Universität Koblenz-Landau

FB 4 Informatik

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Exercises for
"Advances in Theoretical Computer Science"
Exercise sheet 7

Exercise 7.1:
Express in the form $h(\mathbf{n}) = 0$ with h primitive recursive the following conditions:

- n is greater than 20 and is a perfect square, i.e.
  " $n \geq 20$ and $\exists k \leq n : k * k = n$".
- All prime divisors of n_1 are smaller than n_2:
  " $\forall k \leq n_1 \;\; (\mathsf{prime}(k) = 0 \text{ or } |(k, n_1) = 0 \text{ or } k \leq n_2)$".

Note: prime and | are the function defined in the lecture with:
$\mathsf{prime}(k) = 1$ if k is a prime number and $\mathsf{prime}(k) = 0$ if k is not a prime number;
$|(k, n) = 1$ if k divides n and $|(k, n) = 0$ if k does not divide n.

Remark: You are allowed to use all functions that were proved to be primitive recursive in the lecture or in a previous exercise (but especially +, -, *, as well as the functions for sums and products defined in the slides from 28.11.2013, page 48).

Exercise 7.2:
Prove that the following functions are primitive recursive:

(1) $\mathsf{max} : \mathbb{N}^2 \to \mathbb{N}$ defined by $\mathsf{max}(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{otherwise} \end{cases}$

(2) $\mathsf{div} : \mathbb{N}^2 \to \mathbb{N}$ defined by $\mathsf{div}(x, y) = \begin{cases} x + 1 & \text{if } y = 0 \\ \lfloor \frac{x}{y} \rfloor & \text{otherwise} \end{cases}$

(3) $\mathsf{mod} : \mathbb{N}^2 \to \mathbb{N}$ defined by $\mathsf{mod}(x, y) = \begin{cases} x & \text{if } y = 0 \\ x \;\mathsf{mod}\; y & \text{otherwise} \end{cases}$

Here $\lfloor x \rfloor$ is the largest integer which is smaller than or equal to x ("floor x").

Remark: You are allowed to use all functions that were proved to be primitive recursive in the lecture or in a previous exercise.

Exercise 7.3:
Let $d : \mathbb{N} \to \mathbb{N}$ be defined by:

$$d(n) = \begin{cases} \text{number of divisors of } n & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}$$

For instance, $d(0) = 0, d(1) = 1, d(2) = 2, d(3) = 2, d(4) = 3, d(12) = 6$.

(1) Let $d_2 : \mathbb{N}^2 \to \mathbb{N}$ be such that $d_2(n, m)$ is the number of divisors of n which are smaller than or equal to m (we assume that $d_2(0, 0) = 0$). Prove that d_2 is primitive recursive.

(2) Use (1) to show that d is primitive recursive.

Hint: (1) You can e.g. give a definition by primitive recursion for d_2.

Remark: You are allowed to use all functions that were proved to be primitive recursive in the lecture or in a previous exercise.

Exercise 7.4:
Look again at the slides from 5.12.2013 (paages 22 and 27-30) and make sure that you know how to answer the following questions:

(1) Let $D : \mathbb{N}^2 \to \mathbb{N}$ be defined by:

$D(n, i) = k$ if and only if k is the power of the i-th prime number in the prime number decomposition of n

which can also be seen as the smallest natural number k such that $p(i)^{k+1}$ does not divide n (where $p(i)$ is the i-th prime number). By definition we set $D(0, i) = 0$ for every $i \in \mathbb{N}$, and $D(n, 0) = 0$ for every $n \in \mathbb{N}$.

Show that D is primitive recursive.

(2) Let $K^r : \mathbb{N}^r \to \mathbb{N}$ be the Gödelisation function defined by:

$$K^r(n_1, \ldots, n_r) = \prod_{i=1}^{r} p(i)^{n_i} \qquad \text{where } p(i) \text{ is the } i\text{-th prime number}$$

and let $D_1, \ldots, D_r : \mathbb{N} \to \mathbb{N}$ be defined by $D_i(n) = D(n, i)$, where D is as in (1).

(2a) Compute $K^4(1, 2, 0, 1)$ and $K^6(3, 6, 0, 1, 0, 0)$.

(2b) Compute $D_2(36), D_3(12), D_3(50)$.

(2c) Show that K^r is primitive recursive.

(2d) Show that D_i is primitive recursive for every $i \in \{1, \ldots, r\}$.

Remark: You are allowed to use all functions that were proved to be primitive recursive in the lecture or in a previous exercise.

Exercise 7.5:
Let $\mathsf{fib} : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$\begin{aligned} \mathsf{fib}(0) &= 1 \\ \mathsf{fib}(1) &= 1 \\ \mathsf{fib}(n) &= \mathsf{fib}(n-1) + \mathsf{fib}(n-2) \qquad \text{for all } n > 1 \end{aligned}$$

Let $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be defined for every $n \in \mathbb{N}$ by $f(n) = (\mathsf{fib}(n), \mathsf{fib}(n+1))$.

(1) Let $f_p : \mathbb{N} \to \mathbb{N}$ be defined by $f_p(n) = K^2(f(n)) = K^2(\mathsf{fib}(n), \mathsf{fib}(n+1))$, where $K^2 : \mathbb{N}^2 \to \mathbb{N}$ is the Gödelisation function as defined in Exercise 7.4.

Show that f_p is primitive recursive.

(2) Use (1) and the results in Exercise 7.4 to prove that fib is primitive recursive.

Remark: You are allowed to use all functions that were proved to be primitive recursive in the lecture or in a previous exercise.

The submission of the solutions is not compulsory. If you want to submit your solutions, please do so until Tuesday, 10.12.2013, 10:00 s.t.. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name on your solution.
Submission possibilities:

- By e-mail to mbender@uni-koblenz.de with the keyword "Homework ACTCS" in the subject.
- Put it in the box in front of Room B 222.