

# Advanced Topics in Theoretical Computer Science

## Part 3: Recursive functions (3)

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- Register machines (LOOP, WHILE, GOTO)
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- Complexity
- Other computation models: e.g. Büchi Automata,  $\lambda$ -calculus

### 3. Recursive functions

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- Introduction/Motivation
- Primitive recursive functions  $\mapsto \mathcal{P}$
- $\mathcal{P} = \text{LOOP}$
- $\mu$ -recursive functions  $\mapsto F_\mu$
- $F_\mu = \text{WHILE}$
- Summary

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$\mapsto \mathcal{P}$

$\mapsto F_\mu$

# Until now

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# Primitive recursive functions

**Primitive recursion.** If the functions  $g : \mathbb{N}^k \rightarrow \mathbb{N}$  and  $h : \mathbb{N}^{k+2} \rightarrow \mathbb{N} (k \geq 0)$  are primitive recursive, then the following function is also primitive recursive:

$$f : \mathbb{N}^{k+1} \rightarrow \mathbb{N} \text{ with } \begin{aligned} f(\mathbf{n}, 0) &= g(\mathbf{n}) \\ f(\mathbf{n}, m+1) &= h(\mathbf{n}, m, f(\mathbf{n}, m)) \end{aligned}$$

**Notation without arguments:**  $f = \mathcal{PR}[g, h]$

## Definition (Primitive recursive functions)

- **Atomic functions:** The functions null (0), successor (+1) and projection ( $\pi_i^k$  ( $1 \leq i \leq k$ )) are primitive recursive.
- **Composition:** The functions obtained by composition from primitive recursive functions are primitive recursive.
- **Primitive recursion:** The functions obtained by primitive recursion from primitive recursive functions are primitive recursive.

# Examples of primitive recursive functions

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$$f(n) = n + c, \quad \text{for } c \in \mathbb{N}, c \geq 0 \quad + (n, m) = n + m$$

$$(-1)(n) = n - 1 \quad - (n, m) = n - m$$

$$*(n, m) = n * m$$

**Lemma** The set of primitive recursive functions is closed under re-ordering, omitting and repeating of arguments when composing functions.

**Lemma.** Assume  $f : \mathbb{N}^k \rightarrow \mathbb{N}$  is primitive recursive.

Then, for every  $l \in \mathbb{N}$ , the function  $f' : \mathbb{N}^k \times \mathbb{N}^l \rightarrow \mathbb{N}$  defined for every  $\mathbf{n} \in \mathbb{N}^k$  and every  $\mathbf{m} \in \mathbb{N}^l$  by:

$$f'(\mathbf{n}, \mathbf{m}) = f(\mathbf{n})$$

is primitive recursive.

# Primitive recursive functions

**Lemma (Case distinction).** If  $g_i, h_i$  ( $1 \leq i \leq r$ ) are primitive recursive functions, and for every  $n$  there exists a unique  $i$  with  $h_i(n) = 0$ , then the function  $f$  defined by:

$$f(n) = \begin{cases} g_1(n) & \text{if } h_1(n) = 0 \\ \dots & \\ g_r(n) & \text{if } h_r(n) = 0 \end{cases}$$

is primitive recursive.

## Theorem (Sums and products)

If  $g : \mathbb{N}^k \times \mathbb{N} \rightarrow \mathbb{N}$  is a primitive recursive function then the following functions  $f_1, f_2 : \mathbb{N}^k \times \mathbb{N} \rightarrow \mathbb{N}$  are also primitive recursive:

$$f_1(\mathbf{n}, m) = \begin{cases} 0 & \text{if } m = 0 \\ \sum_{i < m} g(\mathbf{n}, i) & \text{if } m > 0 \end{cases} \quad f_2(\mathbf{n}, m) = \begin{cases} 0 & \text{if } m = 0 \\ \prod_{i < m} g(\mathbf{n}, i) & \text{if } m > 0 \end{cases}$$



