Advanced Topics in Theoretical Computer Science

Part 5: Complexity (Part 1)

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Contents

- Recall: Turing machines and Turing computability
- Register machines (LOOP, WHILE, GOTO)
- Recursive functions
- The Church-Turing Thesis
- Computability and (Un-)decidability
- Complexity

Motivation (The pragmatical view)

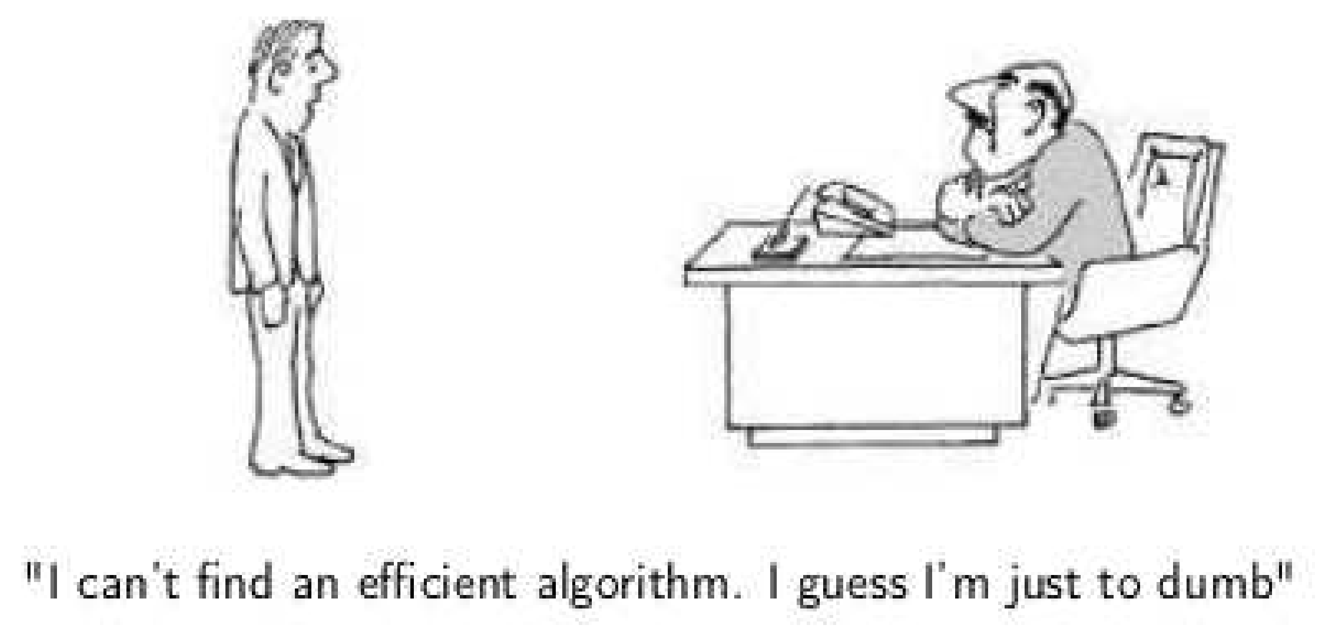
Assume you are employed as software designer.

One day, your boss calls you into his office and and tells you that the company is about to enter a very competitive market, for which it is essential to know how to solve (efficiently) problem X.

Your charge is to find an efficient algorithm for solving this problem.

Motivation (The pragmatical view)

What you certainly don't want:



(Garey, Johnson, 1979)

Motivation (The pragmatical view)

Much better:

"I can't find an efficient algorithm, because no such algorithm is possible!"

(Garey, Johnson, 1979)

Motivation

In this lecture we showed how to prove that certain problems do not have a (terminating) algorithmic solution

$\mapsto$ undecidability results

In the next weeks we will show that even decidable problems are "intractable" in the sense that they have a high complexity.

Unfortunately, proving undecidability or inherent intractability can be just as hard as finding efficient algorithms.

The pragmatical view

However, we will see that you can often answer:

"I can't find an efficient algorithm, but neither can all these famous people."

