#### **Advanced Topics in Theoretical Computer Science**

Part 3: Recursive functions (2)

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- Register machines (LOOP, WHILE, GOTO)
- Recursive functions
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- Complexity
- Other computation models: e.g. Büchi Automata,  $\lambda$ -calculus

## **3. Recursive functions**

- Introduction/Motivation
- Primitive recursive functions
- $\mathcal{P} = \text{LOOP}$
- $\mu$ -recursive functions
- $F_{\mu} = WHILE$
- Summary

 $\mapsto \mathcal{P}$ 

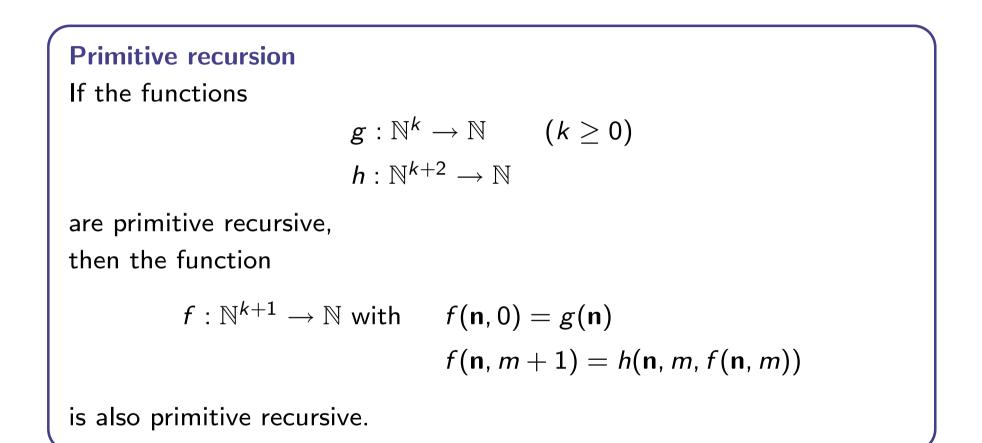
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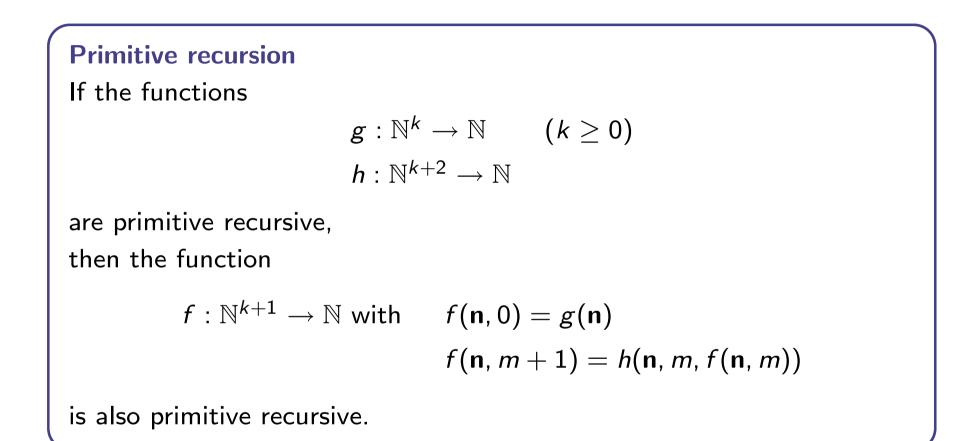
 $\mapsto \mathcal{P}$ 

