Optimization I

Linear Optimization

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Outline

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Organization

Contact

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Organization

Dates

Lecture:

- Monday, from 08:00 – 10:00 in G 409
- Thursday, from 08:00 – 10:00 in G 209

Tutorials:

- Monday, from 16:00 – 18:00 in K 107
- Friday, from 12:00 – 14:00 in E 414

Material

Lecture material (e. g. these slides, exercise sheets, ...) will be available under

http://uni-ko-ld.de/n5

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Organization

Exam

There will be two exams within the next six months:

- Monday, 16.07.2018 in M 001 from 10:00 – 12:00
- At the beginning of October, date and room are not fixed yet.

You must register in KLIPS in order to participate at the exam.
Registration should already be possible from right now.

Register early!

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Introduction

Example 2.1

$\rightsquigarrow$ Handout.

Notes

Definition 2.2 (Linear Program in general form)

$$
\begin{array}{llcl}
\min & c_1 x_1 + \dots + c_n x_n & & \\
\text{s.t.} & a_{i1} x_1 + \dots + a_{in} x_n = b_i & & \forall i = 1, \dots, p \\
& a_{i1} x_1 + \dots + a_{in} x_n \leq b_i & & \forall i = p+1, \dots, q \\
& a_{i1} x_1 + \dots + a_{in} x_n \geq b_i & & \forall i = q+1, \dots, m \\
& x_j \geq 0 & & \forall j = 1, \dots, r \\
& x_j \lesseqgtr 0 & & \forall j = r+1, \dots, n
\end{array}
\tag{G-LP}
$$

- decision variable(s): $x = (x_1, \dots, x_n)^T$
- feasible region:

$$
P := \left\{ x \in \mathbb{R}^n \; : \; \begin{array}{rcll}
\sum_{j=1}^{n} a_{ij} x_j & = & b_i & \forall i = 1, \dots, p \\
\sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & \forall i = p+1, \dots, q \\
\sum_{j=1}^{n} a_{ij} x_j & \geq & b_i & \forall i = q+1, \dots, m \\
x_j & \geq & 0 & \forall j = 1, \dots, r \\
x_j & \lesseqgtr & 0 & \forall j = r+1, \dots, n
\end{array} \right\}
$$

Notes

Definition 2.3 (Linear Program in general form)

$$
\begin{array}{llcl}
\min & c_1 x_1 + \ldots + c_n x_n & & \\
\text{s. t.} & a_{i1} x_1 + \ldots + a_{in} x_n = b_i & \forall i = 1, \ldots, p & \\
& a_{i1} x_1 + \ldots + a_{in} x_n \leq b_i & \forall i = p+1, \ldots, q & \\
& a_{i1} x_1 + \ldots + a_{in} x_n \geq b_i & \forall i = q+1, \ldots, m & \\
& x_j \geq 0 & \forall j = 1, \ldots, r & \\
& x_j \lesseqgtr 0 & \forall j = r+1, \ldots, n &
\end{array} \tag{G-LP}
$$

- feasible solution: $x \in P$
- objective function value of x: $z(x) := c^\top x$
- optimal solution x^*: feasible solution with the best objective function value
- optimal objective function value: objective function value of an optimal solution (if it exists) $z^* = z(x^*) = c^\top x^*$

Notes

Linear programs

Example 2.4

$\rightarrow$ board

Observation 2.5

1. The feasible region is a polyhedron (or polytope).
2. There is an optimal solution in a vertex of the polyhedron.

Notes

Notation 2.6

Denote by $A_{i\bullet}$ the i-th row of the matrix $A \in \mathbb{R}^{m\times n}$ and by $A_{\bullet j}$ its j-th column.

$$A = \left(A_{\bullet 1}, \dots, A_{\bullet j}, \dots, A_{\bullet n}\right) = \begin{pmatrix} A_{1\bullet} \\ \vdots \\ A_{i\bullet} \\ \vdots \\ A_{m\bullet} \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