(Garey, Johnson, 1979)

Motivation

Goals:

- Define formally time and space complexity
- Define a family of "complexity classes": P, NP, PSPACE, ...
- Study the links between complexity classes
- Learn how to show that a problem is in a certain complexity class

  Reductions to problems known to be in the complexity class
- Closure of complexity classes

We will give examples of problems from various areas and study their complexity.

Complexity

- Recall:
    - Big O notation
    - The structure of PSPACE
    - Complete problems; hard problems
    - Examples

Big O notation

Definition. Let $h, f : \mathbb{N} \rightarrow \mathbb{R}$ functions.
The function h is in the class $O(f)$ iff there exists $c \in \mathbb{R}$, $c > 0$ and there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ $|h(n)| \leq c|f(n)|$.

Notation: $f \in O(h)$, sometimes also $f(n) \in O(h(n))$;
by abuse of notation denoted also by $f = O(h)$)

Examples:

$5n + 4 \in O(n)$

$5n + n^2 \not\in O(n)$

$$\left(\begin{array}{c} n \\ 2 \end{array}\right) = \frac{n(n-1)}{2} \in O(n^2)$$

Let p be a polynomial of degree m. Then $p(n) \in O(n^m)$

Big O notation

Computation rules for O

- $f \in O(f)$
- $c \cdot O(f) = O(f)$
- $O(O(f)) = O(f)$
- $O(f) \cdot O(g) = O(f \cdot g)$
- $O(f \cdot g) = |f|O(g)$
- If $|f| \leq |g|$ then $O(f) \subseteq O(g)$

Lemma. The following hold:

- $\forall d > 0,\ n^{d+1} \notin O(n^d)$
- $\forall r > 1 \forall d(r^n \notin O(n^d)$ and $n^d \in O(r^n))$

Complexity

Types of complexity

- Time complexity
- Space complexity

DTIME and NTIME

Basic model: k-DTM or k-NTM M (one tape for the input)

If M makes for every input word of length n at most $T(n)$ steps, then M is $T(n)$-time bounded.

In this case, the language accepted by M has time complexity $T(n)$; (more precisely $\max(n+1, T(n))$.

Definition ($NTIME(T(n)), DTIME(T(n))$)

- $DTIME(T(n))$ class of all languages accepted by $T(n)$-time bounded DTMs.
- $NTIME(T(n))$ class of all languages accepted by $T(n)$-time bounded NTMs.

DSPACE and NSPACE

Basic model: k-DTM or k-NTM M with special tape for the input (is read-only) $+$ k storage tapes (offline DTM) $\mapsto$ needed if $S(n)$ sublinear

If M needs, for every input word of length n, at most $S(n)$ cells on the storage tapes then M is $S(n)$-space bounded.

The language accepted by M has space complexity $S(n)$;
(more precisely $\max(1, S(n))$).

Definition ($NSPACE(S(n)), DSPACE(S(n))$)

- $DSPACE(S(n))$ class of all languages accepted by $S(n)$-space bounded DTMs.
- $NSPACE(S(n))$ class of all languages accepted by $S(n)$-space bounded NTMs.

Example

To which time/space complexity does the following language belong:

$$L_{\text{mirror}} = \{wcw^R \mid w \in \{0, 1\}^*\}$$

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Time: $DTIME(n+1)$: copy input to the right of c in reverse order. When c is found, the rest is compared with the copy of w on the tape.

Space: $DSPACE(n)$: previous DTM

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$$L_{\text{mirror}} = \{wcw^R \mid w \in \{0,1\}^*\}$$

Time: $DTIME(n+1)$: copy input to the right of c in reverse order. When c is found, the rest is compared with the copy of w on the tape.

Space: $DSPACE(n)$: previous DTM

Even better $DSPACE(log(n))$: use two tapes as binary counters.

1. the input is checked for the occurrence of just one c and an equal number of symbols to the left and right of c. This needs only constant space, resp. it can be done with a number of states (and thus needs no space at all).
2. we check the right and left part symbol by symbol: to do this we just have to keep in mind the two positions to be checked (for equality) (and they are coded on the two tapes).

Remember: definition of DSPACE does not count the space used on the input tape.

