal{K} = \{D(A_1), \dots, D(A_n)\}$.*

A similar idea occurs in the work of Jónsson and Tarski [10] on Boolean algebras with operators. We also analyzed other representation theorems, e.g. for orthomodular lattices, which can be used with similar results. A presentation of these results is beyond the scope of the present paper.

4 Automated theorem proving

Translation to clause form. Let $\mathcal{L}$ be a propositional logic with a set Var of propositional variables, and satisfying condition (L), and let $\mathcal{K}_\mathcal{L}$ be as in Definition 2. In what follows we assume that $\mathcal{L}$ is sound and complete w.r.t. $\mathcal{K}_\mathcal{L}$.

Lemma 5 *For every $(X, \leq, \{R_X\}_{R\in\Sigma}) \in \mathcal{K}_\mathcal{L}$ and every $m : \mathsf{Var} \to \mathcal{O}(X)$; for every $H \in Lh, K \in La, R \in Jh, Q \in Mh, R' \in Ja$ and $Q' \in Ma$ the following hold:*

$$
\begin{array}{lll}
X \overset{r}{\models}_{m,x} p & \text{iff} & x \in m(p) \qquad \text{if } p \in \mathsf{Var} \\
X \overset{r}{\models}_{m,x} \phi_1 \vee \phi_2 & \text{iff} & X \overset{r}{\models}_{m,x} \phi_1 \text{ or } X \overset{r}{\models}_{m,x} \phi_2 \\
X \overset{r}{\models}_{m,x} \phi_1 \wedge \phi_2 & \text{iff} & X \overset{r}{\models}_{m,x} \phi_1 \text{ and } X \overset{r}{\models}_{m,x} \phi_2 \\
X \overset{r}{\models}_{m,x} H(\phi) & \text{iff} & X \overset{r}{\models}_{m,H_X(x)} \phi \\
X \overset{r}{\models}_{m,x} K(\phi) & \text{iff} & X \overset{r}{\not\models}_{m,K_X(x)} \phi \\
X \overset{r}{\models}_{m,x} R(\phi_1, \dots, \phi_n) & \text{iff} & \exists x_1, \dots, x_n (R_X(x_1, \dots, x_n, x), \\
 & & X \overset{r}{\models}_{m,x_1} \phi_1, \dots, X \overset{r}{\models}_{m,x_n} \phi_n) \\
X \overset{r}{\models}_{m,x} Q(\phi_1, \dots, \phi_n) & \text{iff} & \forall x_1, \dots, x_n (Q_X(x_1, \dots, x_n, x) \to \\
 & & \exists i \in \{1, \dots, n\}, X \overset{r}{\models}_{m,x_i} \phi_i) \\
X \models_{m,x} R'(\phi_1, \dots, \phi_n) & \text{iff} & \exists x_1, \dots, x_n (R'_X(x_1, \dots, x_n, x), \\
 & & X \overset{r}{\not\models}_{m,x_1} \phi_1, \dots, X \overset{r}{\not\models}_{m,x_n} \phi_n) \\
X \overset{r}{\models}_{m,x} Q'(\phi_1, \dots, \phi_n) & \text{iff} & \forall x_1, \dots, x_n (Q'_X(x_1, \dots, x_n, x) \to \\
 & & \exists i \in \{1, \dots, n\}, X \overset{r}{\not\models}_{m,x_i} \phi_i)
\end{array}
$$

Proof: *(Sketch)* Follows from the definition of $\overset{r}{\models}$ and from the definitions of the operations on $\mathcal{O}(X)$ induced by operations or relations in Lh, La, Jh, Mh, Ja, Ma, as given in Theorem 1. □

If $\mathcal{K}_\mathcal{L}$ is an elementary class (i.e. it is the class of all ordered Σ-relational structures satisfying a set E of first-order formulae), Lemma 5 can be used in order to define a *structure-preserving* transformation method to clause form. This structure-preserving translation to clause form is inspired by the well-known method from classical logic due to Tseitin: for every subformula ψ of ϕ a new (unary) predicate symbol P_ψ is introduced. The fact that this renaming is correct is expressed by a set (Ren) of formulae.