Notes

Graphical Method

Algorithm 2.7 (Graphical solution method for LPs with two variables)

Input: LP of the form: $\max\{c^\top x : Ax \leq b; x_j \geq 0, j = 1, 2\}$

1. Draw the set of feasible solutions P as intersection of the half spaces $A_{i\bullet} x \leq b_i, i = 1, \ldots, m$ and $x_j \geq 0, j = 1, 2$
2. Choose some $z \in \mathbb{R}$ and draw the line $c^\top x = z$
3. Find the uniquely determined line $c^\top x = z^*$ parallel to $c^\top x = z$ that satisfies:
    3.1 $\exists x^* \in P : c^\top x^* = z^*$
    3.2 z^* is the maximum with this property

Output: Every $x^* \in P$ with $c^\top x^* = z^*$ is an optimal solution of the problem and z^* is the optimal objective function value.

Notes

Example 2.8 (2.4, cont.)

Notes

Infeasibility and Unboundedness

Definition 2.9

An LP is called

- *infeasible* if $P = \emptyset$
- *unbounded* if the objective function value is unbounded, i. e., there are feasible solutions with arbitrarily good (large/small) objective function values

Notes

Standard Form

Definition 2.10 (LP in standard form)

$$\begin{aligned} \min\ & c^\top x \\ \text{s. t. } & Ax = b \\ & x \geq 0 \end{aligned} \tag{LP}$$

where $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ is the cost vector and $A \in \mathbb{R}^{m \times n}$, $m \leq n$, with full rank m (otherwise: remove redundant rows).
$P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ is the feasible region or feasible set.

Notes

Theorem 2.11

Every LP in general form (G-LP) *can be transformed into an equivalent LP in standard form* (LP).

Proof.

$\to$ board $\square$

Notes

Linear programs

Example 2.12 (2.4, cont.)

$\longrightarrow$ board

Idea of an Optimization Algorithm for LP:

- move iteratively from one extreme point (vertex) of the feasible region (polyhedron) to another one
- improve the objective function value in every step
- stop if no further improvement is possible

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Notes

Definition 3.1 (Basis, Basic Solution)

Given a LP in standard form

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

A *basis* of A is a set $\mathcal{B} = \{A_{\bullet B(1)}, \ldots, A_{\bullet B(m)}\}$ of m linearly independent columns of A, where $B = \{B(1), \ldots, B(m)\} \subseteq \{1, \ldots, n\}$ is a subset of the column indices.

Notes

Definition 3.1 (Basis, Basic Solution (cont.))

With respect to *a fixed basis* B we define

- Basic variable x_j with $j \in B$
- Vector of basic variables: $x_B = (x_{B(1)}, \dots, x_{B(m)})^\top$
- Let $N := \{1, \dots, n\} \setminus B = \{N(1), \dots, N(n-m)\}$
- Non-basic variable: x_j with $j \in N$
- Vector of non-basic variables: $x_N = (x_{N(1)}, \dots, x_{N(n-m)})^\top$
- $A_B := (A_{\bullet B(1)}, \dots, A_{\bullet B(m)}) \in \mathbb{R}^{m \times m}$ nonsingular submatrix of A
  $A_N := (A_{\bullet N(1)}, \dots, A_{\bullet N(n-m)}) \in \mathbb{R}^{m \times (n-m)}$
- $c_B := (c_{B(1)}, \dots, c_{B(m)})^\top \in \mathbb{R}^m$
  $c_N := (c_{N(1)}, \dots, c_{N(n-m)})^\top \in \mathbb{R}^{n-m}$

Notes

Basic representation

$$
\begin{aligned}
& A\,x = b \\
\iff & x_1\,A_{\bullet 1} + \ldots + x_n\,A_{\bullet n} = b \\
\iff & \left(x_{B(1)} A_{\bullet B(1)} + \cdots + x_{B(m)} A_{\bullet B(m)}\right) \\
& \qquad + \left(x_{N(1)} A_{\bullet N(1)} + \cdots + x_{N(n-m)} A_{\bullet N(n-m)}\right) = b \\
\iff & A_B\,x_B + A_N\,x_N = b \\
\iff & A_B\,x_B = b - A_N\,x_N
\end{aligned}
$$

Since A_B is invertible, we can solve for x_B.

$$
\iff x_B = A_B^{-1} b - A_B^{-1} A_N\,x_N
$$

Basic representation of x w. r. t. B

$\longrightarrow$ Every choice of x_N uniquely determines x_B.