# Bounded $\mu$ operator

**Definition.** Let  $g : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$  be a function. The **bounded  $\mu$  operator** is defined by:

$$\mu_{i < m} i (g(\mathbf{n}, i) = 0) := \begin{cases} i_0 & \text{if } g(\mathbf{n}, i_0) = 0 \text{ and for all } j < i_0 \text{ } g(\mathbf{n}, j) \neq 0 \\ 0 & \text{if } g(\mathbf{n}, j) \neq 0 \text{ for all } 0 \leq j < m \text{ or } m = 0 \end{cases}$$

$\mu_{i < m} i (g(\mathbf{n}, i) = 0)$  is the smallest  $i < m$  such that  $g(\mathbf{n}, i) = 0$

**Theorem.** If  $g : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$  is a primitive recursive function then the function  $f : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$  defined by:

$$f(\mathbf{n}, m) = \mu_{i < m} i (g(\mathbf{n}, i) = 0)$$

is also primitive recursive

# Prime number functions

**Theorem:** The following functions are primitive recursive:

(1) The Boolean function  $| : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$  defined by:

$$|(n, m) = \begin{cases} 1 & \text{if } n \text{ divides } m \\ 0 & \text{otherwise} \end{cases}$$

(2) The Boolean function  $\text{prime} : \mathbb{N} \rightarrow \{0, 1\}$  defined by:

$$\text{prime}(n) = \begin{cases} 1 & \text{if } n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

(3) The function  $p : \mathbb{N} \rightarrow \mathbb{N}$  defined by:  $p(n) = p_n$ , the  $n$ -th prime number.

(4) The function  $D : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  defined by:  $D(n, i) = k$  iff  $k$  is the power of the  $i$ -th prime number in the prime number decomposition of  $n$ .

$$D(n, i) = \max(\{j \mid n \bmod p(i)^j = 0\})$$

# Prime number functions

---

Proof:

(4)  $D : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  defined by:  $D(n, i) = k$  iff  $k$  is the power of the  $i$ -th prime number in the prime number decomposition of  $n$ .

$$D(n, i) = \max(\{j \mid n \bmod p(i)^j = 0\})$$

$$D(0, i) := 0;$$

$$D(n, i) = \min(\{j \leq n \mid |(p(i)^{j+1}, n) = 0\})$$

$$D(n, i) = \mu_{j \leq n} j \ (|(p(i)^{j+1}, n) = 0)$$

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# Goal

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Show that  $\mathcal{P} = \text{LOOP}$

Idea:

To show that  $\mathcal{P} \supseteq \text{LOOP}$  we have to show that every LOOP computable function can be expressed as a primitive recursive function.

For this, we will encode the contents of arbitrarily many registers in one natural number (used as input for this primitive recursive function).

For this encoding we will use Gödelisation. We will need to show that Gödelisation is primitive recursive.

To show that  $\mathcal{P} \subseteq \text{LOOP}$  we have to show that:

- all atomic primitive recursive functions are LOOP computable, and
- LOOP is closed under composition of functions and primitive recursion.

# Gödelisation

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To show: Gödelisation is primitive recursive

Informally:

- Coding number sequences as a number
- Corresponding decoding function (projection)

are primitive recursive

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More precise formulation:

There exist primitive recursive functions

$$K^r : \mathbb{N}^r \rightarrow \mathbb{N} \quad (r \geq 1)$$

$$D_i : \mathbb{N} \rightarrow \mathbb{N} \quad (1 \leq i \leq r)$$

with:

$$D_i(K^r(n_1, \dots, n_r)) = n_i$$



# Gödelisation

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To show: Gödelisation is primitive recursive

Informally:

- Coding number sequences as a number
- Corresponding decoding function (projection)

are primitive recursive

Recall:

Gödelisation: Coding number sequences as a number

Bijection between  $\mathbb{N}^r$  and  $\mathbb{N}$ :  $K^r : \mathbb{N}^r \rightarrow \mathbb{N}$ , defined by:

$$K^r(n_1, \dots, n_r) = \prod_{i=1}^r p(i)^{n_i}.$$

Decoding: The inverses  $D_i : \mathbb{N} \rightarrow \mathbb{N}$  of  $K^r$  defined by  $D_i(n) = D(n, i)$

