Advanced Topics in Theoretical Computer Science

Part 3: Recursive functions (2)

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Contents

3. Recursive functions

- Introduction/Motivation
- Primitive recursive functions $\mapsto \mathcal{P}$
- $\mathcal{P} = \text{LOOP}$
- μ-recursive functions $\mapsto F_\mu$
- $F_\mu = \text{WHILE}$
- Summary

3. Recursive functions

- Introduction/Motivation
- Primitive recursive functions $\mapsto \mathcal{P}$
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- $F_\mu = \text{WHILE}$
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Last time

Primitive recursive functions

Primitive recursion

If the functions

$$g : \mathbb{N}^k \longrightarrow \mathbb{N} \qquad (k \geq 0)$$
$$h : \mathbb{N}^{k+2} \longrightarrow \mathbb{N}$$

are primitive recursive,

then the function

$$f : \mathbb{N}^{k+1} \longrightarrow \mathbb{N} \text{ with } \qquad \begin{aligned} f(\mathbf{n}, 0) &= g(\mathbf{n}) \\ f(\mathbf{n}, m+1) &= h(\mathbf{n}, m, f(\mathbf{n}, m)) \end{aligned}$$

is also primitive recursive.

Primitive recursive functions

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is also primitive recursive.

Notation without arguments: $f = \mathcal{PR}[g, h]$

Primitive recursive functions

Definition (Primitive recursive functions)

- **Atomic functions:** The functions
  - Null 0
  - Successor $+1$
  - Projection π_i^k $(1 \leq i \leq k)$

  are primitive recursive.
- **Composition:** The functions obtained by composition from primitive recursive functions are primitive recursive.
- **Primitive recursion:** The functions obtained by primitive recursion from primitive recursive functions are primitive recursive.

Notation: $\mathcal{P} =$ The set of all primitive recursive functions

Arithmetical functions: definitions

$f(n) = n + c$, **for** $c \in \mathbb{N}, c > 0$

$$f = \underbrace{(+1) \circ \cdots \circ (+1)}_{c \text{ times}}$$

Identity

$$f = \pi^1_1$$

$f(n, m) = n + m$

$$f = \mathcal{PR}[\pi^1_1, (+1) \circ \pi^3_3]$$

$f(n) = n - 1$

$$f = \mathcal{PR}[0, \pi^2_1]$$

$f(n, m) = n - m$

$$f = \mathcal{PR}[\pi^1_1, (-1) \circ \pi^3_3]$$

$f(n, m) = n * m$

$$f = \mathcal{PR}[0, + \circ (\pi^3_3, \pi^3_1)]$$

Re-ordering/Omitting/Repeating Arguments

Lemma The set of primitive recursive functions is closed under:

- Re-ordering
- Omitting
- Repeating

of arguments when composing functions.

Proof: (Idea)

A tuple of arguments $\mathbf{n'} = (n_{i_1}, \ldots, n_{i_k})$ obtained from $\mathbf{n} = (n_1, \ldots, n_k)$ by re-ordering, omitting or repeating components can be represented as:

$$\mathbf{n'} = (\pi^k_{i_1}(\mathbf{n}), \ldots, \pi^k_{i_k}(\mathbf{n}))$$

Today

- More examples
- $\mathcal{P} = \mathsf{LOOP}$

Additional Arguments

Lemma. Assume $f : \mathbb{N}^k \longrightarrow \mathbb{N}$ is primitive recursive.
Then, for every $l \in \mathbb{N}$, the function $f' : \mathbb{N}^k \times \mathbb{N}^l \longrightarrow \mathbb{N}$ defined for every $\mathbf{n} \in \mathbb{N}^k$ and every $\mathbf{m} \in \mathbb{N}^l$ by:

$$f'(\mathbf{n}, \mathbf{m}) = f(\mathbf{n})$$

is primitive recursive.