Notes

Basic solution

Definition 3.2

- $x \in \mathbb{R}^n$ is called *basic solution w. r. t. B*, if

$$x = \binom{x_B}{x_N} : \; x_N = 0, \; x_B = A_B^{-1} b.$$

- A *basic feasible solution (BFS)* is a basic solution with $x_B \geq 0$.

Example 3.3 (Ex. 2.4)

$\rightarrow$ board

Notes

Basic solution

Remark 3.4

Any feasible region of an LP is a *polyhedral set*, i.e., it is the intersection of a finite number of halfspaces:

$$\left\{ \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\} \iff (Ax \leq b \ \wedge \ Ax \geq 0 \wedge x \geq 0)$$

Definition 3.5

An *extreme point* (corner point, vertex) of a polyhedral set $P \subseteq \mathbb{R}^n$ is a point that lies on n linearly independent defining hyperplanes of P.

Notes

Basic solution

Theorem 3.6

Every BFS of (LP) *(in standard form) corresponds to an extreme point of the (polyhedral) feasible set* $P = \{x \in \mathbb{R}^n : \; Ax = b; \; x \geq 0\}$ *and vice versa.*

Proof.

$\rightarrow$ board $\square$

Remark 3.7

There may exist more than one basis corresponding to the same BFS or extreme point.

Definition 3.8

A BFS is called *degenerate* if more than $n - m$ variables of the BFS are equal to 0.

Notes

Optimality condition

Theorem 3.9 (Sufficient optimality condition)

Let x be the basic solution w. r. t. B. If

$$\bar{c}^\top := c^\top - c_B^\top A_B^{-1} A \geq 0,$$

then x is an optimal solution of the LP.
$\bar{c}$ is called vector of reduced costs.

Proof.

$\rightarrow$ board $\square$

Example 3.10

$\rightarrow$ board

Notes

Optimality condition not fulfilled

Assume that the optimality condition is not fulfilled.

I.e. $\bar{c}_{N(s)}^{\top} = c_{N(s)}^{\top} - c_B^{\top} A_B^{-1} A_{\bullet N(s)} < 0$ for some $N(s) \in N$. Consider the proof of Theorem 3.9:

$$c^{\top} x = c_B^{\top} A_B^{-1} b + (c_N^{\top} - c_B^{\top} A_B^{-1} A_N) x_N$$

Then, $c^{\top} x$ can be improved by increasing $x_{N(s)}$ from 0 to $\delta > 0$ (where $x_{N(j)} = 0 \quad \forall N(j) \in N \setminus \{N(s)\})$

But the new solution has to be feasible!

$$x_B = A_B^{-1} b - A_B^{-1} A_{\bullet N(s)}\, x_{N(s)} \overset{!}{\geq} 0$$

Setting $\widetilde{b} := A_B^{-1} b$ and $\widetilde{A}_{\bullet N(s)} := A_B^{-1} A_{\bullet N(s)}$ yields

$$\Longrightarrow x_B = \widetilde{b} - \widetilde{A}_{\bullet N(s)}\, x_{N(s)} \geq 0$$

Notes

Optimality condition not fulfilled

Case 1: $\widetilde{a}_{iN(s)} \leq 0 \quad \forall i = 1, \ldots, m$

In this case, $x_{N(s)} = \delta$ can be arbitrarily increased without violating the feasibility.

In this way, $c^\top x$ can be made arbitrarily small, i. e., the LP is unbounded.