# Gödelisation

---

Bijection between  $\mathbb{N}^r$  and  $\mathbb{N}$ :  $K^r : \mathbb{N}^r \rightarrow \mathbb{N}$ , defined by:

$$K^r(n_1, \dots, n_r) = \prod_{i=1}^r p(i)^{n_i}.$$

$D_i : \mathbb{N} \rightarrow \mathbb{N}$ ,  $1 \leq i \leq r$ , defined by  $D_i(n) = D(n, i)$

**Theorem.**  $K^r$  and  $D_1, \dots, D_r$  are primitive recursive.

# Gödelisation

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Bijection between  $\mathbb{N}^r$  and  $\mathbb{N}$ :  $K^r : \mathbb{N}^r \rightarrow \mathbb{N}$ , defined by:

$$K^r(n_1, \dots, n_r) = \prod_{i=1}^r p(i)^{n_i}.$$

$D_i : \mathbb{N} \rightarrow \mathbb{N}$ ,  $1 \leq i \leq r$ , defined by  $D_i(n) = D(n, i)$

**Theorem.**  $K^r$  and  $D_1, \dots, D_r$  are primitive recursive.

**Lemma.**

- (1)  $D_i(K^r(n_1, \dots, n_r)) = n_i$  for all  $1 \leq i \leq r$ .
- (2)  $K^r(n_1, \dots, n_r) = K^{r+1}(n_1, \dots, n_r, 0)$

In general,  $D_i(K^r(n_1, \dots, n_r)) = 0$  if  $i > r$ .

# Gödelisation

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## Notation:

$$K^r(n_1, \dots, n_r) = \langle n_1, \dots, n_r \rangle$$

$$D_i(n) = (n)_i$$

For  $r = 0$ :

$$\langle \rangle = 1$$

$$(\langle \rangle)_i = 0$$

# Gödelisation: Applications

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## Theorem (Simultaneous Recursion)

If

$$f_1(\mathbf{n}, 0) = g_1(\mathbf{n})$$

...

$$f_r(\mathbf{n}, 0) = g_r(\mathbf{n})$$

$$f_1(\mathbf{n}, m + 1) = h_1(\mathbf{n}, m, f_1(\mathbf{n}, m), \dots, f_r(\mathbf{n}, m))$$

...

$$f_r(\mathbf{n}, m + 1) = h_r(\mathbf{n}, m, f_1(\mathbf{n}, m), \dots, f_r(\mathbf{n}, m))$$

and if  $g_1, \dots, g_r, h_1, \dots, h_r$  are primitive recursive  
then  $f_1, \dots, f_r$  are primitive recursive.

# Example

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Let  $f_1$  and  $f_2$  be defined by simultaneous recursion as follows:

$$f_1(0) = 0$$

$$f_2(0) = 1$$

$$f_1(n+1) = f_2(n)$$

$$f_2(n+1) = f_1(n) + f_2(n)$$

# Example

---

Let  $f_1$  and  $f_2$  be defined by simultaneous recursion as follows:

$$f_1(0) = 0$$

$$g_1 = 0$$

$$f_2(0) = 1$$

$$g_2 = 1$$

$$f_1(n+1) = f_2(n)$$

$$h_1(n, f_1(n), f_2(n)) = f_2(n)$$

$$h_1 = \pi_3^3$$

$$f_2(n+1) = f_1(n) + f_2(n)$$

$$h_2(n, f_1(n), f_2(n)) = f_1(n) + f_2(n)$$

$$h_2 = + \circ (\pi_2^3, \pi_3^3)$$

# Gödelisation: Applications

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## Theorem (Simultaneous Recursion)

If

$$f_1(\mathbf{n}, 0) = g_1(\mathbf{n})$$

...

$$f_r(\mathbf{n}, 0) = g_r(\mathbf{n})$$

$$f_1(\mathbf{n}, m+1) = h_1(\mathbf{n}, m, f_1(\mathbf{n}, m), \dots, f_r(\mathbf{n}, m))$$

...

$$f_r(\mathbf{n}, m+1) = h_r(\mathbf{n}, m, f_1(\mathbf{n}, m), \dots, f_r(\mathbf{n}, m))$$

and if  $g_1, \dots, g_r, h_1, \dots, h_r$  are primitive recursive  
then  $f_1, \dots, f_r$  are primitive recursive.



