Universität Koblenz-Landau

FB 4 Informatik

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Exercises for
"Advances in Theoretical Computer Science"
Exercise sheet 14
Due on 5.02.13, 09:00 s.t.

Exercise 14.1:
Consider the following propositional logic formula:

$$F: \quad (P \vee \neg Q \vee \neg(R \vee \neg S)) \wedge (Q \vee \neg R \vee S)$$

Apply Steps 1-4 on page 28 of the slides from 31.01.2013 to this formula for computing the formula in 3-CNF associated to F (formula which is satisfiable iff F is satisfiable (see supplementary exercise 14.6)).

Exercise 14.2:

(1) Draw the complete graphs with $3, 4$ and 5 vertices.

(2) Consider the undirected graph $G = (V, E)$, where $V = \{a, b, c, d, e, f\}$ and

$$E = \{(a, b), (a, c), (a, e), (a, f), (b, c), (b, d), (b, e), (c, e), (c, f)\}.$$

(Note that in an undirected graph the edge (x, y) is identical to the edge (y, x), i.e. they are not ordered pairs but sets (or 2-multisets) of vertices.)

(a) Draw the graph G.

(b) Does G have a clique of size 3? Does G have a clique of size 4? Does G have a clique of size 5?

Exercise 14.3:
Consider the following formula in 3-CNF:

$$F: \quad (\neg P_1 \vee P_2 \vee P_3) \wedge (P_1 \vee \neg P_2 \vee P_4) \wedge (P_2 \vee \neg P_3 \vee \neg P_4)$$

(1) Is the formula satisfiable? If yes then give a satisfying assignment.

(2) Starting from F construct the pair (G_F, k_F) as explained on pages 38-39 of the slides from 31.01.2012.

(3) Has the graph G_F a clique of size k_F? If so indicate such a clique and reconstruct from it an assignment which makes F true.

Exercise 14.4:
Consider the formula in 3-CNF from the previous exercise.

$$F: \quad (\neg P_1 \vee P_2 \vee P_3) \wedge (P_1 \vee \neg P_2 \vee P_4) \wedge (P_2 \vee \neg P_3 \vee \neg P_4)$$

(1) Starting from F construct the tuple $(b, a_1, \ldots, a_n)$ as explained on pages 42-44 of the slides from 31.01.2012.

(2) Is there a subset I of $\{a_1, \ldots, a_n\}$ which adds up to b? If such a subset exists use it to construct an assignment which makes F true (as explained on page 44).

Exercise 14.5:
The SET PACKING problem is defined as follows:

Given: • $C = \{S_1, \ldots, S_n\}$ where every S_i is a finite set
• $l \geq 1$

Task: Is there a subset D of C with l elements such that the elements of D are pairwise disjoint?

SET PACKING = $\{(C, l) \mid C = \{S_1, \ldots, S_n\}$, every S_i is a finite set and there exists $D \subseteq C$ with l elements such that the elements of D are pairwise disjoint$\}$

(1) Prove that SET PACKING $\in$ NP.

For every pair (G, k), where $G = (V, E)$ is an undirected graph with vertices $\{v_1, \ldots, v_m\}$ and edges in E we associate the pair (C, l), where $l = k$ and $C = \{S_1, \ldots, S_m\}$, with $S_i = \{(v_i, v_j), (v_j, v_i) \mid (v_i, v_j) \notin E\}$.

(2) Estimate the time needed for constructing (C, l) from (G, k).

Prove:

(3) $S_i \cap S_j \neq \emptyset$ if and only if there is no edge between v_i and v_j in G.

(4) If G' is a clique of G with size k, with vertices $\{v_{i_1}, \ldots, v_{i_k}\}$ then the sets in $D = \{S_{i_1}, \ldots, S_{i_k}\}$ are pairwise disjoint.

(5) G hat a clique of size k iff there exists a subset D of C with l elements such that the elements of D are pairwise disjoint.

(6) Infer that Clique (the problem whether a graph has a clique of size k) can be polynomially reduced to SET PACKING.

(7) Is SET PACKING NP-complete? Justify your answer.

Supplementary exercise

Exercise 14.6:
Let F be a propositional formula ($F \neq \perp, F \neq \top$), P a propositional variable not occurring in F, and F' a subformula of F.

We will write F also as $F[F']$ in order to emphasize that F' occurs in F. Let $F[P]$ be the formula obtained from F by replacing the subformula F' with the propositional variable P.

Prove:
The formula $F[P] \wedge (P \leftrightarrow F')$ is satisfiable if and only if $F[F']$ is satisfiable.

Hint: Use structural induction (see below). We check first that the result holds if F is a propositional variable (induction basis). Then we consider a formula F which is not a propositional variable. We assume that the result holds for all proper subformulae of F (induction hypothesis) and show that then the result also holds for F (induction step). For proving this, one has to make a case distinction ($F' = F$ or F' is a proper formula of F?) In the last case, the various ways in which F is built need to be considered ($F = \neg F_1$, or $F = F_1 \circ F_2, \circ \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$).

The structural induction principle (for propositional logic).

Let $\mathcal{B}$ be a property of formulae in propositional logic. Assume that the following hold:

- every propositional variable P has property $\mathcal{B}$;
- $\perp$ and $\top$ have property $\mathcal{B}$;
- if $F = F_1 \ \mathsf{op} \ F_2$ for $\mathsf{op} \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$ and if both F_1 and F_2 have property $\mathcal{B}$ then F has property $\mathcal{B}$;
- if $F = \neg F_1$ and F_1 has property $\mathcal{B}$ then F has property $\mathcal{B}$.

Then property $\mathcal{B}$ holds for all Π-formulae.

The submission of the solutions is not compulsory. If you want to submit your solutions, please do so until 5.2.13, 09:00 s.t.. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name on your solution.
Submission possibilities:

- By e-mail to mbender@uni-koblenz.de with the keyword "Homework ACTCS" in the subject.
- Put it in the box in front of Room B 222.