Questions

Time: Is any language in $DTIME(f(n))$ decided by some DTM?

Space: Is any language in $DSPACE(f(n))$ decided by some DTM?

The functions f are usually very simple functions; in particular they are all computable.

We will consider e.g. powers $f(n) = n^k$.

Questions

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The functions f are usually are very simple functions; in particular they are all computable.

We will consider e.g. powers $f(n) = n^k$.

Time/Space:	What about $NTIME(f(n)), NSPACE(f(n))$
Time vs. Space:	What are the links between $DTIME(f(n)), DSPACE(f(n)), NTIME(f(n)), NSPACE(f(n))$

Questions

Time bounded	What does it mean that a DTM makes at most n steps?
	Strictly speaking, after n steps it should halt or hang.
Halt?	Input is accepted
Hang?	DTM on band which is infinite on both sides cannot hang!

Questions

Time bounded	What does it mean that a DTM makes at most n steps?
	Strictly speaking, after n steps it should halt or hang.
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Stop after n steps

Stop: We understand the following under M makes at most n steps:

- It halts (and accepts the input) within n steps
- It hangs (and does not accept the input) within n steps
- It halts after n steps, but not in halting mode, so it does not accept the input.

Answers

Answers (Informally)

Time: Every language from $DTIME(f(n))$ is decidable:

for an input of length n we wait as long as the value $f(n)$.

If until then no answer "YES" then the answer is "NO".

Space: Every language from $DSPACE(f(n))$ is decidable:

There are only finitely many configurations. We write all configuration

If the TM does not halt then there is a loop. This can be detected.

Answers

Answers (Informally)

NTM vs. DTM: Clearly, $DTIME(f(n)) \subseteq NTIME(f(n))$ and

$$DSPACE(f(n)) \subseteq NSPACE(f(n))$$

If we try to simulate an NTM with a DTM we may need exponentially more time. Therefore:

$$NTIME(f(n)) \subseteq DTIME(2^{h(n)}) \text{ where } h \in O(f).$$

For the space complexity we can show that:

$$NSPACE(f(n)) \subseteq DSPACE(f^2(n))$$

Time vs. Space: Clearly, $DTIME(f(n)) \subseteq DSPACE(f(n))$ and

$$NTIME(f(n)) \subseteq NSPACE(f(n))$$

$DSPACE(f(n))$, $NSPACE(f(n))$ are much larger.

Question

What about constant factors?

Constant factors are ignored. Only the rate of growth of a function in complexity classes is important.

Theorem.
For every $c \in \mathbb{R}^+$ and every storage function $S(n)$ the following hold:

- $DSPACE(S(n)) = DSPACE(cS(n))$
- $NSPACE(S(n)) = NSPACE(cS(n))$

Proof (Idea). One direction is trivial. The other direction can be proved by representing a fixed amount $r > \frac{2}{c}$ of neighboring cells on the tape as a new symbol.

The states of the new machine simulate the movements of the read/write head as transitions. For r-cells of the old machine we use only two: in the most unfavourable case when we go from one block to another.

Time acceleration

Theorem For every $c \in \mathbb{R}^+$ and every time function $T(n)$ with $\lim_{n\to\infty} \frac{T(n)}{n} = \infty$ the following hold:

- $DTIME(T(n)) = DTIME(cT(n))$
- $NTIME(T(n)) = NTIME(cT(n))$

Proof (Idea). One direction is trivial. The other direction can be proved by representing a fixed amount $r > \frac{4}{c}$ of neighboring cells on the tape as a new symbol.

The states of the new machine simulate also now which symbol and which position the read/write head of the initial machine has. When the machine is simulated the new machine needs to make 4 steps instead of r: 2 in order to write on the new fields and 2 in order to move the head on the new field and then back on the old (in the worst case).

Big O notation

Theorem: Let T be a time function with $\lim_{n\to\infty} \frac{T(n)}{n} = \infty$ and S a storage function.

(a) If $f(n) \in O(T(n))$ then $DTIME(f(n)) \subseteq DTIME(T(n))$.

(b) If $g(n) \in O(S(n))$ then $DSPACE(g(n)) \subseteq DSPACE(S(n))$.