[1] The connectives in Σ correspond to versions of non-classical negation, or are generalizations of operators from modal logic such as those modeling knowledge, belief, necessity, or possibility.

[2] Further considerations lead to relaxing the condition that $\leq$ is a partial order to $\leq$ being reflexive and transitive. Then $\mathcal{O}(X)$ is the family of all increasing subsets of X w.r.t. this relation.

Proposition 6 *Let $\phi \in \mathsf{Fma}(\mathcal{L})$. Then $\vdash_{\mathcal{L}} \phi$ iff the following conjunction of formulae is unsatisfiable (where the predicates P_ψ are indexed by subformulae ψ of ϕ):*

$$\left\{\begin{array}{ll}
(\mathsf{Neg}) & \exists x \neg P_\phi(x) \\
(\mathsf{Ren}) & \forall x[P_{\psi_1 \vee \psi_2}(x) \leftrightarrow P_{\psi_1}(x) \vee P_{\psi_2}(x)] \\
 & \forall x[P_{\psi_1 \wedge \psi_2}(x) \leftrightarrow P_{\psi_1}(x) \wedge P_{\psi_2}(x)] \\
 & \forall x[P_{H(\psi)}(x) \leftrightarrow P_\psi(H(x))] \\
 & \forall x[P_{K(\psi)}(x) \leftrightarrow \neg P_\psi(K(x))] \\
 & \forall x[P_{R(\psi_1,\dots,\psi_n)}(x) \leftrightarrow \\
 & \quad \exists x_1,\dots,x_n(R(x_1,\dots,x_n,x) \wedge \bigwedge_{i=1}^n P_{\psi_i}(x_i))] \\
 & \forall x[P_{Q(\psi_1,\dots,\psi_n)}(x) \leftrightarrow \\
 & \quad \forall x_1,\dots,x_n(Q(x_1,\dots,x_n,x) \rightarrow \bigvee_{i=1}^n P_{\psi_i}(x_i))] \\
 & \forall x[P_{R'(\psi_1,\dots,\psi_n)}(x) \leftrightarrow \\
 & \quad \exists x_1,\dots,x_n(R'(x_1,\dots,x_n,x) \wedge \bigwedge_{i=1}^n \neg P_{\psi_i}(x_i))] \\
 & \forall x[P_{Q'(\psi_1,\dots,\psi_n)}(x) \leftrightarrow \\
 & \quad \forall x_1,\dots,x_n(Q'(x_1,\dots,x_n,x) \rightarrow \bigvee_{i=1}^n \neg P_{\psi_i}(x_i))] \\
(\mathsf{Str}) & \forall x\; x \leq x \\
 & \forall x,y,z\,[(x \leq y \wedge y \leq z) \rightarrow x \leq z] \\
 & \forall x,y\,[(x \leq y \wedge P_\psi(x)) \rightarrow P_\psi(y)] \\
 & E \quad \text{(the set of first-order formulae that characterize } \mathcal{K}_{\mathcal{L}}\text{)}
\end{array}\right.$$

Proof: *(Sketch)* It follows from the fact that for every Kripke-style model (X, m), where $X \in \mathcal{K}_{\mathcal{L}}$ and $m : \mathsf{Var} \to \mathcal{O}(X)$ such that $X \not\models_m \phi$ we can construct a first-order model of $(\mathsf{Neg}) \cup (\mathsf{Ren}) \cup (\mathsf{Str})$, and vice versa. □

The set of formulae $(\mathsf{Neg}) \cup (\mathsf{Ren}) \cup (\mathsf{Str})$ can be transformed to clause form in the usual way. Proposition 7 gives an estimate of the number of clauses generated from a given formula ϕ by using Proposition 6.

