Universität Koblenz-Landau

FB 4 Informatik

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Exercises for
“Advances in Theoretical Computer Science”
Exercise sheet 11
Due on 15.1.13, 09:00 s.t.

In the lecture from 13.12.2012 we sketched a possibility of associating with every Turing Machine M a unique Gödel number $\langle M \rangle \in \mathbb{N}$ such that the coding function and the decoding function are primitive recursive. Similarly, we could associate with every configuration of a given TM a unique Gödel number for the configuration such that coding and decoding are primitive recursive.

The construction uses the following encoding of words as natural numbers: If $\Sigma = \{a_0, a_1, \dots, a_m\}$ and $w = a_{i_1} \dots a_{i_n}$ is a word over Σ then $\langle w \rangle_l = \langle i_1, \dots, i_n \rangle = \prod_{j=1}^{n} p(j)^{i_j}$.

Therefore, we can represent w.l.o.g. words as natural numbers and languages as sets of natural numbers.

Notation: In what follows we will denote by M_n the Turing machine with Gödel number n and with $L(M)$ the language accepted by the Turing machine M.

Exercise 11.1:
Let $K = \{n \mid M_n \text{ halts on } n\}$.

- Prove that K is undecidable.
- Prove that K is acceptable.
- Prove that the complement of K is not acceptable.

Exercise 11.2:
Let L_1, L_2 be two languages (regarded here as sets of natural numbers). We say that L_1 can be reduced to L_2 (denoted by writing $L_1 \leq L_2$) if there exists a TM computable function $f : \mathbb{N} \to \mathbb{N}$ with the property that:

$$\forall n \in \mathbb{N} \qquad n \in L_1 \quad \text{if and only if} \quad f(n) \in L_2.$$

Prove that the relation $\leq$ is transitive, i.e. that if L_1, L_2 and L_3 are languages (regarded here as sets of natural numbers) such that $L_1 \leq L_2$ and $L_2 \leq L_3$ then $L_1 \leq L_3$.

Exercise 11.3:
Prove that the following problems are undecidable using a reduction to an undecidable problem.

- $P_1 = \{n \mid L(M_n) = \emptyset\}$
- $P_2 = \{n \mid L(M_n) \text{ is finite }\}$
- $P_3 = \{\langle n, m\rangle \mid L(M_n) \cap L(M_m) = \emptyset\}$

You are allowed to use the undecidability of the following problems: $HALT = \{\langle n, m\rangle \mid M_n \text{ halts on input } m\}$, $K = \{n \mid M_n \text{ halts on input } n\}$ or $H_0 = \{n \mid M_n \text{ halts on input } 0\}$ or their complements. If you have proven the (un-)decidability of P_i then you may use this result for the following tasks.

Exercise 11.4:
Prove that it is undecidable whether a WHILE program which computes a partial function $f : \mathbb{N} \to \mathbb{N}$ terminates on input n.

Hint: One can give e.g. a proof by contradiction using the fact that the class of WHILE-computable functions coincides with the class of TM-computable functions.

Exercise 11.5:
Prove that the following problems are undecidable using the theorem of Rice.

- $L_1 = \{n \mid M_n \text{ accepts an infinite language }\}$
- $L_2 = \{n \mid M_n \text{ accepts a finite language }\}$
- $L_3 = \{n \mid M_n \text{ accepts a decidable language }\}$
- Let $k \in \mathbb{N}$ and $L_4 = \{n \mid M_n \text{ accepts only words which have length greater than } k\}$
- $L_5 = \{n \mid L(M_n) \text{ is context sensitive }\}$
- $L_6 = \{n \mid \text{the language accepted by } M_n \text{ is regular }\}$
- $L_7 = \{n \mid M_n \text{ halts on all inputs } w \in \Sigma^*\}$

The submission of the solutions is not compulsory. If you want to submit your solutions, please do so until 15.1.13, 09:00 s.t.. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name on your solution.
Submission possibilities:

- By e-mail to mbender@uni-koblenz.de with the keyword "Homework ACTCS" in the subject.
- Put it in the box in front of Room B 222.