Advanced Topics in Theoretical Computer Science

Part 3: Recursive Functions (1)

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Recursive functions

Motivation

Functions as model of computation (without an underlying machine model)

Idea

- Simple ("atomic") functions are computable
- "Combinations" of computable functions are computable

(We consider functions $f : \mathbb{N}^k \to \mathbb{N}$, $k \geq 0$)

Recursive functions

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- Simple ("atomic") functions are computable
- "Combinations" of computable functions are computable

(We consider functions $f : \mathbb{N}^k \to \mathbb{N}$, $k \geq 0$)

Questions

- Which are the atomic functions?
- Which combinations are possible?

Recursive functions: Atomic functions

The following functions are primitive recursive and μ-recursive:

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The constant null

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$$+1 : \mathbb{N}^1 \to \mathbb{N} \text{ with } +1(n) = n + 1 \text{ for all } n \in \mathbb{N}$$

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Projection function

$$\pi_i^k : \mathbb{N}^k \to \mathbb{N} \text{ with } \pi_i^k(n_1, \ldots, n_k) = n_i$$

Recursive functions

Notation:

We will write $\mathbf{n}$ for the tuple $(n_1, \ldots, n_k)$, $k \geq 0$.

Recursive functions: Composition

Composition:

If the functions: $g : \mathbb{N}^r \to \mathbb{N}$ $\quad r \geq 1$

$h_1 : \mathbb{N}^k \to \mathbb{N}, \dots, h_r : \mathbb{N}^k \to \mathbb{N}$ $\quad k \geq 0$

are primitive recursive resp. μ-recursive, then

$$f : \mathbb{N}^k \to \mathbb{N}$$

defined for every $\mathbf{n} \in \mathbb{N}^k$ by:

$$f(\mathbf{n}) = g(h_1(\mathbf{n}), \dots, h_r(\mathbf{n}))$$

is also primitive recursive resp. μ-recursive.

Notation without arguments: $f = g \circ (h_1, \dots, h_r)$

Primitive recursive functions

Until now:

- **Atomic functions** (Null, Successor, Projections)
- **Composition**

Next:

- **Primitive recursion**

Definition of primitive recursive functions

Primitive recursive functions

Primitive recursion

If the functions

$$g : \mathbb{N}^k \to \mathbb{N} \qquad (k \geq 0)$$
$$h : \mathbb{N}^{k+2} \to \mathbb{N}$$

are primitive recursive,

then the function

$$f : \mathbb{N}^{k+1} \to \mathbb{N} \text{ with } \quad \begin{aligned} f(\mathbf{n}, 0) &= g(\mathbf{n}) \\ f(\mathbf{n}, m+1) &= h(\mathbf{n}, m, f(\mathbf{n}, m)) \end{aligned}$$

is also primitive recursive.

Primitive recursive functions

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is also primitive recursive.

Notation without arguments: $f = \mathcal{PR}[g, h]$

Primitive recursive functions

Definition (Primitive recursive functions)

- **Atomic functions:** The functions
  - Null 0
  - Successor $+1$
  - Projection $\pi_i^k \quad (1 \leq i \leq k)$

  are primitive recursive.
- **Composition:** The functions obtained by composition from primitive recursive functions are primitive recursive.
- **Primitive recursion:** The functions obtained by primitive recursion from primitive recursive functions are primitive recursive.

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- **Primitive recursion:** The functions obtained by primitive recursion from primitive recursive functions are primitive recursive.

Notation: $\mathcal{P} =$ The set of all primitive recursive functions

Arithmetical functions: definitions

$f(n) = n + c$

$f(n) = n$

$f(n, m) = n + m$

$f(n, m) = n - 1$

$f(n, m) = n - m$

$f(n, m) = n * m$

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Identity

$$f : \mathbb{N} \to \mathbb{N}, \text{ with } f(n) = n$$

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$$f = \pi^1_1$$

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$$f(n, m + 1) = (+1)(f(n, m))$$

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$$\begin{array}{lll} f(n, 0) = n & g(n) = n & g = \pi_1^1 \\ f(n, m+1) = (+1)(f(n, m)) & h(n, m, k) = +1(k) & h = (+1) \circ \pi_3^3 \end{array}$$

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$$f = \mathcal{PR}[\pi_1^1, (+1) \circ \pi_3^3]$$

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$f(n + 1) = n$

Arithmetical functions: definitions

$f(n) = n - 1$

$f(0) = 0$	$g() = 0$	$g = 0$
$f(n+1) = n$	$h(n, k) = n$	$h = \pi_1^2$

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$f(n, 0) = n$	$g(n) = n$	$g = \pi_1^1$
$f(n, m+1) = f(n, m) - 1$	$h(n, m, k) = k - 1$	$h = (-1) \circ \pi_3^3$

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$f(n, m) = n * m$

$$\begin{array}{lll} f(n,0) = 0 & g(n) = 0 & g = 0 \\ f(n, m+1) = f(n,m) + n & h(n,m,k) = k + n & h = + \circ (\pi_3^3, \pi_1^3) \end{array}$$

$$f = \mathcal{PR}[0, + \circ (\pi_3^3, \pi_1^3)]$$

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