# Gödelisation: Applications

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**Proof:** We define a new function  $f$  by:

$$f(\mathbf{n}, m) = \langle f_1(\mathbf{n}, m), \dots, f_r(\mathbf{n}, m) \rangle$$

$f$  can be computed by primitive recursion as follows:

$$\begin{aligned} f(\mathbf{n}, 0) &= \langle g_1(\mathbf{n}), \dots, g_r(\mathbf{n}) \rangle \\ f(\mathbf{n}, m+1) &= \langle h_1(\mathbf{n}, m, (f(\mathbf{n}, m))_1, \dots, (f(\mathbf{n}, m))_r), \dots, \\ &\quad h_r(\mathbf{n}, m, (f(\mathbf{n}, m))_1, \dots, (f(\mathbf{n}, m))_r) \rangle \end{aligned}$$

$K^r \circ (g_1, \dots, g_r)$  and  $K^r \circ (h_1, \dots, h_r)$  are primitive recursive.

For all  $1 \leq i \leq r$ ,  $f_i(\mathbf{n}, m) = D_i(f(\mathbf{n}, m))$ .

Since  $f_i = D_i \circ f$  is primitive recursive, it follows that  $f_i$  is primitive recursive for all  $1 \leq i \leq r$ .

$$\mathcal{P} = \text{LOOP}$$

---

**Theorem ( $\mathcal{P} = \text{LOOP}$ ).** The set of all LOOP computable functions is equal to the set of all primitive recursive functions

Proof (Idea)

1.  $\mathcal{P} \subseteq \text{LOOP}$

$$\mathcal{P} = \text{LOOP}$$

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**Theorem ( $\mathcal{P} = \text{LOOP}$ ).** The set of all LOOP computable functions is equal to the set of all primitive recursive functions

Proof (Idea)

1.  $\mathcal{P} \subseteq \text{LOOP}$

- 1a: We show that all atomic primitive recursive functions are LOOP computable
- 1b: We show that LOOP is closed under composition of functions
- 1c: We show that LOOP is closed under primitive recursion

$$\mathcal{P} = \text{LOOP}$$

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**Theorem ( $\mathcal{P} = \text{LOOP}$ ).** The set of all LOOP computable functions is equal to the set of all primitive recursive functions

Proof (Idea)

### 1. $\mathcal{P} \subseteq \text{LOOP}$

1a: All atomic primitive recursive functions are LOOP computable

0 :	$\mathbf{x}_1 := \mathbf{x}_1 - 1$	//NOP
+1 :	$\mathbf{x}_2 := \mathbf{x}_1 + 1$	
$\pi_j^k$	$\mathbf{x}_{k+1} := \mathbf{x}_j$	

# $\mathcal{P} = \text{LOOP}$

---

Proof (ctd) 1b: LOOP is closed under composition of functions

Let  $f : \mathbb{N}^k \rightarrow \mathbb{N}$  with  $f(\mathbf{n}) = h(g_1(\mathbf{n}), \dots, g_r(\mathbf{n}))$

Assume that:

- $P_h$  computes  $h$
- $P_{g_j}$  computes  $g_j$  ( $1 \leq j \leq r$ )

Idea:  $f$  is computed by the program  $P_f$ :

$$P'_{g_1}; \dots; P'_{g_r}; P'_h$$

where  $P'_{g_i}$  differs from  $P_{g_i}$  (and  $P'_h$  from  $P_h$ ) only up to the fact that registers have been renamed/the contents stored in them copied.

# $\mathcal{P} = \text{LOOP}$

---

Proof (ctd) 1b: LOOP is closed under composition of functions

Let  $f : \mathbb{N}^k \rightarrow \mathbb{N}$  with  $f(\mathbf{n}) = h(g_1(\mathbf{n}), \dots, g_r(\mathbf{n}))$

Assume that:

- $P_h$  computes  $h$
- $P_{g_j}$  computes  $g_j$  ( $1 \leq j \leq r$ )

More precisely:  $P'_{g_i}$ : obtained from  $P_{g_i}$  by renaming register  $x_{k+i}$  to  $x_{k+r+i}$ .  
 $\mapsto$  keep free registers  $x_{k+1}, \dots, x_{k+r}$  for writing result of  $P_{g_1}, \dots, P_{g_r}$

$P'_h$ : obtained from  $P_h$  by renaming  $x_j$  to  $x_{k+j}$ .