Proposition 7 *Let ϕ be a formula of $\mathcal{L}$ containing logical connectives in $\{\vee, \wedge\} \cup Lh \cup La \cup Jh \cup Mh \cup Ja \cup Ma$. Let l be the number of subformulae of ϕ, and let m be the highest among the arities of the operators occurring in ϕ. Then the number of clauses generated using the rules in Proposition 6 is at most $l(m+2)+s$, where s is the number of clauses induced by the formulae in* $(\mathsf{Str}) \cup (\mathsf{Neg})$.

Proof: It follows from the fact that the formulae in (Ren) introduce, for every operation in $\{\vee, \wedge\}$, 3 clauses; for every operation in $Lh \cup La$ 2 clauses; and for every n-ary operation in $Jh \cup Mh \cup Ja \cup Ma$, $(n+2)$ clauses. □

Translation to clause form for finitely-valued logics. If $\mathcal{L}$ is a finitely-valued logic having as algebra of truth values a distributive lattice with operators the procedure described above can be further extended. The Priestley dual $D(A)$ of A can be seen as a finite Kripke-style frame with respect to which, by Corollary 4, $\mathcal{L}$ is sound and complete. Even a version of first-order finitely-valued logics with quantifiers $\forall, \exists$ (interpreted as generalized meets resp. joins in A), defined by following the ideas in [1], is tractable this way. The rules for translating to clause form quantified formulae essentially use the fact that every element in the lattice of truth values can be uniquely written as a join of join-irreducible elements [9, 21, 23]; this argument cannot be applied if the algebra of truth values is infinite. For details cf. [21, 23].

For finitely-valued logics the use of quantification over the possible worlds can be avoided: the existential and universal quantifier can be replaced with joins resp. meets over all the elements of $D(A)$, which is a finite set. The advantage is that (for propositional logics) in the clause generation process, we remain inside propositional logic. The disadvantage is that, in the presence of logical connectives in Σ of arity greater than 1, it may lead to long clause forms.

Theorem proving. The translation to clause form described in this section is a translation to classical first-order logic; thus general methods for automated theorem proving in classical logic can be applied to the resulting sets of clauses.

5 Examples

We now illustrate the construction of classes of Kripke-style models and the translation to clause form presented above by some examples: logics based on (finite) p-lattices, logics based on implicative algebras, logics based on Ockham algebras, relevant logics. Due to space limitations, only one example is presented in detail; for the others the main ideas (especially concerning the use of Priestley duality for constructing classes of Kripke models) are only sketched. Other examples, such as SHn-logics or $SHKn$-logics, are presented in detail in [21].

5.1 Logics based on p-lattices

A distributive p-lattice is an algebra $(A, \vee, \wedge, *, 0, 1)$ of type $(2,2,1,0,0)$ such that $(A, \vee, \wedge, 0, 1)$ is a distributive lattice with 0 and 1, and $*$ is an operation of pseudocomplementation, i.e. $a^* = \max\{b \in A \mid a \wedge b = 0\}$.

It is known that the distributive p-lattices form a variety B_ω, and that the subvarieties of B_ω form a chain $\mathsf{B}_{-1} \subset \mathsf{B}_0 \subset \dots \subset \mathsf{B}_n \subset \dots \subset \mathsf{B}_\omega$, where B_{-1} is the trivial variety, B_0 the variety of Boolean algebras, and B_n $(n \geq 1)$ is generated by $P_n = 2^n \oplus 1$ (the n-th power of the 2-element chain to which a new top element is adjoined).