#### Last time

## **Primitive recursive functions**

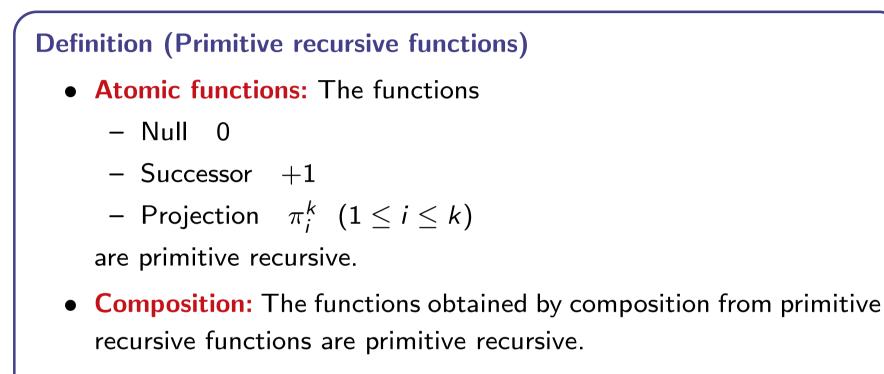


## **Primitive recursive functions**



**Notation without arguments:**  $f = \mathcal{PR}[g, h]$ 

## **Primitive recursive functions**



• **Primitive recursion:** The functions obtained by primitive recursion from primitive recursive functions are primitive recursive.

**Notation:**  $\mathcal{P} =$  The set of all primitive recursive functions

#### **Arithmetical functions: definitions**

f(n) = n + c, for  $c \in \mathbb{N}, c > 0$  $f = \underbrace{(+1) \circ \cdots \circ (+1)}_{c \text{ times}}$ 

#### Identity

$$f=\pi_1^1$$

$$f(n, m) = n + m$$

$$f = \mathcal{PR}[\pi_1^1, (+1) \circ \pi_3^3]$$

$$f(n) = n - 1$$

$$f = \mathcal{PR}[0, \pi_1^2]$$

$$f(n, m) = n - m$$

$$f = \mathcal{PR}[\pi_1^1, (-1) \circ \pi_3^3]$$

$$f(n, m) = n * m$$

$$f = \mathcal{PR}[0, + \circ (\pi_3^3, \pi_1^3)]$$

# **Re-ordering/Omitting/Repeating Arguments**

Lemma The set of primitive recursive functions is closed under:

- Re-ordering
- Omitting
- Repeating

of arguments when composing functions.

#### Proof: (Idea)

A tuple of arguments  $\mathbf{n'} = (n_{i_1}, \ldots, n_{i_k})$  obtained from  $\mathbf{n} = (n_1, \ldots, n_k)$  by re-ordering, omitting or repeating components can be represented as:

$$\mathbf{n'} = (\pi_{i_1}^k(\mathbf{n}), \dots, \pi_{i_k}^k(\mathbf{n}))$$

# Today

- More examples
- $\bullet \ \mathcal{P} = \mathsf{LOOP}$

#### **Additional Arguments**

**Lemma.** Assume  $f : \mathbb{N}^k \to \mathbb{N}$  is primitive recursive. Then, for every  $I \in \mathbb{N}$ , the function  $f' : \mathbb{N}^k \times \mathbb{N}^l \to \mathbb{N}$  defined for every  $\mathbf{n} \in \mathbb{N}^k$  and every  $\mathbf{m} \in \mathbb{N}^l$  by:

$$f'(\mathbf{n},\mathbf{m}) = f(\mathbf{n})$$

is primitive recursive.

#### **Additional Arguments**

**Lemma.** Assume  $f : \mathbb{N}^k \to \mathbb{N}$  is primitive recursive. Then, for every  $I \in \mathbb{N}$ , the function  $f' : \mathbb{N}^k \times \mathbb{N}^l \to \mathbb{N}$  defined for every  $\mathbf{n} \in \mathbb{N}^k$  and every  $\mathbf{m} \in \mathbb{N}^l$  by:

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#### Proof:

Case 1: k = 0, i.e. f is a constant. Then f' can be expressed by primitive recursion:

$$f'(n) = f f' = \mathcal{PR}[f, \pi_2^2]$$
  
$$f'(n+1) = f'(n) = \pi_2^2(n, f'(n))$$

Case 2:  $k' \neq 0$ . Let  $\mathbf{n} = (n_1, ..., n_k, m_1, ..., m_l)$ Then  $f'(\mathbf{n}) = f(\pi_1^{k+l}(\mathbf{n}), ..., \pi_k^{k+l}(\mathbf{n})) = f \circ \pi^{k+1}$ . Lemma (Case distinction is primitive recursive)

- If  $g_i$ ,  $h_i$   $(1 \le i \le r)$  are primitive recursive functions, and
  - for every *n* there exists a unique *i* with  $h_i(n) = 0$

then the function f defined by:

$$f(n) = \begin{cases} g_1(n) & \text{if } h_1(n) = 0 \\ \dots & \\ g_r(n) & \text{if } h_r(n) = 0 \end{cases}$$

is primitive recursive.

