Universität Koblenz-Landau

FB 4 Informatik

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Exercises for
"Advances in Theoretical Computer Science"
Exercise sheet 9
Due on 18.12.12, 09:00 s.t.

Exercise 9.1:

Prove that function $\mathsf{log} : \mathbb{N}^2 \rightarrow \mathbb{N}$ defined by $\mathsf{log}(n, m) = \lfloor \mathsf{log}_n(m) \rfloor$ is μ-recursive.

Remark: You are allowed to use all functions that were proved to be primitive and/or μ-recursive in the lecture or in a previous exercise.

Exercise 9.2:

Which functions are computed by:

(1) $f_1 = \mu c_1^2$

(2) $f_2 = \mu g$, where $g(n, i) = \begin{cases} n + 1 & \text{if } i = 0 \\ \mu j(j + 1 + n = 0) & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{cases}$

(3) $f_3 = \mu g$, where $g(n, i) = \begin{cases} n + 1 & \text{if } i = 0 \\ \mu j(j + 1 - n = 0) & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{cases}$

Exercise 9.3:

Recall the proof of the fact that $LOOP \subseteq \mathcal{P}$ for the case when P is loop x_i do P_1 end. We assumed that f_{P_1} be a primitive recursive function which computes what P_1 computes.

We used the following functions:

- $f_1(n) = \langle (n)_1, \dots, (n)_l, (n)_{l+1}, \dots (n)_{l+j-1}, (n)_i, (n)_{l+j+1}, \dots \rangle$
- $f_2(n) = \begin{cases} n & \text{if } (n)_{l+j} = 0 \\ f_{P_1}(f_2(\langle \dots, (n)_{l+j} - 1, \dots \rangle)) & \text{otherwise} \end{cases}$

Prove that f_2 is primitive recursive.

Hint: Fill in the details left open in the proof given on the slides of the lecture from 13.12.2012.

Exercise 9.4:
Consider the definition of the Ackermann function given in the lecture:

$$\begin{aligned} A_0(x) &= \begin{cases} 1 & \text{is } x = 0 \\ 2 & \text{is } x = 1 \\ x+2 & \text{otherwise} \end{cases} \\ A_{n+1}(0) &= A_n(1) \\ A_{n+1}(x+1) &= A_n(A_{n+1}(x)) \\ A(x) &= A_x(x) \end{aligned}$$

Prove:

(1) $A_1(n) > 2n$ for every $n \in \mathbb{N}$
(2) $A_2(n) > 2^{n+1}$ for every $n \in \mathbb{N}$
(3) $m < A_n(m)$ for all $n, m \in \mathbb{N}$
(4) $A_n(m) < A_n(m+1)$ for all $n, m \in \mathbb{N}$
(5) $A_{n+1}(m) \geq A_n(m+1)$ for all $n, m \in \mathbb{N}$
(6) $A_n(m) \leq A_{n+1}(m)$ for all $n, m \in \mathbb{N}$.

Prove (possibly using some of the inequalities above) that:

(7) $0 < A_0(0)$ for all $n \in \mathbb{N}$
(8) $\pi_i^r(n_1, \ldots, n_r) < A_0(\sum_{i=1}^r n_i)$
(9) $n + 1 < A_1(n)$

Remark: The results (1)–(9) are the easy part in the proof of the fact that the Ackermann function is not primitive recursive. The proof of the following facts is a bit more difficult (and it is not part of this exercise).

For every $m \in \mathbb{N}$, let $B_m = \{f \mid f$ primitive recursive and for all $n_1, \ldots, n_r \in \mathbb{N}$ where r is the arity of f, $f(n_1, \ldots, n_r) < A_m(\sum_{i=1}^r n_i)$

- If $f, g_1, \ldots, g_r \in B_m$ and $h = f \circ (g_1, \ldots, g_m)$ then there exists a natural number m' (depending on m and r) such that $h \in B_{m'}$.
- If $g, h \in B_m$ and f is defined by primitive recursion from g and h then $f \in B_{m+4}$.

Prove:

(10) For every primitive recursive function f there exists $m \in \mathbb{N}$ with $f \in B_m$.
(11) The Ackermann function A defined by $A(n) = A_n(n)$ is not primitive recursive.

The submission of the solutions is not compulsory. If you want to submit your solutions, please do so until 18.12.12, 09:00 s.t.. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name on your solution.
Submission possibilities:

- By e-mail to mbender@uni-koblenz.de with the keyword "Homework ACTCS" in the subject.
- Put it in the box in front of Room B 222.