Advanced Topics in Theoretical Computer Science

Part 5: Complexity (Part 1)

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Contents

- Recall: Turing machines and Turing computability

- Register machines (LOOP, WHILE, GOTO)

- Recursive functions

- The Church-Turing Thesis

- Computability and (Un-)decidability

- Complexity

Motivation (The pragmatical view)

Assume you are employed as software designer.

One day, your boss calls you into his office and and tells you that the company is about to enter a very competitive market, for which it is essential to know how to solve (efficiently) problem X.

Your charge is to find an efficient algorithm for solving this problem.

Motivation (The pragmatical view)

What you certainly don't want:

"I can't find an efficient algorithm. I guess I'm just to dumb"
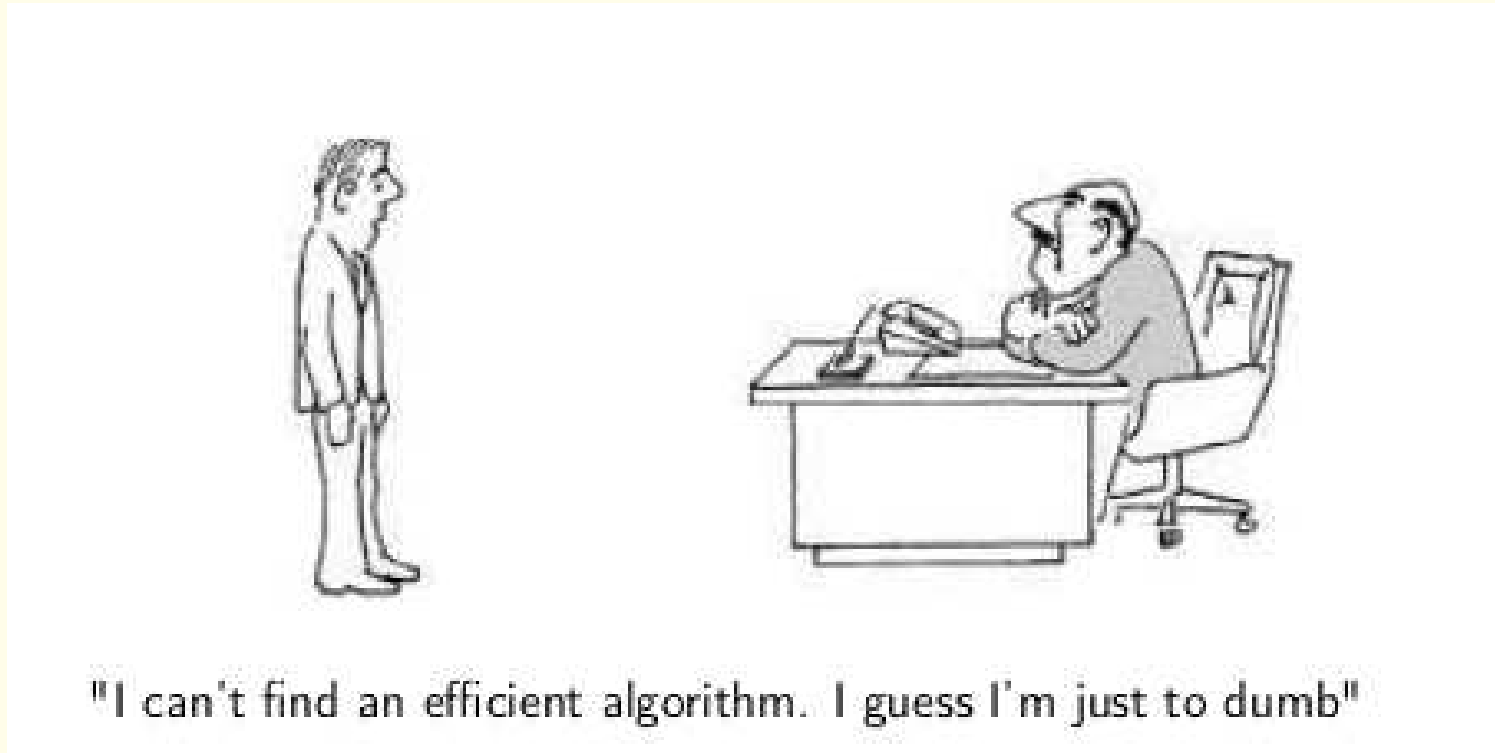
(Garey, Johnson, 1979)

Motivation (The pragmatical view)

Much better:

"I can't find an efficient algorithm, because no such algorithm is possible!"

(Garey, Johnson, 1979)

Motivation

In this lecture we showed how to prove that certain problems do not have a (terminating) algorithmic solution
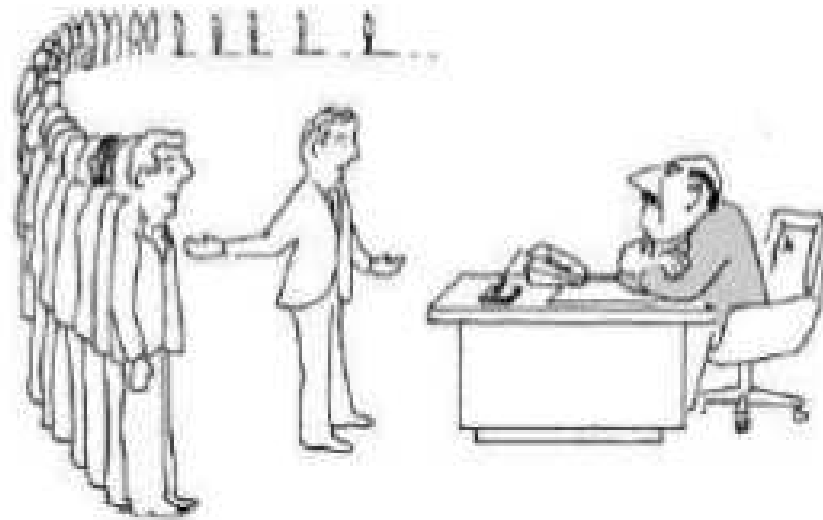
$\mapsto$ undecidability results

In the next weeks we will show that even decidable problems are "intractable" in the sense that they have a high complexity.