Proof:  $f(n) = g_1(n) * (1 - h_1(n)) + \cdots + g_r(n) * (1 - h_r(n))$ 

#### **Sums and products**

**Theorem** If  $\varphi : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$  is a

If  $g : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$  is a primitive recursive function then the following functions  $f_1, f_2 : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$  are also primitive recursive:

$$f_1(\mathbf{n}, m) = \begin{cases} 0 & \text{if } m = 0\\ \sum_{i < m} g(\mathbf{n}, i) & \text{if } m > 0 \end{cases}$$

$$f_2(\mathbf{n}, m) = \begin{cases} 0 & \text{if } m = 0\\ 0 & \text{if } m = 0\\ \prod_{i < m} g(\mathbf{n}, i) & \text{if } m > 0 \end{cases}$$

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**Proof**:  $f_1$  and  $f_2$  can be written using primitive recursion and case distinction:

$$egin{aligned} f_1({f n},0) &= 0 \ f_1({f n},m+1) &= f_1({f n},m) + g({f n},m) \end{aligned}$$

Theorem

If  $g : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$  is a primitive recursive function then the following functions  $f_1, f_2 : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$  are also primitive recursive:

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**Proof**:  $f_1$  and  $f_2$  can be written using primitive recursion and case distinction:

$$f_1(\mathbf{n}, 0) = 0$$
  

$$f_1(\mathbf{n}, m+1) = f_1(\mathbf{n}, m) + g(\mathbf{n}, m)$$
  

$$f_2(\mathbf{n}, 0) = 1$$
  

$$f_2(\mathbf{n}, m+1) = f_2(\mathbf{n}, m) * g(\mathbf{n}, m)$$

**Definition.** Let  $g : \mathbb{N}^{k+1} \to \mathbb{N}$  be a function. The bounded  $\mu$  operator is defined as follows:  $\mu_{i < m} i (g(\mathbf{n}, i) = 0) := \begin{cases} i_0 & \text{if } g(\mathbf{n}, i_0) = 0 \\ & \text{and for all } j < i_0 & g(\mathbf{n}, j) \neq 0 \\ 0 & \text{if } g(\mathbf{n}, j) \neq 0 \text{ for all } 0 \le j < m \\ & \text{or } m = 0 \end{cases}$ 

 $\mu_{i < m}$  i  $(g(\mathbf{n}, i) = 0)$  is the smallest i < m such that  $g(\mathbf{n}, i) = 0$ 

**Theorem.** If  $g : \mathbb{N}^{k+1} \to \mathbb{N}$  is a primitive recursive function then the function  $f : \mathbb{N}^{k+1} \to \mathbb{N}$  defined by:

$$f(\mathbf{n},m) = \mu_{i < m} i (g(\mathbf{n},i) = 0)$$

is also primitive recursive

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**Proof**: We can define *f* as follows:

$$f(\mathbf{n}, 0) = 0$$

$$f(\mathbf{n}, m+1) = \begin{cases} 0 & \text{if } m = 0 \\ m & \text{if } g(\mathbf{n}, m) = f(\mathbf{n}, m) = 0 \land g(\mathbf{n}, 0) \neq 0 \land m > 0 \\ f(\mathbf{n}, m) \text{ otherwise} \end{cases}$$