Theorem 3.11 (Criterion for unbounded LPs)

If x is a basic solution and if

$$\bar{c}_{N(s)} < 0 \quad \textit{and} \quad A_B^{-1} A_{\bullet N(s)} \leq 0$$

for some $N(s) \in N$, then the LP is unbounded.

Notes

Case 2: $\exists i \in \{1, \ldots, m\} : \widetilde{a}_{iN(s)} > 0$

$$x_B \overset{!}{\geq} 0 \iff x_{B(i)} = \widetilde{b}_i - \widetilde{a}_{iN(s)} \cdot x_{N(s)} \geq 0 \quad \forall i \in \{1, \ldots, m\}$$
$$\implies x_{N(s)} \leq \frac{\widetilde{b}_i}{\widetilde{a}_{iN(s)}} \quad \forall i \in \{1, \ldots, m\} \text{ with } \widetilde{a}_{iN(s)} > 0$$

$\rightarrow$ we increase $x_{N(s)}$ as much as possible i. e., we choose:

$$x_{N(s)} := \min \left\{ \frac{\widetilde{b}_i}{\widetilde{a}_{iN(s)}} : \ \widetilde{a}_{iN(s)} > 0 \right\} \qquad \text{(min ratio rule)}$$

Notes

Case 2: $\exists i \in \{1, \ldots, m\} : \widetilde{a}_{iN(s)} > 0$ **(cont.)**

Let $i = r$ be an index where this minimum is attained. The *new solution* is then given by:

$$
\begin{aligned}
x_{N(s)} &= \frac{\widetilde{b}_r}{\widetilde{a}_{rN(s)}} \\
x_{N(j)} &= 0 \qquad \forall j \in N \setminus \{N(s)\} \\
x_{B(i)} &= \widetilde{b}_i - \widetilde{a}_{iN(s)} \cdot x_{N(s)} = \widetilde{b}_i - \widetilde{a}_{iN(s)} \frac{\widetilde{b}_r}{\widetilde{a}_{rN(s)}}, \quad i \in \{1, \ldots, m\} \\
x_{B(r)} &= 0
\end{aligned}
$$

$\longrightarrow$ **Basis exchange:** $\begin{cases} x_{B(r)} \text{ \textit{leaves} the basis} \\ x_{N(s)} \text{ \textit{enters} the basis} \end{cases}$

Notes

Example 3.12 (Ex. 3.2)

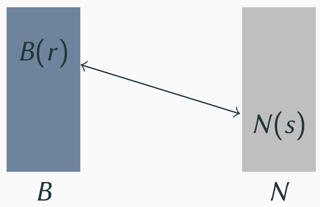
$\rightarrow$ board

Notes

Basis exchange

Let B be a basis and let $B' := \{B(1), \ldots, B(r-1), N(s), B(r+1), \ldots, B(m)\}$ be the index set after the basis exchange.

$\longrightarrow$ *Is B' again a basis and $x_{B'}$ a basic solution?*



Notes

Basis exchange

Lemma 3.13 (Steinitz exchange lemma)

Let $B = \{A_1, \dots, A_m\}$ be a basis of $\mathbb{R}^m$ and let

$$A_s = \sum_{i=1}^{m} \lambda_i A_i \quad (s \notin \{1, \dots, m\})$$

Then it holds that $B' := \{A_1, \dots, A_{r-1}, A_s, A_{r+1}, \dots, A_m\}$ is basis of $\mathbb{R}^m$ iff $\lambda_r \neq 0$.

Proof.

see linear algebra lecture □

Notes

Basis exchange

Theorem 3.14

If B is a basis of A, then B' is also a basis of A.

Proof.

$\rightarrow$ board $\square$

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Idea of the Simplex algorithm

Idea:

As long as the optimality condition is not fulfilled

choose some index $N(s)$ with $\bar{c}_{N(s)} < 0$
if the criterion for unboundedness is fulfilled

$\rightarrow$ LP is unbounded (Thm. 3.11)

else

apply the min ratio rule
make a basis exchange

Here: Efficient organization of the basis exchange and the optimality test.
Let the basis B of A and the basic feasible solution $x = \binom{x_B}{x_N}$ be given.