Unfortunately, proving undecidability or inherent intractability can be just as hard as finding efficient algorithms.

The pragmatical view

However, we will see that you can often answer:

"I can't find an efficient algorithm, but neither can all these famous people."

(Garey, Johnson, 1979)

Motivation

Goals:

- Define formally time and space complexity

- Define a family of "complexity classes": P, NP, PSPACE, ...

- Study the links between complexity classes

- Learn how to show that a problem is in a certain complexity class

    Reductions to problems known to be in the complexity class

- Closure of complexity classes

We will give examples of problems from various areas and study their complexity.

Complexity

- Recall:

    – Big O notation

    – The structure of PSPACE

    – Complete problems; hard problems

    – Examples

Big O notation

Definition. Let $h, f : \mathbb{N} \to \mathbb{R}$ functions.

The function h is in the class $O(f)$ iff there exists $c \in \mathbb{R}$, $c > 0$ and there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ $|h(n)| \leq c|f(n)|$.

Notation: $f \in O(h)$, sometimes also $f(n) \in O(h(n))$;
by abuse of notation denoted also by $f = O(h))$

Examples:

$5n + 4 \in O(n)$

$5n + n^2 \notin O(n)$

$$\binom{n}{2} = \frac{n(n-1)}{2} \in O(n)$$

Let p be a polynomial of degree m. Then $p(n) \in O(n^m)$

Big O notation

Computation rules for O

- $f \in O(f)$

- $c \cdot O(f) = O(f)$

- $O(O(f)) = O(f)$

- $O(f) \cdot O(g) = O(f \cdot g)$

- $O(f \cdot g) = |f|O(g)$

- If $|f| \leq |g|$ then $O(f) \subseteq O(g)$

Lemma. The following hold:
- $\forall d > 0,\ n^{d+1} \notin O(n^d)$
- $\forall r > 1 \forall d (r^n \notin O(n^d)$ and $n^d \in O(r^n))$

Complexity

Types of complexity

- Time complexity

- Space complexity

DTIME and NTIME

Basic model: k-DTM or k-NTM M (one tape for the input)

If M makes for every input word of length n at most $T(n)$ steps, then M is $T(n)$-time bounded.

In this case, the language accepted by M has time complexity $T(n)$; (more precisely $\max(n+1, T(n))$.

Definition ($NTIME(T(n)), DTIME(T(n))$)

- $DTIME(T(n))$ class of all languages accepted by $T(n)$-time bounded DTMs.

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DSPACE and NSPACE

Basic model: k-DTM or k-NTM M with special tape for the input (is read-only) + k storage tapes (offline DTM) $\mapsto$ needed if $S(n)$ sublinear

If M needs, for every input word of length n, at most $S(n)$ cells on the storage tapes then M is $S(n)$-space bounded.

The language accepted by M has space complexity $S(n)$; (more precisely $\max(1, S(n))$).

Definition $(NSPACE(S(n)), DSPACE(S(n)))$

- $DSPACE(S(n))$ class of all languages accepted by $S(n)$-space bounded DTMs.

- $NSPACE(S(n))$ class of all languages accepted by $S(n)$-space bounded NTMs.

Example

To which time/space complexity does the following language belong:

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Time: $DTIME(n + 1)$: copy input to the right of c in reverse order. When c is found, the rest is compared with the copy of w on the tape.

Space: $DSPACE(n)$: previous DTM

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$$L_{\mathrm{mirror}} = \{wcw^R \mid w \in \{0,1\}^*\}$$

Time: $DTIME(n+1)$: copy input to the right of c in reverse order. When c is found, the rest is compared with the copy of w on the tape.

Space: $DSPACE(n)$: previous DTM

Even better $DSPACE(log(n))$: use two tapes as binary counters (length of word before/after c). Remember: the definition of DSPACE does not count the space used on the input tape.

Questions

Time: Is any language in $DTIME(f(n))$ decided by some DTM?

Space: Is any language in $DSPACE(f(n))$ decided by some DTM?

The functions f are usually very simple functions; in particular they are all computable.

We will consider e.g. powers $f(n) = n^k$.

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We will consider e.g. powers $f(n) = n^k$.

Time/Space: What about $NTIME(f(n)), NSPACE(f(n))$

Time vs. Space: What are the links between $DTIME(f(n)), DSPACE(f(n)),$

                             $NTIME(f(n)), NSPACE(f(n))$

Questions

Time bounded   What does it mean that a DTM makes at most n steps?

Strictly speaking, after n steps it should halt or hang.

Halt?   Input is accepted

Hang?   DTM on band which is infinite on both sides cannot hang!

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Stop after n steps

Stop: We understand the following under M makes at most n steps:

- It halts (and accepts the input) within n steps
- It hangs (and does not accept the input) within n steps
- It halts after n steps, but not in halting mode, so it does

  not accept the input.

Answers

Answers (Informally)

Time: Every language from $DTIME(f(n))$ is decidable:

for an input of length n we wait as long as the value $f(n)$.

If until then no answer "YES" then the answer is "NO".

Space: Every language from $DSPACE(f(n))$ is decidable:

There are only finitely many configurations. We write all configurations.

If the TM does not halt then there is a loop. This can be detected.

Answers

… to be continued next time