Lemma 8 *If a logic $\mathcal{L}$ is sound and complete w.r.t. B_ω, it is sound and complete w.r.t. the class of all Kripke models $((X, \leq), m)$, where $\leq$ is a partial order[3] on X, m is a meaning function, and the satisfiability relation is defined as in Lemma 5 for $\phi_1 \vee \phi_2, \phi_1 \wedge \phi_2$, and, additionally, by:*

$$X \overset{r}{\models}_{m,x} \phi^* \quad \textit{iff} \quad \forall y (x \leq y \to X \overset{r}{\not\models}_{m,y} \phi).$$

Proof: Follows from the duality theorem for p-lattices or Heyting algebras (cf. [18]) and Theorem 3, and, in particular, from the fact that for every meaning function $m : \mathsf{Var} \to \mathcal{O}(X)$, and every $x \in X$, $x \in \overline{m}(\phi^*) = \neg(\overline{m}(\phi)) = \{z \mid \forall y, z \leq y \to y \notin \overline{m}(\phi)\}$ iff $\forall y (x \leq y \to y \notin \overline{m}(\phi))$. □

[3] One can in fact prove soundness and completeness w.r.t. the class of the Kripke frames $(X, \leq)$, where $\leq \subseteq X \times X$ is reflexive and transitive.

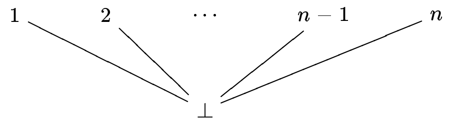

Figure 1. $V_n = D(P_n)$

For the translation to clause form, the semantics of the logical connectives $\vee, \wedge$ is used, as explained in Section 4. Additionally, for formulae starting with the connective $*$ the following formula has to be added to (Ren):

$$\forall x[P_{\phi^*}(x) \leftrightarrow (\forall y(x \leq y \rightarrow \neg P_\phi(y)))].$$

Logics based on finite distributive p-lattices. Concerning logics having P_n as an algebra of truth values, we make the following remarks about the use of the Priestley representation: the Priestley dual of $P_n = 2^n \oplus 1$ is the space V_n, represented in Figure 1. Thus, for every n, P_n is isomorphic to the lattice of order filters of V_n, where for every $U \subseteq \mathcal{O}(V_n)$, $U^* = X \setminus \downarrow U$. The finiteness of V_n leads to the following simplified rule for the pseudocomplementation $*$, without quantification over the elements of V_n:

$$P_{\phi^*}(\perp) \leftrightarrow \neg P_\phi(\perp) \wedge \neg P_\phi(1) \wedge \ldots \wedge \neg P_\phi(n)$$
$$P_{\phi^*}(i) \leftrightarrow \neg P_\phi(i) \text{ (for every } i \in \{1, \ldots, n\}).$$

Proposition 9 *The number of clauses generated from a formula ϕ in a logic based on B_ω is $\mathcal{O}(l)$, where l is the length of ϕ. In a logic based on P_n, the number of clauses generated from a formula ϕ is $\mathcal{O}(nl)$ and all clauses are ground.*

Proof: Since the formulae for renaming introduce at most 3 new clauses, in the case of logics based on B_ω, the number of clauses generated by the above procedure is $\mathcal{O}(l)$. In the case of logics based on P_n, due to the particular form of V_n, for every $x \in V_n \backslash \{\perp\}$ there is exactly one $y \in V_n$ with $y \geq x$. For $x = \perp$ there are exactly $n+1$ such elements. Thus, in this case the number of clauses generated from a formula ϕ is $1+3(k_1+k_2)(n+1)+k_3((n+2)+2n)$, where k_1, k_2, k_3 are the numbers of occurrences of $\vee, \wedge$, resp. $*$ in ϕ. It is easy to see that all these clauses are ground. □

Proposition 10 *Validity of propositional formulae in logics based on B_ω and P_n is decidable.*

Proof: If we analyze the form of the clauses in (Str) $\cup$ (Neg) $\cup$ (Ren) for a formula in a logic based on B_ω, we notice that we are in the situation studied in [4]. Hence, the validity of formulae in logics based on B_ω is decidable by ordered chaining with selection. The propositional logics based on P_n for some n are of course decidable. □

For first-order logics based on P_n one can prove the decidability of certain fragments by using classical decidability results from first-order logic. For further details cf. [22].

5.2 Other classes of logics

Intuitionistic logics. By the duality theorem for Heyting algebras (cf. e.g. [5]) and Theorem 3 it follows that intuitionistic logics are sound and complete with respect to the class of all Kripke models $(X, \leq)$, where $\leq$ is a partial order on X, a well-known result.

