

Prof. Dr. Viorica Sofronie-Stokkermans
Dipl. Inf. Markus Bender

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**Exercises for
“Advances in Theoretical Computer Science”
Exercise sheet 15**

The structure, content, type of exercises and distribution of points in this last exercise sheet give an indication about how the written exam(s) could be structured. Please note that the real exams will also contain other types of exercises. For preparing please look at all the exercises on the homework sheets which were discussed in the exercise session.

The submission of the solutions is not compulsory. If you want to submit your solutions, please do so until Wednesday, 13.02.2013 at 9:00. (Joint solutions prepared by up to three persons are allowed.) We can discuss the solutions in the Question/Answer session on Friday, 15.02.2013.

Please note that since this is only an example of how the written exam could look, we did not check if it is suitable for 120 minutes (but we will do so for the real exam). Therefore please do not draw any conclusion out of the time it took you to solve this exercise sheet.

1 Multiple choice (8+12+16+4 = 40p)

For each correct cross you get 2 points. For each wrong cross you lose 1 point. The minimum for each subtask is 0 points, i.e. you cannot carry negative points into the next subtask.

1.1 Register machines

Please indicate (by crossing the right box) whether the following statements are true or false:

In the LOOP program “loop x_1 do S end” the number of times S is executed might be influenced by the fact that S changes x_1 .	true <input type="checkbox"/>
	false <input type="checkbox"/>
The if-then-else instruction can be simulated with a WHILE program.	true <input type="checkbox"/>
	false <input type="checkbox"/>
Every WHILE program terminates.	true <input type="checkbox"/>
	false <input type="checkbox"/>
For every WHILE program there exists an equivalent GOTO program.	true <input type="checkbox"/>
	false <input type="checkbox"/>

1.2 Primitive recursive functions, μ -recursive functions

Please indicate (by crossing the right box) whether the following statements are true or false:

For every $k \in \mathbb{N}$, the function $f : \mathbb{N}^k \rightarrow \mathbb{N}$ defined by $f(n_1, \dots, n_k) = \prod_{i=1}^k n_i$ is primitive recursive.	true <input type="checkbox"/>
	false <input type="checkbox"/>
Every total function is primitive recursive.	true <input type="checkbox"/>
	false <input type="checkbox"/>
For every primitive recursive function there exists a LOOP program which computes it.	true <input type="checkbox"/>
	false <input type="checkbox"/>
Every μ -recursive function is primitive recursive.	true <input type="checkbox"/>
	false <input type="checkbox"/>
Every μ -recursive function is total.	true <input type="checkbox"/>
	false <input type="checkbox"/>
Every WHILE computable function is μ -recursive.	true <input type="checkbox"/>
	false <input type="checkbox"/>

1.3 Complexity theory

Please indicate (by crossing the right box) whether the following statements are true or false:

Assume f is a computable function. Then every language in $DSPACE(f(n))$ is decidable.	true <input type="checkbox"/>
	false <input type="checkbox"/>
Assume f is a computable function. Then $NSPACE(f(n)) \subseteq NTIME(f(n))$.	true <input type="checkbox"/>
	false <input type="checkbox"/>
Every problem in NP can be solved deterministically in exponential time.	true <input type="checkbox"/>
	false <input type="checkbox"/>
Every NP-hard problem is in NP.	true <input type="checkbox"/>
	false <input type="checkbox"/>
If there exists an NP-hard problem which is in P then $P = NP$.	true <input type="checkbox"/>
	false <input type="checkbox"/>
Let L_1, L_2 be languages with $L_1 \preceq_{\text{pol}} L_2$ and $L_2 \in P$. Then $L_1 \in P$.	true <input type="checkbox"/>
	false <input type="checkbox"/>
3-CNF-SAT is NP-complete.	true <input type="checkbox"/>
	false <input type="checkbox"/>
DNF-SAT = $\{w \mid w \text{ satisfiable propositional formula in DNF}\}$ is NP-complete.	true <input type="checkbox"/>
	false <input type="checkbox"/>

1.4 Decidability

Please indicate (by crossing the right box) whether the following statements are true or false:

The question whether a Turing machine halts on a certain input is decidable.	true <input type="checkbox"/>
	false <input type="checkbox"/>
The question whether a Turing machine halts on input 0 is undecidable.	true <input type="checkbox"/>
	false <input type="checkbox"/>

2 Register machines $((4 + 6) + (2 + 4 + 4) = 20\text{p})$

Exercise 2.1

The following instructions will be considered to be GOTO instructions in this exercise:

$x_i := c$	$x_i := c \text{ op } x_j$	goto l	Here: x_i, x_j, x_k are registers
$x_i := x_j$	$x_i := x_j \text{ op } c$	if $x_i = 0$ goto l	c is a constant
	$x_i := x_j \text{ op } x_k$		op $\in \{+, -, *\}$
			and l is a label.

A GOTO program has the form

$$l_1 : B_1, \dots, l_k : B_k \quad (k \geq 1)$$

where B_1, \dots, B_k are GOTO instructions and l_1, \dots, l_k are labels.

Let $\text{sqrt} : \mathbb{N} \rightarrow \mathbb{N}$ be defined by

$$\text{sqrt}(n) = \lfloor \sqrt{n} \rfloor$$

- (1) Write a pseudocode program for sqrt which only uses the arithmetical operators $+, -, *$ in assignments and only uses comparisons such as $\leq, <, \geq, >, =$ in tests or as loop/for/while conditions.
- (2) Write a GOTO program which computes sqrt .