Notes

Simplex tableau

Initial tableau:

$$
T = \begin{array}{cccc|c}
-z & x_1 & \dots & x_n & \\
\hline
1 & c_1 & \dots & c_n & 0 \\
\hline
0 & a_{11} & \dots & a_{1n} & b_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & a_{m1} & \dots & a_{mn} & b_m
\end{array}
= \begin{array}{|c|c|c|}
\hline
1 & c & 0 \\
\hline
0 & A & b \\
\hline
\end{array} \in \mathbb{R}^{(m+1)\times(n+2)}
$$

For the basis B we define:

$$
T_B := \begin{array}{|c|c|}
\hline
1 & c_B^\top \\
\hline
0 & A_B \\
\hline
\end{array} \in \mathbb{R}^{(m+1)\times(m+1)}
$$

Then it holds (since A_B is regular):

$$
T_B^{-1} = \begin{array}{|c|c|}
\hline
1 & -c_B^\top A_B^{-1} \\
\hline
0 & A_B^{-1} \\
\hline
\end{array}
$$

Notes

Simplex tableau

Simplex tableau w. r. t. B:

$$T(B) := T_B^{-1} T = \begin{array}{|c|c|c|} \hline 1 & c^\top - c_B^\top A_B^{-1} A & -c_B^\top A_B^{-1} b \\ \hline 0 & & \\ \vdots & A_B^{-1} A & A_B^{-1} b \\ 0 & & \\ \hline \end{array}$$

$$= (t_{ij})_{\substack{i=0,\ldots,m \\ j=0,\ldots,n+1}} = \begin{array}{|c|ccc|c|} \hline 1 & t_{01} & \ldots & t_{0n} & t_{0n+1} \\ \hline 0 & t_{11} & \ldots & t_{1n} & t_{1n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & t_{m1} & \ldots & t_{mn} & t_{mn+1} \\ \hline \end{array}$$

Since T_B^{-1} is invertible, $T(B)$ represents the same system of linear equations as T.

Notes

Interpretation of $T(B)$

Basic variable: Let $j = B(i) \in B$:

- $A_B^{-1} A_{\bullet j} = e_i$ i-th unit vector
- $c_j - c_B^\top A_B^{-1} A_{\bullet j} = c_j - c_B^\top e_i = c_j - c_{B(i)} = c_j - c_j = 0$
  $\Rightarrow T(B)$ contains unit vectors $\begin{pmatrix} 0 \\ e_i \end{pmatrix}$ in columns corresponding basic variable $x_{B(i)} = x_j$

Non-basic variable: Let $j = N(i) \in N$[1]:

- $A_B^{-1} A_{\bullet j} = A_B^{-1} A_{\bullet N(i)} = \widetilde{A}_{\bullet N(i)} = \begin{pmatrix} \widetilde{a}_{1N(i)} \\ \vdots \\ \widetilde{a}_{mN(i)} \end{pmatrix}$
- $c_j - c_B^\top A_B^{-1} A_{\bullet j} = c_{N(i)} - c_B^\top A_B^{-1} A_{\bullet N(i)} = \overline{c}_{N(i)}$
  reduced cost of the non-basic variable $x_{N(i)} = x_j$

Vector $A_B^{-1} b$ (last column)

- Values of the basic variable $x_B = A_B^{-1} b$
- simultaneously: $\widetilde{b} = \begin{pmatrix} \widetilde{b}_1 \\ \vdots \\ \widetilde{b}_m \end{pmatrix} = A_B^{-1} b$

[1] We adapt the notation from Theorem 3.11.

Notes

Interpretation of $T(B)$

Scalar $-c_B^\top A_B^{-1} b$: ("top right")

negative objective function value of the current basic solution:
$c^\top x = c_B^\top x_B + c_N^\top x_N = c_B^\top A_B^{-1} b$

Every row in the intial tableau corresponds to one constraint.
Known from linear algebra: elementary row operations (like e. g. to multiply T_B^{-1} with T) transform the system of linear equations into an equivalent system of linear equations.

