Universität Koblenz-Landau

FB 4 Informatik

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Exercises for "Advances in Theoretical Computer Science" Exercise sheet 7

Due on 4.12.12, 09:00 s.t.

Exercise 7.1:
Prove that the following functions are primitive recursive:

(1) $\mathsf{max} : \mathbb{N}^2 \to \mathbb{N}$ defined by $\mathsf{max}(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{otherwise} \end{cases}$

(2) $\mathsf{div} : \mathbb{N}^2 \to \mathbb{N}$ defined by $\mathsf{div}(x, y) = \begin{cases} x + 1 & \text{if } y = 0 \\ \lfloor \frac{x}{y} \rfloor & \text{otherwise} \end{cases}$

(3) $\mathsf{mod} : \mathbb{N}^2 \to \mathbb{N}$ defined by $\mathsf{mod}(x, y) = \begin{cases} x & \text{if } y = 0 \\ x \ \mathsf{mod} \ y & \text{otherwise} \end{cases}$

Here $\lfloor x \rfloor$ is the largest integer which is smaller than or equal to x ("floor x").

Remark: You are allowed to use all functions that were proved to be primitive recursive in the lecture or in a previous exercise.

Exercise 7.2:
Let $d : \mathbb{N} \to \mathbb{N}$ be defined by:

$$d(n) = \begin{cases} \text{number of divisors of } n & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}$$

For instance, $d(0) = 0, d(1) = 1, d(2) = 2, d(3) = 2, d(4) = 3, d(12) = 6$.

(1) Let $d_2 : \mathbb{N}^2 \to \mathbb{N}$ be such that $d_2(n, m)$ is the number of divisors of n which are smaller than or equal to m (we assume that $d_2(0, 0) = 0$). Prove that d_2 is primitive recursive.

(2) Use (1) to show that d is primitive recursive.

Hint: (1) You can e.g. give a definition by primitive recursion for d_2.

Remark: You are allowed to use all functions that were proved to be primitive recursive in the lecture or in a previous exercise.

Exercise 7.3:
Let $D : \mathbb{N}^2 \to \mathbb{N}$ be defined by:

$$D(n,i) = k \quad \text{if and only if} \quad \begin{array}{l} k \text{ is the power of the } i\text{-th prime number} \\ \text{in the prime number decomposition of } n \end{array}$$

which can also be seen as the smallest natural number k such that $p(i)^{k+1}$ does not divide n (where $p(i)$ is the i-th prime number). By definition we set $D(0,i) = 0$ for every $i \in \mathbb{N}$, and $D(n,0) = 0$ for every $n \in \mathbb{N}$.

Show that D is primitive recursive.

Remark: You are allowed to use all functions that were proved to be primitive recursive in the lecture or in a previous exercise.

Exercise 7.4:
Let $K^r : \mathbb{N}^r \to \mathbb{N}$ be the Gödelisation function defined by:

$$K^r(n_1, \ldots, n_r) = \prod_{i=1}^{r} p(i)^{n_i} \qquad \text{where } p(i) \text{ is the } i\text{-th prime number}$$

and let $D_1, \ldots, D_r : \mathbb{N} \to \mathbb{N}$ be defined by $D_i(n) = D(n,i)$, where D is as in Exercise 7.3.

(1) Show that K^r is primitive recursive.

(2) Show that D_i is primitive recursive for every $i \in \{1, \ldots, r\}$.

Remark: You are allowed to use all functions that were proved to be primitive recursive in the lecture or in a previous exercise.

Exercise 7.5:
Let $\mathsf{fib} : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$\begin{array}{rcll} \mathsf{fib}(0) & = & 1 & \\ \mathsf{fib}(1) & = & 1 & \\ \mathsf{fib}(n) & = & \mathsf{fib}(n-1) + \mathsf{fib}(n-2) & \text{for all } n > 1 \end{array}$$

Let $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be defined for every $n \in \mathbb{N}$ by $f(n) = (\mathsf{fib}(n), \mathsf{fib}(n+1))$.

(1) Let $f_p : \mathbb{N} \to \mathbb{N}$ be defined by $f_p(n) = K^2(f(n)) = K^2(\mathsf{fib}(n), \mathsf{fib}(n+1))$, where $K^2 : \mathbb{N}^2 \to \mathbb{N}$ is the Gödelisation function as defined in Exercise 7.4.

Show that f_p is primitive recursive.

(2) Use (1) and the results in Exercise 7.4 to prove that fib is primitive recursive.

Remark: You are allowed to use all functions that were proved to be primitive recursive in the lecture or in a previous exercise.

The submission of the solutions is not compulsory. If you want to submit your solutions, please do so until 4.12.12, 09:00 s.t.. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name on your solution.
Submission possibilities:

- By e-mail to mbender@uni-koblenz.de with the keyword “Homework ACTCS” in the subject.
- Put it in the box in front of Room B 222.