$P_f :$   $P'_{g_1}; x_{k+1} := x_{k+r+1}; x_{k+r+1} := 0; \dots$

$P'_{g_r}; x_{k+r} := x_{k+r+1}; x_{k+r+1} := 0;$

$P'_h; x_{k+1} := x_{k+r+1}; x_{k+2} := 0; \dots; x_{k+r+1} := 0$

# $\mathcal{P} = \text{LOOP}$

---

Proof (ctd) 1c: LOOP is closed under primitive recursion

Assume that  $f : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$  is such that:

$$f(\mathbf{n}, 0) = g(\mathbf{n})$$

$$f(\mathbf{n}, m + 1) = h(\mathbf{n}, m, f(\mathbf{n}, m))$$

Then  $f$  is computed by the following LOOP Program:

```
x_store_m := x_{k+1};                                     // Number of loops (m)
x_{k+1} := 0;                                              // Actual value of m (at the beginning 0)
P'_g;                                                    // Computes f(n, 0); result in x_{k+2}
loop x_store_m do
  P_h;                                                    // Computes f(n, x_{k+1} + 1) = h(n, m, f(n, m))
  x_{k+2} := x_{k+2+1};                                   // x_{k+2} = f(n, x_{k+1} + 1)
  x_{k+2+1} := 0;
  x_{k+1} := x_{k+1} + 1                                  // m = m + 1
end;
x_store_m := 0
```

# $\mathcal{P} = \text{LOOP}$

---

Proof (ctd) **1c: LOOP is closed under primitive recursion**

Assume that  $f : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$  is such that:

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x_{k+1} := 0;                                          // Actual value of m (at the beginning 0)
P'_g;                                                // Computes f(n, 0); result in x_{k+2}
loop x_store_m do
  P_h;
  x_{k+2} := x_{k+2+1};
  x_{k+2+1} := 0;
  x_{k+1} := x_{k+1} + 1
end;
x_store_m := 0
```

where  $P'_g$  differs from  $P_g$  only in the fact that some registers have been renamed (e.g. output in  $x_{k+2}$ , not in  $x_{k+1}$ )



$$\mathcal{P} = \text{LOOP}$$

---

**Theorem ( $\mathcal{P} = \text{LOOP}$ ).** The set of all LOOP computable functions is equal to the set of all primitive recursive functions

Proof (Idea)

## 2. $\text{LOOP} \subseteq \mathcal{P}$

Let  $P$  be a LOOP program which:

- uses registers  $x_1, \dots, x_l$
- has  $m$  loop instructions

We construct a primitive recursive function  $f_P$  which “simulates”  $P$

$$f(\langle n_1, \dots, n_l, h_1, \dots, h_m \rangle) = \langle n'_1, \dots, n'_l, h_1, \dots, h_m \rangle$$

if and only if:

$P$  started with  $n_i$  in register  $x_i$  terminates with  $n'_i$  in  $x_i$  ( $1 \leq i \leq l$ ).

In  $h_j$  it is “recorded” how long loop  $j$  should still run.

# $\mathcal{P} = \text{LOOP}$

---

**Proof** (ctd) At the beginning and at the end of the simulation of  $P$  we have

$$h_1 = 0, \dots, h_m = 0.$$

Let  $g : \mathbb{N}^k \rightarrow \mathbb{N}$  be the function computed by the LOOP program  $P$ .

- At the beginning the registers  $x_1, \dots, x_k$  contain the input  $n_1, \dots, n_k$  and all the other registers contain the value 0.
- At the end of the execution of  $P$  the registers  $x_1, \dots, x_k$  contain the input  $n_1, \dots, n_k$ , register  $x_{k+1}$  contains the value  $n_{k+1} = g(n_1, \dots, n_k)$  and all the other registers contain the value 0;

The function  $g$  is then primitive recursive, since:

$$\begin{aligned} g(n_1, \dots, n_k) &= n_{k+1} = (\underbrace{\langle n_1, \dots, n_k, n_{k+1}, 0, \dots, 0 \rangle}_{\text{contents of registers } x_1, \dots, x_l} \underbrace{\langle 0, \dots, 0 \rangle}_{h_1, \dots, h_m})_{k+1} \\ &= (f_P(\langle n_1, \dots, n_k, 0, 0, \dots \rangle))_{k+1} \end{aligned}$$

# $\mathcal{P} = \text{LOOP}$

---

Proof (ctd) **Construction of  $f_P$ :**

2a:  $P$  is  $x_i := x_i + 1$

$$f_P(n) = \langle (n)_1, \dots, (n)_{i-1}, (n)_i + 1, (n)_{i+1}, \dots \rangle$$

$P$  is  $x_i := x_i - 1$

$$f_P(n) = \langle (n)_1, \dots, (n)_{i-1}, (n)_i - 1, (n)_{i+1}, \dots \rangle$$

# $\mathcal{P} = \text{LOOP}$

---

Proof (ctd) **Construction of  $f_P$ :**

2a:  $P$  is  $x_i := x_i + 1$

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$P$  is  $x_i := x_i - 1$

$$f_P(n) = \langle (n)_1, \dots, (n)_{i-1}, (n)_i - 1, (n)_{i+1}, \dots \rangle$$

2b:  $P$  is  $P_1; P_2$

$$f_P = f_{P_2} \circ f_{P_1} \quad \text{i.e. } f_P(n) = f_{P_2}(f_{P_1}(n))$$

# $\mathcal{P} = \text{LOOP}$

---

Proof (ctd) **Construction of  $f_P$ :**

2c:  $P$  is loop  $x_i$  do  $P_1$  end

Let  $f_{P_1}$  be the p.r. function which computes what  $P_1$  computes.

Initialize the  $j$ -th loop:

$$f_1(n) = \langle (n)_1, \dots, (n)_l, (n)_{l+1}, \dots, (n)_{l+j-1}, \textcolor{red}{(n)}_i, (n)_{l+j+1}, \dots \rangle$$

$$f_1 = n * p(l+j)^{(n)}_i.$$

Let the  $j$ -th loop run:

$$f_2(n) = \begin{cases} n & \text{if } (n)_{l+j} = 0 \\ f_{P_1}(f_2(\langle \dots, \textcolor{red}{(n)}_{l+j} - \textcolor{red}{1}, \dots \rangle)) & \text{otherwise} \end{cases}$$

Then:

$$f_P = f_2 \circ f_1$$

# $\mathcal{P} = \text{LOOP}$

---

Proof (ctd) **Construction of  $f_P$ :**

2c:  $P$  is loop  $x_i$  do  $P_1$  end

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Initialize the  $j$ -th loop:

$$f_1(n) = \langle (n)_1, \dots, (n)_l, (n)_{l+1}, \dots, (n)_{l+j-1}, \textcolor{red}{(n)}_i, (n)_{l+j+1}, \dots \rangle$$

$$f_1 = n * p(l+j)^{(n)}_i.$$

Let the  $j$ -th loop run:

$$f_2(n) = \begin{cases} n & \text{if } (n)_{l+j} = 0 \\ f_{P_1}(f_2(n \text{ DIV } p(l+j))) & \text{otherwise} \end{cases}$$

Then:

$$f_P = f_2 \circ f_1$$

# $\mathcal{P} = \text{LOOP}$

---

**Proof** (ctd) We show that  $f_2$  is primitive recursive.

Let  $F : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  be defined by:

$$F(n, 0) = n$$

$$F(n, m + 1) = f_{p_1}(F(n, m) \text{ DIV } p(l + j))$$

Then  $F \in \mathcal{P}$ .

$$f_2 = F(n, D(n, l + j)).$$

Since  $f_1, f_2$  are primitive recursive, so is  $f_P = f_2 \circ f_1$ .

### 3. Recursive functions

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- Introduction/Motivation
- Primitive recursive functions  $\mapsto \mathcal{P}$
- $\mathcal{P} = \text{LOOP}$
- $\mu$ -recursive functions  $\mapsto F_\mu$
- $F_\mu = \text{WHILE}$
- Summary