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Proof:

Case 1: $k = 0$, i.e. f is a constant. Then f' can be expressed by primitive recursion:

$$f'(n) = f \qquad\qquad f' = \mathcal{PR}[f, \pi^2_2]$$

$$f'(n+1) = f'(n) = \pi^2_2(n, f'(n))$$

Case 2: $k' \neq 0$. Let $\mathbf{n} = (n_1, \ldots, n_k, m_1, \ldots, m_l)$

Then $f'(\mathbf{n}) = f(\pi^{k+l}_1(\mathbf{n}), \ldots, \pi^{k+l}_k(\mathbf{n})) = f \circ \pi^{k+1}$.

Case distinction

Lemma (Case distinction is primitive recursive)

If
- g_i, h_i $(1 \le i \le r)$ are primitive recursive functions, and
- for every n there exists a unique i with $h_i(n) = 0$

then the function f defined by:

$$f(n) = \begin{cases} g_1(n) & \text{if } h_1(n) = 0 \\ \dots & \\ g_r(n) & \text{if } h_r(n) = 0 \end{cases}$$

is primitive recursive.

Proof: $f(n) = g_1(n) * (1 - h_1(n)) + \cdots + g_r(n) * (1 - h_r(n))$

Sums and products

Theorem

If $g : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$ is a primitive recursive function then the following functions $f_1, f_2 : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$ are also primitive recursive:

$$
\begin{aligned}
f_1(\mathbf{n}, m) &= \begin{cases} 0 & \text{if } m = 0 \\ \sum_{i<m} g(\mathbf{n}, i) & \text{if } m > 0 \end{cases} \\
f_2(\mathbf{n}, m) &= \begin{cases} 0 & \text{if } m = 0 \\ \prod_{i<m} g(\mathbf{n}, i) & \text{if } m > 0 \end{cases}
\end{aligned}
$$

Sums and products

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Proof: f_1 and f_2 can be written using primitive recursion and case distinction:

$$f_1(\mathbf{n}, 0) = 0$$

$$f_1(\mathbf{n}, m+1) = f_1(\mathbf{n}, m) + g(\mathbf{n}, m)$$

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$$f_2(\mathbf{n}, 0) = 1$$

$$f_2(\mathbf{n}, m+1) = f_2(\mathbf{n}, m) * g(\mathbf{n}, m)$$

Bounded μ operator

Definition.
Let $g : \mathbb{N}^{k+1} \longrightarrow \mathbb{N}$ be a function.
The bounded μ operator is defined as follows:

$$\mu_{i<m}\ i\ (g(\mathbf{n}, i) = 0) := \begin{cases} i_0 & \text{if } g(\mathbf{n}, i_0) = 0 \\ & \text{and for all } j < i_0 \ \ g(\mathbf{n}, j) \neq 0 \\ 0 & \text{if } g(\mathbf{n}, j) \neq 0 \text{ for all } 0 \leq j < m \\ & \text{or } \ m = 0 \end{cases}$$

$\mu_{i<m}\ i\ (g(\mathbf{n}, i) = 0)$ is the smallest $i < m$ such that $g(\mathbf{n}, i) = 0$

Bounded μ operator

Theorem.
If $g : \mathbb{N}^{k+1} \longrightarrow \mathbb{N}$ is a primitive recursive function
then the function $f : \mathbb{N}^{k+1} \longrightarrow \mathbb{N}$ defined by:

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is also primitive recursive

Proof: We can define f as follows:

$$\begin{aligned} f(\mathbf{n}, 0) &= 0 \\ f(\mathbf{n}, m+1) &= \begin{cases} 0 & \text{if } m = 0 \\ m & \text{if } g(\mathbf{n}, m) = f(\mathbf{n}, m) = 0 \wedge g(\mathbf{n}, 0) \neq 0 \wedge m > 0 \\ & \\ f(\mathbf{n}, m) & \text{otherwise} \end{cases} \end{aligned}$$

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Prime number functions

Theorem: The following functions are primitive recursive:

(1) The Boolean function $| : \mathbb{N} \times \mathbb{N} \longrightarrow \{0, 1\}$ defined by:

$$|(n, m) = \begin{cases} 1 & \text{if } n \text{ divides } m \\ 0 & \text{otherwise} \end{cases}$$

(2) The Boolean function $\mathrm{prime} : \mathbb{N} \longrightarrow \{0, 1\}$ defined by:

$$\mathrm{prime}(n) = \begin{cases} 1 & \text{if } n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

(3) The function $p : \mathbb{N} \longrightarrow \mathbb{N}$ defined by: $p(n) = p_n$, the n-th prime number.