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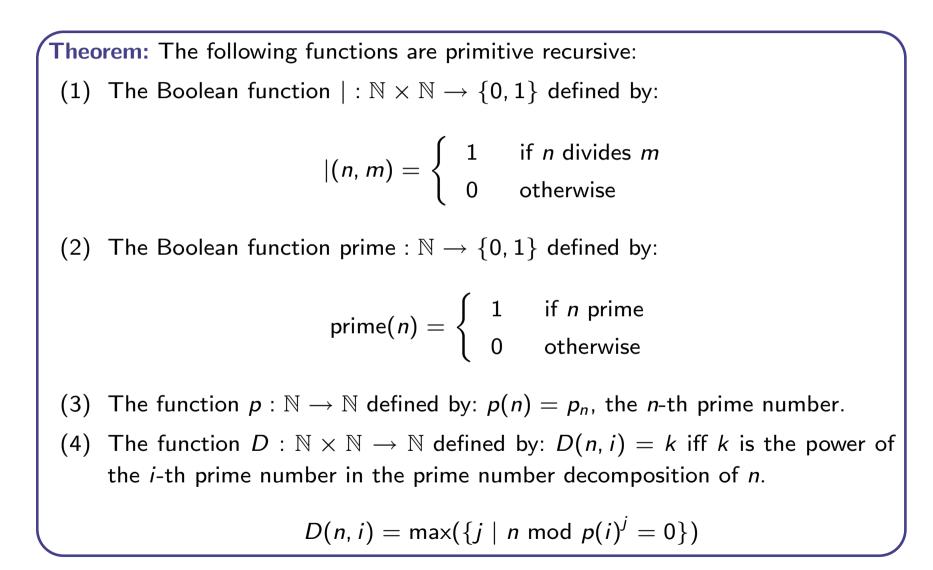
$$f(\mathbf{n}, m) = \mu_{i < m} i (g(\mathbf{n}, i) = 0)$$

is also primitive recursive

**Proof**: We can define *f* as follows:

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Proof:

(1)  $|: \mathbb{N} \times \mathbb{N} \to \{0, 1\}$  defined by:  $|(n, m) = \begin{cases} 1 & \text{if } n \text{ divides } m \\ 0 & \text{otherwise} \end{cases}$ 

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 defined by:  
 $|(n, m) = \begin{cases} 1 & \text{if } n \text{ divides } m \\ 0 & \text{otherwise} \end{cases}$   
 $|(n, m) = 1 \text{ iff } \exists z (n * z = m) \text{ iff } \prod_{z \le m} (n * z - m) + (m - n * z) = 0.$ 

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 defined by:  
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$$|(n, m) = 1 - \prod_{z \le m} (n * z - m) + (m - n * z)$$

Proof:

(2) prime :  $\mathbb{N} \to \{0, 1\}$  defined by: prime(n) =  $\begin{cases} 1 & \text{if } n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$ 

Proof:

(2) prime :  $\mathbb{N} \to \{0, 1\}$  defined by: prime(n) =  $\begin{cases} 1 & \text{if } n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$ 

prime(n) = 1 iff  $(n \ge 2$  and  $\forall y < n(y = 0 \lor y = 1 \lor | (y, n) = 0)$ 

 $prime(n) = 1 - ((2 - n) + \sum_{y < n} (|(y, n) * y * ((y - 1) + (1 - y))))$ 

Proof:

- (3) The function  $p : \mathbb{N} \to \mathbb{N}$  defined by:  $p(n) = p_n$ , the *n*-th prime number.
- p(0) = 0 and p(1) = 2.
- p(n+1) is the smallest number *i* which is larger than p(n) and is prime.

Proof:

- (3) The function  $p : \mathbb{N} \to \mathbb{N}$  defined by:  $p(n) = p_n$ , the *n*-th prime number.
- p(0) = 0 and p(1) = 2.

p(n+1) is the smallest number *i* which is larger than p(n) and is prime.

We also have an upper bound for the number i.

Recall the proof of the fact that the set of prime numbers is infinite.

 $i \leq p(n)! + 1$ 

$$p(n+1) = \mu_{i \le p(n)!+1} i [((1 - prime(i)) + ((p(n)+1) - i)) = 0]$$

Proof:

(4)  $D : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$  defined by: D(n, i) = k iff k is the power of the *i*-th prime number in the prime number decomposition of n.

 $D(n,i) = \max(\{j \mid n \mod p(i)^j = 0\})$ 

D(0, i) := 0;  $D(n, i) = \min(\{j \le n \mid |(p(i)^{j+1}, n) = 0\})$  $D(n, i) = \mu_{j \le n} j (|(p(i)^{j+1}, n) = 0)$