Advanced Topics in Theoretical Computer Science

Part 3: Recursive Functions (1)

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Recursive functions

Motivation

Functions as model of computation (without an underlying machine model)

Idea

- Simple ("atomic") functions are computable
- "Combinations" of computable functions are computable

(We consider functions $f : \mathbb{N}^k \to \mathbb{N}$, $k \geq 0$)

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- Simple ("atomic") functions are computable
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Questions

- Which are the atomic functions?
- Which combinations are possible?

Recursive functions: Atomic functions

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The constant null

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Projection function

$$\pi_i^k : \mathbb{N}^k \to \mathbb{N} \text{ with } \pi_i^k(n_1, \ldots, n_k) = n_i$$

Recursive functions

Notation:

We will write **n** for the tuple $(n_1, \ldots, n_k)$, $k \geq 0$.

Recursive functions: Composition

Composition:

If the functions: $g : \mathbb{N}^r \to \mathbb{N}$ $\quad r \geq 1$

$h_1 : \mathbb{N}^k \to \mathbb{N}, \dots, h_r : \mathbb{N}^k \to \mathbb{N}$ $\quad k \geq 0$

are primitive recursive resp. μ-recursive, then

$$f : \mathbb{N}^k \to \mathbb{N}$$

defined for every $\mathbf{n} \in \mathbb{N}^k$ by:

$$f(\mathbf{n}) = g(h_1(\mathbf{n}), \dots, h_r(\mathbf{n}))$$

is also primitive recursive resp. μ-recursive.

Notation without arguments: $f = g \circ (h_1, \dots, h_r)$

Primitive recursive functions

Until now:

- **Atomic functions** (Null, Successor, Projections)
- **Composition**

Next:

- **Primitive recursion**

Definition of primitive recursive functions

Primitive recursive functions

Primitive recursion

If the functions

$$g : \mathbb{N}^k \to \mathbb{N} \qquad (k \geq 0)$$
$$h : \mathbb{N}^{k+2} \to \mathbb{N}$$

are primitive recursive,

then the function

$$f : \mathbb{N}^{k+1} \to \mathbb{N} \text{ with } \quad \begin{aligned} f(\mathbf{n}, 0) &= g(\mathbf{n}) \\ f(\mathbf{n}, m+1) &= h(\mathbf{n}, m, f(\mathbf{n}, m)) \end{aligned}$$

is also primitive recursive.

Primitive recursive functions

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Notation without arguments: $f = \mathcal{PR}[g, h]$

Primitive recursive functions

Definition (Primitive recursive functions)

- **Atomic functions:** The functions
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  - Successor $+1$
  - Projection π_i^k $(1 \leq i \leq k)$

  are primitive recursive.
- **Composition:** The functions obtained by composition from primitive recursive functions are primitive recursive.
- **Primitive recursion:** The functions obtained by primitive recursion from primitive recursive functions are primitive recursive.

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- **Primitive recursion:** The functions obtained by primitive recursion from primitive recursive functions are primitive recursive.

Notation: $\mathcal{P} =$ The set of all primitive recursive functions

Arithmetical functions: definitions

$f(n) = n + c$

$f(n) = n$

$f(n, m) = n + m$

$f(n) = n - 1$

$f(n, m) = n - m$

$f(n, m) = n * m$

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Identity

$$f : \mathbb{N} \to \mathbb{N}, \text{ with } f(n) = n$$

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$$\begin{array}{lll} f(n,0) = n & g(n) = n & g = \pi^1_1 \\ f(n, m+1) = (+1)(f(n,m)) & h(n,m,k) = +1(k) & h = (+1) \circ \pi^3_3 \end{array}$$

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$f(n, 0) = n$	$g(n) = n$	$g = \pi_1^1$
$f(n, m + 1) = (+1)(f(n, m))$	$h(n, m, k) = +1(k)$	$h = (+1) \circ \pi_3^3$

$$f = \mathcal{PR}[\pi_1^1, (+1) \circ \pi_3^3]$$

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$f(0) = 0 \qquad g() = 0 \qquad g = 0$

$f(n+1) = n \qquad h(n,k) = n \qquad h = \pi^2_1$

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$f(n, 0) = n$	$g(n) = n$	$g = \pi^1_1$
$f(n, m+1) = f(n, m) - 1$	$h(n, m, k) = k - 1$	$h = (-1) \circ \pi^3_3$

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$f(n, m) = n * m$

$f(n, 0) = 0$	$g(n) = 0$	$g = 0$
$f(n, m+1) = f(n, m) + n$	$h(n, m, k) = k + n$	$h = + \circ (\pi^3_3, \pi^3_1)$

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Re-ordering/Omitting/Repeating Arguments

Lemma The set of primitive recursive functions is closed under:

- Re-ordering
- Omitting
- Repeating

of arguments when composing functions.

