Example 1 (page 1)

Let $f: \mathbb{N} \to \mathbb{N}$ be defined as follows $f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f(n-1) + f(n-2) & \text{if } n \geq 2 \end{cases}$

Is f primitive recursive?

Idea: Define auxiliary functions $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ as follows:

0	1	2	3	4	5	6	7	8 …	$n-2$	$n-1$	n	$n+1$	$n+2$
$f(0)$	$f(1)$	$f(2)$	…						$f(n-2)$	$f(n-1)$	$f(n)$,	$f(n+1)$ –	$f(n+2)$
$f_1(0)$	$f_1(1)$	$f_1(2)$	…						$f_1(n-2)$	$f_1(n-1)$	$f_1(n)$,	$f_1(n+1)$	$f_1(n+2)$
	$f_2(0)$	$f_2(1)$	…						$f_2(n-3)$	$f_2(n-2)$	$f_2(n-1)$,	$f_2(n)$	$f_2(n+1)$
													(*)

$f_1(0) = 0$ $\qquad g_1 = 0$

$f_2(0) = 1$ $\qquad g_2 = 0$

$f_1(n+1) = f_2(n)$ (**) $\qquad h_1(m, k_1, k_2) = k_2 \quad h_1 = \pi^3_3$

$f_2(n+1) = f_2(n) + f_1(n)$ $\qquad h_2(m, k_1, k_2) = k_2 + k_1 = +\circ(\pi^3_3, \pi^3_2)$

$$\boxed{\begin{aligned} \text{Goal: } f_2(n+1) &\overset{(*)}{=} f(n+2) = f(n+1) + f(n) \\ &= f_1(n+1) + f_1(n) \\ &\overset{(**)}{=} f_2(n) + f_1(n). \end{aligned}}$$

& Explanation.

f_1, f_2 are defined by simultaneous recursion using primitive recursive functions g_1, g_2, h_1, h_2

$\Rightarrow f_1, f_2$ are primitive recursive.

$f = f_1 \Rightarrow f$ is primitive recursive.

Example 2 (pages 2 and 3)

Let $f: \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$f(n) = \begin{cases} 8 & \text{if } n=0 \\ 11 & \text{if } n=1 \\ 3 & \text{if } n=2 \\ (f(n-1) * f(n-3)) - (f(n-2) + f(n-1)) & \text{if } n \geq 3 \end{cases}$$

Is f primitive recursive? Justify your answer.

Idea: Define auxiliary functions f_1, f_2, f_3.

0	1	2	3	4	5	...	n-1	n	n+1	n+2	n+3
$f(0)$	$f(1)$	$f(2)$	-	-	-		$f(n-1)$	$f(n)$	$f(n+1)$	$f(n+2)$	$\boxed{f(n+3)}$
$f_1(0)$	$f_1(1)$	$f_1(2)$	-	-	-		$f_1(n-1)$	$f_1(n)$	$f_1(n+1)$	$f_1(n+2)$	$\boxed{f_1(n+3)}$
	$f_2(0)$	$f_2(1)$	-	-	-		$f_2(n-2)$	$f_2(n-1)$	$f_2(n)$	$f_2(n+1)$	$\boxed{f_2(n+2)}$
		$f_3(0)$	-	-	-		$f_3(n-3)$	$f_3(n-2)$	$f_3(n-1)$	$f_3(n)$	$\boxed{f_3(n+1)}$

(*)

$f_1(0) = 8 \qquad g_1 = 8$

$f_2(0) = 11 \qquad g_2 = 11$

$f_3(0) = 3 \qquad g_3 = 3$

$f_1(n+1) = f_2(n)$ (**) $\qquad h_1(n, k_1, k_2, k_3) = k_2 \quad h_1 = \pi^4_3$

$f_2(n+1) = f_3(n)$ (***) $\qquad h_2(n, k_1, k_2, k_3) = k_3 \quad h_2 = \pi^4_4$

$f_3(n+1) = (f_3(n) * f_1(n)) - (f_2(n) + f_3(n))$ $\qquad h_3(n, k_1, k_2, k_3) = (k_3 \cdot k_1) - (k_2 + k_3)$

$\Rightarrow h_3 = -\circ(*\circ(\pi^4_4, \pi^4_2), +\circ(\pi^4_3, \pi^4_4))$

Goal and explanation:

$$f_3(n+1) \underset{(*)}{=} f(n+3) \underset{\text{def. of } f}{=} (f(n+2) * f(n)) - (f(n+1) + f(n+2))$$

$= (f_1(n+2) * f_1(n)) - (f_1(n+1) + f_1(n+2))$

because $f(n+2) = f_1(n+2)$

$\underset{(**)}{=} (f_2(n+1) * f_1(n)) - (f_2(n) + f_2(n+1))$

$\underset{(***)}{=} (f_3(n) * f_1(n)) - (f_2(n) + f_3(n))$

Solution continues on next page

f_1, f_2, f_3 are defined by simultaneous recursion using primitive recursive functions g_1, g_2, g_3 and h_1, h_2, h_3

$\Rightarrow f_1, f_2, f_3$ are primitive recursive.

$f = f_1 \Rightarrow f$ is primitive recursive.

End of solution to Exercise 2.