Advanced Topics in Theoretical Computer Science

Part 3: Recursive Functions (2)

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- Recursive functions
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3. Recursive functions

- Introduction/Motivation
- Primitive recursive functions $\mapsto \mathcal{P}$
- $\mathcal{P} = \text{LOOP}$
- μ-recursive functions $\mapsto F_\mu$
- $F_\mu = \text{WHILE}$
- Summary

Recursive functions: Atomic functions

The following functions are primitive recursive and μ-recursive:

The constant null

$$0 : \mathbb{N}^0 \to \mathbb{N} \text{ with } 0() = 0$$

Successor function

$$+1 : \mathbb{N}^1 \to \mathbb{N} \text{ with } +1(n) = n + 1 \text{ for all } n \in \mathbb{N}$$

Projection function

$$\pi_i^k : \mathbb{N}^k \to \mathbb{N} \text{ with } \pi_i^k(n_1, \ldots, n_k) = n_i$$

Recursive functions

Notation:

We will write $\mathbf{n}$ for the tuple $(n_1, \ldots, n_k)$, $k \geq 0$.

Recursive functions: Composition

Composition:

If the functions: $g : \mathbb{N}^r \to \mathbb{N}$ $\quad r \geq 1$

$h_1 : \mathbb{N}^k \to \mathbb{N}, \dots, h_r : \mathbb{N}^k \to \mathbb{N}$ $\quad k \geq 0$

are primitive recursive resp. μ-recursive, then

$$f : \mathbb{N}^k \to \mathbb{N}$$

defined for every $\mathbf{n} \in \mathbb{N}^k$ by:

$$f(\mathbf{n}) = g(h_1(\mathbf{n}), \dots, h_r(\mathbf{n}))$$

is also primitive recursive resp. μ-recursive.

Notation without arguments: $f = g \circ (h_1, \dots, h_r)$

Primitive recursive functions

Primitive recursion

If the functions

$$g : \mathbb{N}^k \to \mathbb{N} \qquad (k \geq 0)$$
$$h : \mathbb{N}^{k+2} \to \mathbb{N}$$

are primitive recursive,

then the function

$$f : \mathbb{N}^{k+1} \to \mathbb{N} \text{ with } \qquad \begin{aligned} f(\mathbf{n}, 0) &= g(\mathbf{n}) \\ f(\mathbf{n}, m+1) &= h(\mathbf{n}, m, f(\mathbf{n}, m)) \end{aligned}$$

is also primitive recursive.

Primitive recursive functions

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is also primitive recursive.

Notation without arguments: $f = \mathcal{PR}[g, h]$

Primitive recursive functions

Definition (Primitive recursive functions)

- **Atomic functions:** The functions
  - Null 0
  - Successor $+1$
  - Projection $\pi_i^k \; (1 \leq i \leq k)$

  are primitive recursive.
- **Composition:** The functions obtained by composition from primitive recursive functions are primitive recursive.
- **Primitive recursion:** The functions obtained by primitive recursion from primitive recursive functions are primitive recursive.

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- **Composition:** The functions obtained by composition from primitive recursive functions are primitive recursive.
- **Primitive recursion:** The functions obtained by primitive recursion from primitive recursive functions are primitive recursive.

Notation: $\mathcal{P} =$ The set of all primitive recursive functions

Arithmetical functions: definitions

$f(n) = n + c$, **for** $c \in \mathbb{N}, c > 0$

$$f = \underbrace{(+1) \circ \cdots \circ (+1)}_{c \text{ times}}$$

$f(n) = n$

$$f = \pi^1_1$$

$f(n, m) = n + m$

$$f = \mathcal{PR}[\pi^1_1, (+1) \circ \pi^3_3]$$

$f(n) = n - 1$

$$f = \mathcal{PR}[0, \pi^2_1]$$

$f(n, m) = n - m$

$$f = \mathcal{PR}[\pi^1_1, (-1) \circ \pi^3_3]$$

$f(n, m) = n * m$

$$f = \mathcal{PR}[0, + \circ (\pi^3_3, \pi^3_1)]$$

Defining new primitive recursive functions

Re-ordering/Omitting/Repeating Arguments

Lemma The set of primitive recursive functions is closed under:

- Re-ordering
- Omitting
- Repeating

of arguments when composing functions.

