Universität Koblenz-Landau
** **FB 4 Informatik**

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Exercises for
"Advances in Theoretical Computer Science"
Exercise sheet **

1 Recursive functions

In the next exercises we use the following notation:

- $\circ$ is function composition.
- if $j \leq k$, π^k_j is the projection function defined by $\pi^k_j(n_1, \dots, n_k) = n_j$.
- $(+1) : \mathbb{N} \to \mathbb{N}$ is defined by: $(+1)(n) = n + 1$.
- $(-1) : \mathbb{N} \to \mathbb{N}$ is defined by: $(-1)(n) = \begin{cases} 0 & \text{if } n = 0 \\ n-1 & \text{otherwise} \end{cases}$.
- $* : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined for all $n_1, n_2 \in \mathbb{N}$ by: $*(n_1, n_2) = n_1 * n_2$.
- $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined for all $n_1, n_2 \in \mathbb{N}$ by: $+(n_1, n_2) = n_1 + n_2$.
- $- : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined for all $n_1, n_2 \in \mathbb{N}$ by $-(n_1, n_2) = n_1 - n_2 = \begin{cases} 0 & \text{if } n_1 \leq n_2 \\ n_1 - n_2 & \text{otherwise} \end{cases}$.
- for all $s, k \in \mathbb{N}$, $c^k_s : \mathbb{N}^k \to \mathbb{N}$ is defined for all $\mathbf{n} \in \mathbb{N}^k$ by: $c^k_s(\mathbf{n}) = s$.

In what follows we will assume known that all the functions above are primitive recursive. We also assume known that definitions by case distinction and definitions using the bounded μ operator (as defined in the lecture, in which only primitive functions are used) define primitive recursive functions.

Exercise 1.1 ((1+1+4) + 2 + (1+4)=13p)

(1) Consider the following primitive recursive function:

$$f_1 = \mathcal{PR}[+ \circ ((-1) \circ c^1_6, \pi^1_1), (+1) \circ \pi^3_3]$$

(a) Which is the arity of f_1?

(b) What does f_1 compute if all arguments are equal to 1?

(c) Which (concrete) function is computed by f_1?

(2) Let $f_2 : \mathbb{N} \to \mathbb{N}$ be defined by $f_2(m) = \mu_{i<m} i(m - i = 0)$. Which (concrete) function is computed by f_2?

(3) Consider the following μ-recursive function:

$$f_3 = \mu g, \text{ where } g(n,i) = \begin{cases} n & \text{if } i = 0 \\ \mu j(j+3-n=0) & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{cases}$$

(a) Which is the arity of f_3?

(b) Which is the (concrete) function computed by f_3?

Exercise 2.2 (5p)

Let $f : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ f(n-1) + 2 * f(n-2) + 3 * f(n-3) & \text{if } n \geq 3 \end{cases}$$

Prove that f is primitive recursive.
You can use any of the results proved in the lecture for this.

Exercise 2.1 ((1+1+4) + (2+2) + (1+4)=15p)

(1) Consider the following primitive recursive function:

$$f_1 = \mathcal{PR}[* \circ ((-1) \circ \pi_1^1, c_5^1), * \circ ((+1) \circ \pi_2^3, \pi_3^3)]$$

(a) Which is the arity of f_1?

(b) What does f_1 compute if all arguments are equal to 1?

(c) Which (concrete) function is computed by f_1?

(2) Which (concrete) functions are computed by:

(a) $f_2 : \mathbb{N} \to \mathbb{N}$ defined by $f_2(n) = \mu\ i\ \ ((i+2) - n = 0)$?

(b) $f_3 : \mathbb{N} \to \mathbb{N}$ defined by $f_3(n) = \mu_{i<n}\ i\ \ ((i+2) - n = 0)$?

(3) Consider the following μ-recursive function:

$$f_4 = \mu g, \text{ where } g(n,i) = \begin{cases} n & \text{if } i = 0 \\ \mu\ j\ \ ((j+2) - n = 0) & \text{if } i = 1 \\ \mu_{j<n}\ j\ \ ((j+2) - n = 0) & \text{if } i \geq 2 \end{cases}$$

(a) Which is the arity of f_4?

(b) Which is the (concrete) function computed by f_4?

Exercise 2.2 (5p)

Let $f : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ f(n-1) * (f(n-2) + 1) * (f(n-3) + 2) & \text{if } n \geq 3 \end{cases}$$

Prove that f is primitive recursive.
You can use any of the results proved in the lecture for this.