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of arguments when composing functions.

Proof: (Idea)

A tuple of arguments $\mathbf{n'} = (n_{i_1}, \ldots, n_{i_k})$ obtained from $\mathbf{n} = (n_1, \ldots, n_k)$ by re-ordering, omitting or repeating components can be represented as:

$$\mathbf{n'} = (\pi^k_{i_1}(\mathbf{n}), \ldots, \pi^k_{i_k}(\mathbf{n}))$$

Additional Arguments

Lemma. Assume $f : \mathbb{N}^k \to \mathbb{N}$ is primitive recursive.
Then, for every $p \in \mathbb{N}$, the function $f' : \mathbb{N}^k \times \mathbb{N}^p \to \mathbb{N}$ defined for every $\mathbf{n} \in \mathbb{N}^k$ and every $\mathbf{m} \in \mathbb{N}^p$ by:

$$f'(\mathbf{n}, \mathbf{m}) = f(\mathbf{n})$$

is primitive recursive.

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is primitive recursive.

Proof:

Case 1: $k = 0$, i.e. f is a constant. Then $f^1 : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$ with $f^1(\mathbf{n}, m) = f(\mathbf{n})$ for all $m \in \mathbb{N}$ can be expressed by primitive recursion as follows:

$$f^1(0) = f \qquad\qquad f^1 = \mathcal{PR}[f, \pi^2_2]$$

$$f^1(n+1) = f^1(n) = \pi^2_2(n, f^1(n))$$

By iterating this construction p times we obtain extensions $f^2, f^3, \ldots, f^p$ with $2, 3, \ldots p$ additional arguments. The function f' is f^p.

Case 2: $k \neq 0$. Let $\mathbf{n} = (n_1, \ldots, n_k, m_1, \ldots, m_p)$

Then $f'(\mathbf{n}) = f(\pi^{k+p}_1(\mathbf{n}), \ldots, \pi^{k+p}_k(\mathbf{n})) = f \circ (\pi^{k+p}_1, \ldots, \pi^{k+p}_k)$.

Case distinction

Lemma (Case distinction is primitive recursive)

If
- g_i, h_i $(1 \leq i \leq r)$ are primitive recursive functions, and
- for every n there exists a unique i with $h_i(n) = 0$

then the function f defined by:

$$f(n) = \begin{cases} g_1(n) & \text{if } h_1(n) = 0 \\ \dots & \\ g_r(n) & \text{if } h_r(n) = 0 \end{cases}$$

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is primitive recursive.

Proof: $f(n) = g_1(n) * (1 - h_1(n)) + \cdots + g_r(n) * (1 - h_r(n))$

Sums and products

Theorem

If $g : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$ is a primitive recursive function then the following functions $f_1, f_2 : \mathbb{N}^k \times \mathbb{N} \to \mathbb{N}$ are also primitive recursive:

$$f_1(\mathbf{n}, m) = \begin{cases} 0 & \text{if } m = 0 \\ \sum_{i<m} g(\mathbf{n}, i) & \text{if } m > 0 \end{cases}$$

$$f_2(\mathbf{n}, m) = \begin{cases} 1 & \text{if } m = 0 \\ \prod_{i<m} g(\mathbf{n}, i) & \text{if } m > 0 \end{cases}$$

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Proof: f_1 and f_2 can be written using primitive recursion and case distinction:

$$f_1(\mathbf{n}, 0) = 0$$

$$f_1(\mathbf{n}, m+1) = f_1(\mathbf{n}, m) + g(\mathbf{n}, m)$$

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$$f_2(\mathbf{n}, 0) = 1$$

$$f_2(\mathbf{n}, m+1) = f_2(\mathbf{n}, m) * g(\mathbf{n}, m)$$