(4) The function $D : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ defined by: $D(n, i) = k$ iff k is the power of the i-th prime number in the prime number decomposition of n.

$$D(n, i) = \max(\{j \mid n \bmod p(i)^j = 0\})$$

Prime number functions

Proof:

(1) $| : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ defined by:

$$|(n, m) = \begin{cases} 1 & \text{if } n \text{ divides } m \\ 0 & \text{otherwise} \end{cases}$$

Prime number functions

Proof:

(1) $| : \mathbb{N} \times \mathbb{N} \to \{0,1\}$ defined by:

$$|(n,m) = \begin{cases} 1 & \text{if } n \text{ divides } m \\ 0 & \text{otherwise} \end{cases}$$

$|(n,m) = 1$ iff $\exists z(n * z = m)$ iff $\prod_{z \leq m}(n * z - m) + (m - n * z) = 0$.

Prime number functions

Proof:

(1) $\mid : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ defined by:

$$\mid(n, m) = \begin{cases} 1 & \text{if } n \text{ divides } m \\ 0 & \text{otherwise} \end{cases}$$

$\mid(n, m) = 1$ iff $\exists z(n * z = m)$ iff $\prod_{z \leq m} (n * z - m) + (m - n * z) = 0$.

$\mid(n, m) = 1 - \prod_{z \leq m} (n * z - m) + (m - n * z)$

Prime number functions

Proof:

(2) prime : $\mathbb{N} \to \{0, 1\}$ defined by:

$$\text{prime}(n) = \begin{cases} 1 & \text{if } n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

Prime number functions

Proof:

(2) prime : $\mathbb{N} \to \{0,1\}$ defined by:

$$\mathrm{prime}(n) = \begin{cases} 1 & \text{if } n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

$\mathrm{prime}(n) = 1$ iff $(n \geq 2$ and $\forall y < n(y = 0 \vee y = 1 \vee |(y, n) = 0)$

$\mathrm{prime}(n) = 1 - ((2 - n) + \sum_{y<n}(|(y, n) * y * ((y - 1) + (1 - y))))$

Prime number functions

Proof:

(3) The function $p : \mathbb{N} \longrightarrow \mathbb{N}$ defined by: $p(n) = p_n$, the n-th prime number.

$p(0) = 0$ and $p(1) = 2$.

$p(n+1)$ is the smallest number i which is larger than $p(n)$ and is prime.

Prime number functions

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$p(0) = 0$ and $p(1) = 2$.

$p(n+1)$ is the smallest number i which is larger than $p(n)$ and is prime.

We also have an upper bound for the number i.

Recall the proof of the fact that the set of prime numbers is infinite.

$$i \leq p(n)! + 1$$

$$p(n+1) = \mu_{i \leq p(n)!+1} \ i \ [((1 - \text{prime}(i)) + ((p(n)+1) - i)) = 0]$$

Prime number functions

Proof:

(4) $D : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ defined by: $D(n, i) = k$ iff k is the power of the i-th prime number in the prime number decomposition of n.

$$D(n, i) = \max(\{j \mid n \bmod p(i)^j = 0\})$$

$D(0, i) := 0;$

$D(n, i) = \min(\{j \leq n \mid \quad |(p(i)^{j+1}, n) = 0\})$

$D(n, i) = \mu_{j \leq n}\, j\ (|(p(i)^{j+1}, n) = 0)$