Logics based on Ockham algebras. Ockham algebras are distributive lattices with 0, 1, and one unary lattice antimorphism $\sim$. They are models for logics endowed with a negation operator that satisfies the De Morgan laws. A Priestley duality for Ockham algebras is given in [25]. The corresponding Priestley spaces are endowed with a unary, continuous, order-reversing map g. By Theorem 1 and Theorem 3, the logics which are sound and complete with respect to the variety of Ockham algebras have as Kripke-style models ordered spaces $(X, \leq, g)$ endowed with an order-reversing map (the condition that $\leq$ is a partial order can be relaxed to the condition that $\leq$ is reflexive and transitive).

Łukasiewicz logics of order n. We showed that, by using the Priestley-style duality for implicative algebras due to Martínez [12], in the case of Łukasiewicz algebras of order n we obtain a translation to clause form similar to the translation to mixed integer programming given in [6].

Relevance logics. Urquhart [26] gave an algebraic semantics for *relevance logics* in terms of *relevant algebras* (a class of distributive lattices with operators) and established a Priestley duality for this type of algebras. We showed that for every space X obeying all those properties of the Priestley duals of relevant algebras which do not involve topological notions, $\mathcal{O}(X)$ (with operations appropriately defined) is a relevant algebra. Thus, by Theorem 3, such spaces provide a family of Kripke-style models for relevant logics, which coincides with the Kripke models for a version of relevance logics studied by Routley and Meyer (cf. [19]). Theorem 3 shows that soundness and completeness with respect to this class of Kripke-style models can be proved by completely algebraical means, by using the soundness and completeness with respect to the class of relevant algebras.

6 Related Work

The work reported here was inspired by the papers of Baaz and Fermüller [1] and Hähnle [7] on resolution-based theorem proving in finite-valued quantificational first-order logics. It extends the ideas in [21] where we gave a method for translation to a signed clause form and signed resolution methods for (first-order) finitely-valued logics having a finite DLO A as algebra of truth values. The signed literals used in [21, 23] were of the form $\boxed{\alpha}L^t$ and $\boxed{\alpha}L^f$, where $\alpha \in D(A)$. Our method extended existing methods for automated theorem proving in *regular logics* [7] (there, the literals were of the form $\boxed{\geq i}L$ and resp. $\boxed{< i}L$, for some truth value i) to cases when the set of truth values is not linearly ordered. In this paper we replaced the notation above by $L(\alpha)$ resp. $\neg L(\alpha)$ (the translation is one to classical logic [22]), and showed that, using additional theoretical considerations, the ideas we had for finitely-valued logics can be extended to more general propositional logics.

Among existing methods exploiting the structure of the algebra of truth values, we mention [16, 13, 15], who gave theorem proving systems for m-valued Post logics and for algorithmic logics, Salzer [20] who studied operators and distribution quantifiers in finitely-valued logics based on semi-lattices, and Hähnle [9] who derived tableau-style axiomatizations of distribution quantifiers by using Birkhoff's representation theorem for finite distributive lattices. Methods for automated proving in paraconsistent or annotated logics (cf. e.g. [11]) also have similarities with our approach in the case of finitely-valued logics. Known methods from modal logic seem to occur as particular cases of general concepts such as those considered here (see also [5], [14]).

7 Conclusions

In this paper we showed that the Priestley duality for distributive lattices with operators provides a theoretical tool for clarifying the link between algebraic and Kripke-style models for certain classes of non-classical logics. Based on these results, we gave a method for translation to clause form and automated theorem proving for some of these classes of non-classical logics. We expect that this will enable us to use recent results in the algebraic investigation of various classes of distributive lattices with operators for obtaining Kripke models and automated theorem proving procedures for logics with interesting practical applications. Until now we only studied the existence of decision procedures in certain cases; in the future we intend to continue our work in this direction.

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