Exercise 2.2

Let P be the following GOTO program:

```
1 :  $x_5 := x_1$ ;  
2 :  $x_4 := x_2$ ;  
3 : if  $x_1 = 0$  goto 11;  
4 :  $x_2 := x_4$ ;  
5 : if  $x_2 = 0$  goto 9;  
6 :  $x_3 := x_3 + 1$ ;  
7 :  $x_2 := x_2 - 1$ ;  
8 : if  $x_6 = 0$  goto 5;  
9 :  $x_1 := x_1 - 1$ ;  
10 : if  $x_6 = 0$  goto 3;  
11 :  $x_2 := x_4$ ;  
12 :  $x_1 := x_5$ ;  
13 :  $x_4 := 0$ ;  
14 :  $x_5 := 0$ 
```

- (1) Which value does P compute on input $x_1 = 2, x_2 = 3$?
- (2) Which function $f : \mathbb{N}^2 \rightarrow \mathbb{N}$ is computed by P ?
- (3) Use the transformation presented in the lecture to construct a WHILE-IF program with only one WHILE loop which has the same semantics as P .

3 Recursive functions (((1+1+5) + (1+5)) + 7 = 20p)

In the next exercises we use the following notation:

- \circ is function composition.
- if $j \leq k$, π_j^k is the projection function defined by $\pi_j^k(n_1, \dots, n_k) = n_j$.
- $(+1) : \mathbb{N} \rightarrow \mathbb{N}$ is defined by: $(+1)(n) = n + 1$.
- $(-1) : \mathbb{N} \rightarrow \mathbb{N}$ is defined by: $(-1)(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{otherwise} \end{cases}$.
- $*$: $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is defined for all $n_1, n_2 \in \mathbb{N}$ by: $*(n_1, n_2) = n_1 * n_2$.
- $+$: $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is defined for all $n_1, n_2 \in \mathbb{N}$ by: $+(n_1, n_2) = n_1 + n_2$.
- $-$: $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is defined for all $n_1, n_2 \in \mathbb{N}$ by $-(n_1, n_2) = n_1 - n_2 = \begin{cases} 0 & \text{if } n_1 \leq n_2 \\ n_1 - n_2 & \text{otherwise} \end{cases}$.
- for all $s, k \in \mathbb{N}$, $c_s^k : \mathbb{N}^k \rightarrow \mathbb{N}$ is defined for all $\mathbf{n} \in \mathbb{N}^k$ by: $c_s^k(\mathbf{n}) = s$.

In what follows we will assume known that all the functions above are primitive recursive.

Exercise 3.1

(1) Consider the following primitive recursive function:

$$f_1 = \mathcal{PR}[(+1) \circ (+ \circ (c_4^0, 0)), * \circ ((-1) \circ \pi_1^2, \pi_2^2)]$$

- Which is the arity of f_1 ?
- What does f_1 compute if all arguments are equal to 1?
- Which (concrete) function is computed by f_1 ?

(2) Consider the following μ -recursive function:

$$f_2 = \mu g, \text{ where } g(n, i) = \begin{cases} n + 2 & \text{if } i = 0 \\ \mu_j(j + 2 - n = 0) & \text{if } i = 1 \\ 2 & \text{if } i \geq 2 \end{cases}$$

- Which is the arity of f_2 ?
- Which is the (concrete) function computed by f_2 ?

Exercise 3.2

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ f(n-1) * f(n-2) & \text{if } n \geq 2 \end{cases}$$

Prove that f is primitive recursive.

You can use any of the results proved in the lecture for this.

4 Decidability, Rice's theorem (4+6 = 10p)

Exercise 4.1

- State the theorem of Rice.
- Prove that the following problem is undecidable using the theorem of Rice:

$$L = \{n \mid M_n \text{ accepts only words which have length greater than } 10\}$$

5 Complexity (2+4+2+2 = 10p)

Exercise 5.1

A zoo acquires for the first time n animals x_1, \dots, x_n . A list L of enemies is provided, containing sets $\{x_i, x_j\}$ consisting of two animals which cannot be placed in the same cage. The zoo has k cages.

We are interested in the problem of deciding whether the animals can be placed in the zoo such that they are all safe. Thus,

$$\text{ZOO} = \{(L, n, k) \mid \begin{array}{l} \text{the } n \text{ animals with enemy list } L \text{ can be placed} \\ \text{on } k \text{ cages such that they are all safe.} \end{array} \}$$

- (1) One of the two triples (L_i, n_i, k_i) is an instance of ZOO. Which is this?

$(\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}\}, 4, 3)$	<input type="checkbox"/>
$(\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}, 4, 3)$	<input type="checkbox"/>

- (2) Let f be the function which associates with every undirected graph $G = (V, E)$ – where $V = \{v_1, \dots, v_m\}$ – the tuple (L_G, n_G, k_G) where:

$$L_G = E, n_G = m, k_G = 3$$

Prove that f defines a polynomial reduction of 3-colorability to ZOO.

- (3) Is ZOO in NP? Briefly justify your answer (you do not need to construct a Turing machine for this).
- (4) We know that the 3-colorability problem is an NP-complete problem. Prove or refute the following: ZOO is an NP-complete problem.

The submission of the solutions is not compulsory. If you want to submit your solutions, please do so until 13.02.13, 09:00 s.t.. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name on your solution.

Submission possibilities:

- By e-mail to mbender@uni-koblenz.de with the keyword “Homework ACTCS” in the subject.
- Put it in the box in front of Room B 222.