Universität Koblenz-Landau

FB 4 Informatik

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Exercises for
"Advances in Theoretical Computer Science"
Exercise sheet 9

Exercise 9.1:

Let $g : \mathbb{N} \to \mathbb{N}$ be defined by: $g(n) = f(n, n+1) - 1$, and $f : \mathbb{N}^2 \to \mathbb{N}$ be defined by:

$$\begin{aligned} f(n, 0) &= 0 \\ f(n, k+1) &= f(n, k) + (1 - (k^2 - n)) \end{aligned}$$

(1) Show that g is primitive recursive.

(2) Compute $g(5)$ and $g(10)$.

(3) Give a LOOP program which computes g.

(4) Can you describe what mathematical function is computed by g?

(5) Give a primitive recursive function which computes $\lfloor \mathsf{log}_2(n) \rfloor$. (For $n = 0$, the value of the function should be 0, not $-\infty$.)

Hint: Modify the functions g and f in a suitable way.

Remark: You are allowed to use all functions that were proved to be primitive recursive in the lecture or in a previous exercise.

Exercise 9.2:

Prove that function $\mathsf{log} : \mathbb{N}^2 \to \mathbb{N}$ defined by $\mathsf{log}(n, m) = \lfloor \mathsf{log}_n(m) \rfloor$ is μ-recursive.

Remark: You are allowed to use all functions that were proved to be primitive and/or μ-recursive in the lecture or in a previous exercise.

Exercise 9.3:

Which functions are computed by:

(1) $f_1 = \mu c_1^2$

(2) $f_2 = \mu g$, where $g(n, i) = \begin{cases} n + 1 & \text{if } i = 0 \\ \mu j(j + 1 + n = 0) & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{cases}$

(3) $f_3 = \mu g$, where $g(n,i) = \begin{cases} n+1 & \text{if } i = 0 \\ \mu j((j+1) - n = 0) & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{cases}$

Exercise 9.4:
Consider the definition of the Ackermann function given in the lecture:

$$\begin{aligned} A(0,y) &= y+1 \\ A(x+1,0) &= A(x,1) \\ A(x+1,y+1) &= A(x, A(x+1,y)) \end{aligned} \qquad Ack(x) = A(x,x)$$

We assume that the following properties of the function A are known (proving these facts is not part of this exercise).

(1) $A(1,y) = y+2$ for every $y \in \mathbb{N}$
(2) $A(2,y) > 2y$ for every $y \in \mathbb{N}$
(3) $A(3,y) > 2^{y+1}$ for every $y \in \mathbb{N}$
(4) $y < A(x,y)$ for all $x, y \in \mathbb{N}$
(5) $A(x,y) < A(x,y+1)$ for all $x, y \in \mathbb{N}$
(6) $A(x,y+1) \leq A(x+1,y)$ for all $x, y \in \mathbb{N}$
(7) $A(x,y) < A(x+1,y)$ for all $x, y \in \mathbb{N}$.
(8) $A(x,2y) < A(x+3,y)$ for all $x, y \in \mathbb{N}$.

Prove (possibly using some of the inequalities above) that:

(9) $0 < A(0,0)$ for all $n \in \mathbb{N}$
(10) $\pi_i^r(n_1, \ldots, n_r) < A(0, \sum_{i=1}^r n_i)$ for all $n_1, \ldots, n_r \in \mathbb{N}$
(11) $n+1 < A(1,n)$ for all $n \in \mathbb{N}$

Remark: The results (1)–(11) are the easy part in the proof of the fact that the Ackermann function is not primitive recursive. The proof of the following facts (12) and (13) is a bit more difficult (and it is not part of this exercise).

For every $m \in \mathbb{N}$, let $B_m = \{f \mid f$ primitive recursive and for all $n_1, \ldots, n_r \in \mathbb{N}$ where r is the arity of $f, f(n_1, \ldots, n_r) < A(m, \sum_{i=1}^r n_i)$

(12) If $f, g_1, \ldots, g_r \in B_m$ and $h = f \circ (g_1, \ldots, g_m)$ then there exists a natural number m' (depending on m and r) such that $h \in B_{m'}$.
(13) If $g, h \in B_m$ and f is defined by primitive recursion from g and h then $f \in B_{m+4}$.

Prove:

(14) For every primitive recursive function f there exists $m \in \mathbb{N}$ with $f \in B_m$.
(15) The Ackermann function Ack defined by $Ack(n) = A(n,n)$ is not primitive recursive.

The submission of the solutions is not compulsory. If you want to submit your solutions, please do so until Tuesday, 14.1.2014, 10:00 s.t.. Joint solutions prepared by up to three persons are allowed. Please do not forget to write your name on your solution.
Submission possibilities:

- By e-mail to mbender@uni-koblenz.de with the keyword “Homework ACTCS” in the subject.
- Put it in the box in front of Room B 222.