Notes

Example 4.1

$\rightarrow$ board

Notes

Simplex tableau

Definition 4.2

$T(B)$ is called:

feasible $:\Leftrightarrow t_{in+1} \geq 0 \quad \forall i = 1, \ldots, m$
$(\Leftrightarrow x_{B(i)} \geq 0 \quad \forall i = 1, \ldots, m$; feasibility)

optimal $:\Leftrightarrow t_{in+1} \geq 0 \quad \forall i = 1, \ldots, m$ and $t_{0j} \geq 0 \quad \forall j = 1, \ldots, n$
$(\Leftrightarrow \overline{c} \geq 0)$

unbounded $:\Leftrightarrow \exists N(s) \in N$:

1. $t_{0N(s)} = \overline{c}_{N(s)} < 0$
2. $t_{iN(s)} = \widetilde{a}_{iN(s)} \leq 0 \quad \forall i = 1, \ldots, m$

Note: These definitions are consistent with the previous definitions of feasibility, the optimality condition and the criterion for unbounded LPs.

Notes

Simplex algorithm

Basis exchange in the tableau:

- Choose a non-basic column $N(s)$ with $t_{0N(s)} < 0$ (negative reduced cost)
- Apply the min ratio rule to this column:

$$\delta := \min \left\{ \frac{t_{in+1}}{t_{iN(s)}} : t_{iN(s)} > 0, i \in \{1, \dots, m\} \right\}$$

Let $\delta = \frac{t_{rn+1}}{t_{rN(s)}}$ with $r \in \{1, \dots, m\}$

Notes

Definition 4.3

A basis exchange $B' := B \setminus \{B(r)\} \cup \{N(s)\}$ is called *pivot operation*. The column $N(s)$ of $T(B)$ is caled *pivot column* and the row r *pivot row*. The entry $t_{rN(s)}$ is called *pivot element*.

Notes

Simplex algorithm

Open question: How is the pivot operation in the tableau $T(B)$ realized? How does this lead to $T(B')$?

- $T(B')$ contains the vector $\binom{0}{e_r}$ in column $N(s)$
- Transform the column $N(s)$ of $T(B)$ into the r-th unit vector by applying elementary row transformations.

Notes

Simplex algorithm

Result: $T(B') = (\bar{t}_{ij})_{\substack{i=0,\ldots,m \\ j=0,\ldots,n+1}}$ with

- $\bar{t}_{rj} = \frac{t_{rj}}{t_{rN(s)}} \quad \forall j = 0, \ldots, n+1$
- $\bar{t}_{ij} = t_{ij} - \frac{t_{iN(s)}}{t_{rN(s)}} \cdot t_{rj} \quad \forall i = 0, \ldots, m, i \neq r \quad \forall j = 0, \ldots, n+1$

$\Rightarrow$ $\bar{t}_{rN(s)} = 1$ and
$\bar{t}_{iN(s)} = t_{iN(s)} - \frac{t_{iN(s)}}{t_{rN(s)}} \cdot t_{rN(s)} = 0$

Notes

Algorithm 4.4 (Simplex algorithm)

Input: LP in standard form: $\min\{c^\top x : A\,x = b,\ x \geq 0\}$ and BFS $x = \binom{x_B}{x_N}$ w. r. t. to a given basis B.

Determine the Simplex tableau $T(B)$

while $\exists j \in \{1, \ldots, n\} :\ t_{0j} < 0$ **do**   *// Tableau not optimal*

    Choose j with $t_{0\,j} < 0$

    if $t_{i\,j} \leq 0\ \forall i \in \{1, \ldots, m\}$ **then**

        return "LP is unbounded."

    else

        Find $r \in \{1, \ldots, m\}$ with

$$\frac{t_{r\,n+1}}{t_{r\,j}} = \min\left\{\frac{t_{i\,n+1}}{t_{i\,j}} : t_{i\,j} > 0, i \in \{1, \ldots, m\}\right\}$$

        and make a pivot operation with pivot element $t_{r\,j}$.

return $x^* = \binom{x_B}{x_N}$ with $x_{B(i)} = t_{i\,n+1}$ $(i = 1, \ldots, m)$, $x_N = 0$ and objective function value $z^* = c^\top x^* = -t_{0\,n+1}$ is an optimal solution of LP.