Additional Arguments

Lemma. Assume $f : \mathbb{N}^k \to \mathbb{N}$ is primitive recursive.
Then, for every $p \in \mathbb{N}$, the function $f' : \mathbb{N}^k \times \mathbb{N}^p \to \mathbb{N}$ defined for every $\mathbf{n} \in \mathbb{N}^k$ and every $\mathbf{m} \in \mathbb{N}^p$ by:

$$f'(\mathbf{n}, \mathbf{m}) = f(\mathbf{n})$$

is primitive recursive.

Defining new primitive recursive functions

Case distinction

Lemma (Case distinction is primitive recursive)

If
- g_i, h_i $(1 \leq i \leq r)$ are primitive recursive functions, and
- for every n there exists a unique i with $h_i(n) = 0$

then the following function f is primitive recursive:

$$f(n) = \begin{cases} g_1(n) & \text{if } h_1(n) = 0 \\ \dots & \\ g_r(n) & \text{if } h_r(n) = 0 \end{cases}$$

Sums and products

Theorem

If $g : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$ is a primitive recursive function then the following functions $f_1, f_2 : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$ are also primitive recursive:

$$f_1(\mathbf{n}, m) = \begin{cases} 0 & \text{if } m = 0 \\ \sum_{i<m} g(\mathbf{n}, i) & \text{if } m > 0 \end{cases} \qquad f_2(\mathbf{n}, m) = \begin{cases} 1 & \text{if } m = 0 \\ \prod_{i<m} g(\mathbf{n}, i) & \text{if } m > 0 \end{cases}$$

Bounded μ operator

Definition.

Let $g : \mathbb{N}^{k+1} \to \mathbb{N}$ be a function.

The bounded μ operator is defined as follows:

$$\mu_{i<m}\ i\ (g(\mathbf{n}, i) = 0) := \begin{cases} i_0 & \text{if } g(\mathbf{n}, i_0) = 0 \\ & \text{and for all } j < i_0 \;\; g(\mathbf{n}, j) \neq 0 \\ 0 & \text{if } g(\mathbf{n}, j) \neq 0 \text{ for all } 0 \leq j < m \\ & \text{or } m = 0 \end{cases}$$

$\mu_{i<m}\ i\ (g(\mathbf{n}, i) = 0)$ is the smallest $i < m$ such that $g(\mathbf{n}, i) = 0$

Bounded μ operator

Theorem.
If $g : \mathbb{N}^{k+1} \to \mathbb{N}$ is a primitive recursive function then the function $f : \mathbb{N}^{k+1} \to \mathbb{N}$ defined by:

$$f(\mathbf{n}, m) = \mu_{i<m}\ i\ (g(\mathbf{n}, i) = 0)$$

is also primitive recursive

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Theorem.
If $g : \mathbb{N}^{k+1} \to \mathbb{N}$ is a primitive recursive function then the function $f : \mathbb{N}^{k+1} \to \mathbb{N}$ defined by:

$$f(\mathbf{n}, m) = \mu_{i<m}\ i\ (g(\mathbf{n}, i) = 0)$$

is also primitive recursive

Proof: We can define f as follows:

$$\begin{aligned} f(\mathbf{n}, 0) &= 0 \\ f(\mathbf{n}, m+1) &= \begin{cases} 0 & \text{if } m = 0 \\ m & \text{if } g(\mathbf{n}, m) = f(\mathbf{n}, m) = 0 \wedge g(\mathbf{n}, 0) \neq 0 \wedge m > 0 \\ & \\ f(\mathbf{n}, m) & \text{otherwise} \end{cases} \end{aligned}$$

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then the function $f : \mathbb{N}^{k+1} \to \mathbb{N}$ defined by:

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is also primitive recursive

Proof: We can define f as follows:

$$\begin{aligned}
f(\mathbf{n}, 0) &= 0 \\
f(\mathbf{n}, m+1) &= \begin{cases} 0 & \text{if } m = 0 \\ m & \text{if } g(\mathbf{n}, m) = f(\mathbf{n}, m) = 0 \wedge g(\mathbf{n}, 0) \neq 0 \wedge m > 0 \\ & \quad \text{i.e. if } g(\mathbf{n}, m) + f(\mathbf{n}, m) + (1 - g(\mathbf{n}, 0)) + (1 - m) = 0 \\ f(\mathbf{n}, m) & \text{otherwise} \end{cases}
\end{aligned}$$

Prime number functions

Theorem: The following functions are primitive recursive:

(1) The Boolean function $| : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ defined by:

$$|(n, m) = \begin{cases} 1 & \text{if } n \text{ divides } m \\ 0 & \text{otherwise} \end{cases}$$

(2) The Boolean function prime : $\mathbb{N} \to \{0, 1\}$ defined by:

$$\text{prime}(n) = \begin{cases} 1 & \text{if } n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

(3) The function $p : \mathbb{N} \to \mathbb{N}$ defined by: $p(n) = p_n$, the n-th prime number.

(4) The function $D : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by: $D(n, i) = k$ iff k is the power of the i-th prime number in the prime number decomposition of n.

$$D(n, i) = \max(\{j \mid n \bmod p(i)^j = 0\})$$

Prime number functions

Proof:

(1) $| : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ defined by:

$$|(n, m) = \begin{cases} 1 & \text{if } n \text{ divides } m \\ 0 & \text{otherwise} \end{cases}$$

Prime number functions

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$$|(n, m) = \begin{cases} 1 & \text{if } n \text{ divides } m \\ 0 & \text{otherwise} \end{cases}$$

$|(n, m) = 1$ iff $\exists z(n * z = m)$ iff $\prod_{z \leq m}(n * z - m) + (m - n * z) = 0$.

Prime number functions

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$$|(n, m) = \begin{cases} 1 & \text{if } n \text{ divides } m \\ 0 & \text{otherwise} \end{cases}$$

$|(n, m) = 1$ iff $\exists z(n * z = m)$ iff $\prod_{z \leq m}(n * z - m) + (m - n * z) = 0$.

$|(n, m) = 1 - \prod_{z \leq m}(n * z - m) + (m - n * z)$

Prime number functions

Proof:

(2) prime : $\mathbb{N} \to \{0, 1\}$ defined by:

$$\text{prime}(n) = \begin{cases} 1 & \text{if } n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

Prime number functions

Proof:

(2) prime : $\mathbb{N} \to \{0,1\}$ defined by:

$$\text{prime}(n) = \begin{cases} 1 & \text{if } n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

$\text{prime}(n) = 1$ iff $(n \geq 2$ and $\forall y < n(y = 0 \vee y = 1 \vee |(y, n) = 0)$

$\text{prime}(n) = 1 - ((2 - n) + \sum_{y<n}(|(y, n) * y * ((y - 1) + (1 - y))))$

Prime number functions

Proof:

(3) The function $p : \mathbb{N} \to \mathbb{N}$ defined by: $p(n) = p_n$, the n-th prime number.

$p(0) = 0$ and $p(1) = 2$.

$p(n+1)$ is the smallest number i which is larger than $p(n)$ and is prime.

Prime number functions

Proof:

(3) The function $p : \mathbb{N} \to \mathbb{N}$ defined by: $p(n) = p_n$, the n-th prime number.

$p(0) = 0$ and $p(1) = 2$.

$p(n+1)$ is the smallest number i which is larger than $p(n)$ and is prime.

We also have an upper bound for the number i.

Recall the proof of the fact that the set of prime numbers is infinite.

$$i \leq p(n)! + 1$$

$$p(n+1) = \mu_{i \leq p(n)!+1}\ i\ [((1 - \mathrm{prime}(i)) + ((p(n)+1) - i)) = 0]$$

Prime number functions

Proof:

(4) $D : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by: $D(n, i) = k$ iff k is the power of the i-th prime number in the prime number decomposition of n.

$$D(n, i) = \max(\{j \mid n \text{ mod } p(i)^j = 0\})$$

$D(0, i) := 0;$

$D(n, i) = \min(\{j \leq n \mid \ \ |(p(i)^{j+1}, n) = 0\})$

$D(n, i) = \mu_{j \leq n}\, j\ (|(p(i)^{j+1}, n) = 0)$

3. Recursive functions

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- $\mathcal{P} =$ LOOP
- μ-recursive functions $\mapsto F_\mu$
- $F_\mu =$ WHILE
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Goal

Show that $\mathcal{P} = \text{LOOP}$

Idea:

To show that $\mathcal{P} \supseteq \text{LOOP}$ we have to show that every LOOP computable function can be expressed as a primitive recursive function.

For this, we will encode the contents of arbitrarily many registers in one natural number (used as input for this primitive recursive function).

For this encoding we will use Gödelisation. We will need to show that Gödelisation is primitive recursive.

To show that $\mathcal{P} \subseteq \text{LOOP}$ we have to show that:
– all atomic primitive recursive functions are LOOP computable, and
– LOOP is closed under composition of functions and primitive recursion.

Gödelisation

To show: Gödelisation is primitive recursive

Informally:

- Coding number sequences as a number
- Corresponding decoding function (projection)

are primitive recursive

Gödelisation

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- Corresponding decoding function (projection)

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More precise formulation:

There exist primitive recursive functions

$$K^r : \mathbb{N}^r \to \mathbb{N} \qquad (r \geq 1)$$

$$D_i : \mathbb{N} \to \mathbb{N} \qquad (1 \leq i \leq r)$$

with:

$$D_i(K^r(n_1, \ldots, n_r)) = n_i$$

Gödelisation

To show: Gödelisation is primitive recursive

Informally:
- Coding number sequences as a number
- Corresponding decoding function (projection)

are primitive recursive

Recall:

Gödelisation: Coding number sequences as a number

Bijection between $\mathbb{N}^r$ and $\mathbb{N}$: $K^r : \mathbb{N}^r \to \mathbb{N}$, defined by:

$$K^r(n_1, \ldots, n_r) = \prod_{i=1}^{r} p(i)^{n_i}.$$

Decoding: The inverses $D_i : \mathbb{N} \to \mathbb{N}$ of K^r defined by $D_i(n) = D(n, i)$

Gödelisation

Bijection between $\mathbb{N}^r$ and $\mathbb{N}$: $K^r : \mathbb{N}^r \to \mathbb{N}$, defined by:

$$K^r(n_1, \ldots, n_r) = \prod_{i=1}^{r} p(i)^{n_i}.$$

$D_i : \mathbb{N} \to \mathbb{N}$, $1 \leq i \leq r$, defined by $D_i(n) = D(n, i)$

Theorem. K^r and $D_1, \ldots, D_r$ are primitive recursive.

Gödelisation

Bijection between $\cup_{r\geq 1}\mathbb{N}^r$ and $\mathbb{N}$: $K^r : \mathbb{N}^r \to \mathbb{N}$, defined by:

$$K^r(n_1, \ldots, n_r) = \prod_{i=1}^{r} p(i)^{n_i}.$$

$D_i : \mathbb{N} \to \mathbb{N}$, $1 \leq i \leq r$, defined by $D_i(n) = D(n, i)$

Theorem. K^r and $D_1, \ldots, D_r$ are primitive recursive.

Lemma.
(1) $D_i(K^r(n_1, \ldots, n_r)) = n_i \quad$ for all $1 \leq i \leq r$.
(2) $K^r(n_1, \ldots, n_r) = K^{r+1}(n_1, \ldots, n_r, 0)$

In general, $D_i(K^r(n_1, \ldots, n_r)) = 0$ if $i > r$.

Gödelisation

Notation:

$$
\begin{aligned}
K^r(n_1, \ldots, n_r) &= \langle n_1, \ldots, n_r \rangle \\
D_i(n) &= (n)_i
\end{aligned}
$$

For $r = 0$:

$\langle\rangle = 1$

$(\langle\rangle)_i = 0$