Notes

Simplex algorithm

Example 4.5

$\rightarrow$ board

Notes

Simplex algorithm

Remark 4.6

Open questions:

1. Why can we restrict ourselves to basic feasible solutions?
   $\rightarrow$ Fundamental theorem of linear programming
2. How do we find a feasible start basic solution?
   $\rightarrow$ 2-phase-method
3. Is the simplex algorithm finite?
   $\rightarrow$ degeneracy, cycling of the simplex, Bland's pivot rule

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Fundamental theorem of linear programming

Given: LP in standard form:

$$\begin{aligned} \min\ & c^\top x \\ \text{s.t. } & x \in P = \{x \in \mathbb{R}^n : Ax = b\} \\ & x \geq 0 \end{aligned}$$

Goal: Show that every feasible LP with bounded objective function value has an *optimal basic solution*.

It follows: Justification that the simplex algorithm restricts to *basic feasible solutions* (i. e., we do not overlook optimal solutions).

Notes

Fundamental theorem of linear programming

Theorem 5.1

If $P := \{x \in \mathbb{R}^n \colon Ax = b, x \geq 0\} \neq \emptyset$ (i. e. there exists a feasible solution), then there exists a basic feasible solution in P.

Proof.

$\rightarrow$ board $\square$

Notes

Fundamental theorem of linear programming

Theorem 5.2 (Fundamental theorem of linear programming)

If $P := \{x \in \mathbb{R}^n \colon Ax = b, x \geq 0\} \neq \emptyset$ *and if the corresponding LP* $\min\{c^\top x \colon Ax = b, x \geq 0\}$ *is bounded, then there exists an optimal basic feasible solution.*

Proof.

$\rightarrow$ board $\square$

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Finding a basic feasible solution

Goal: Find a feasible start basis (and basic solution) or show the infeasibility of the LP!

Structure of the 2-Phase-Method:

Phase 1 Determination of a feasible start basis or proof of infeasibility

Phase 2 ($\widehat{=}$ Simplex Algorithm, 4.4) Determination of an optimal basic feasible solution or proof of unboundedness.

Notes

Finding a basic feasible solution

Case (a) original constraints have the form

$$\overline{A}_{i\bullet} x \leq b_i \text{ with } b_i \geq 0 \quad \forall i = 1, \dots, m$$

$\Rightarrow$ introduce slack variables

$$A x = b \text{ with } A = (\overline{A} \mid I), x = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})$$

$\Rightarrow$ select the columns

$$B = \{n + 1, \dots, n + m\}$$

corresponding to slack variables as basis columns.

Then, $x = \binom{x_B}{x_N}$ with $x_{B(i)} = x_{n+1} = b_i$ is a b.f.s. (see Example 2.4).

Notes

Finding a basic feasible solution

Case (b) $A\,x = b$, $b_i \lesseqgtr 0$, $i = 1, \dots, m$

- Transform the system in such a way, such that as many variables as possible can be identified as slack variables of a constraint
- We introduce *artificial variables* $\hat{x}_i$ for the remaining constraints and obtain:

$$\sum_{j=1}^{n} a_{ij} x_j + \hat{x}_i = b_i$$

- Note that this changes the feasible set of the LP! A solution of it corresponds only to a feasible solution $(x_1, \dots, x_n)$ of the original LP if all artifical variables equal zero.

Notes

Example 6.1

$\rightarrow$ board

Notes

2-Phase-Method - Phase 1

Idea: Enforce that all artificial variables have the value 0 and pivot them out of the basis.

For this purpose, minimize the *auxiliary objective function*

$$h(\widetilde{x}) := \sum \hat{x}_j \quad \text{s.t. } \widetilde{A}\,\widetilde{x} = b, \quad \widetilde{x} \geq 0$$

For this problem, we know a basic feasible solution!

Case 1: $h(\widetilde{x}^*) = \sum \hat{x}_j^* > 0$

Then, there exists no solution for $\widetilde{A}\,\widetilde{x} = b$, $\widetilde{x} \geq 0$ with $\hat{x}_i = 0$ for all artifical variables. Hence, there exists no solution for the original system $A\,x = b, x \geq 0$, i. e., the LP is infeasible.

Notes

2-Phase-Method - Phase 1

Case 2: $h(\tilde{x}^*) = \sum \hat{x}_j^* = 0$

$\Rightarrow \hat{x}_i^* = 0 \quad \forall i$

$\Rightarrow$ there exists a solution of the original system

Case 2a): All $\hat{x}_i^*$ are non-basic variables:

$\Rightarrow$ all basic variables in the optimal solution of the auxiliary LP are original variables

$\Rightarrow$ a basic feasible solution of the original LP is known

Case 2b): There exists a basic variable $\hat{x}_i^* = \hat{x}_{B(\ell)} = 0$:

(Note: Such a basic solution is called "degenerate" $\rightarrow$ see next chapter)

Pivot the artificial variable out of the basis, i. e., pivot with $t_{\ell j} \neq 0$, which belongs to an original variable.

Notes

Algorithm 6.2 (2-Phase method)

Input: LP in standard form $\min\{c^\top x : Ax = b,\ x \geq 0\}$ with $b \geq 0$ (multiply $A_{i\bullet}x = b_i$ by (-1) if $b_i < 0$).

1: Let $I \subseteq \{1, \ldots, m\}$ be the index set of equations $A_{i\bullet}x = b_i$, in which a variable $x_{s(i)}$ occurs exclusively, with $a_{is(i)} > 0$.

2: **for all** $i \in \{1, \ldots, |\bar{I}|\}$ with $\bar{I} := \{1, \ldots, m\} \setminus I$ **do**

3: Introduce an artificial variable $\hat{x}_{n+i} \geq 0$.

4: **if** $\bar{I} \neq \varnothing$ **then** *// Phase 1*

5: $\tilde{A} := (\tilde{a}_{ij})$

$$\tilde{a}_{ij} := \begin{cases} a_{ij} & \forall (i,j) :\ i \in \bar{I},\ j \in \{1, \ldots, n\} \\ \frac{a_{ij}}{a_{is(i)}} & \forall (i,j) :\ i \in I,\ j \in \{1, \ldots, n\} \\ 1 & \forall (\bar{I}(k), n+k) :\ k \in \{1, \ldots, |\bar{I}|\} \\ 0 & \text{else} \end{cases}$$

$$\tilde{b}_i := \begin{cases} b_i & \forall i \in \bar{I} \\ \frac{b_i}{a_{is(i)}} & \forall i \in I \end{cases}$$

Notes

$$\tilde{x} := (x_1, \ldots, x_n, \hat{x}_{n+1}, \ldots, \hat{x}_{n+|\bar{I}|})^\top$$

$B := \{s(i) : \; i \in I\} \cup \{n+1, \ldots, n+|\bar{I}|\}$, $\tilde{x} = \binom{\tilde{x}_B}{\tilde{x}_N}$ with $\tilde{x}_B = \tilde{b}$

Determine optimal solution $\tilde{x}^*$ of the LP $\min \left\{ \sum_{i \in \bar{I}} \hat{x}_{i+n} \; : \tilde{A}\,\tilde{x} = \tilde{b}, \; \tilde{x} \geq 0 \right\}$

if $\sum_{i \in \bar{I}} \hat{x}_{i+n} > 0$ **then**

  return LP is infeasible.

Pivot all artificial variables out of the basis.

Remove columns $n+1$ bis $n+|\bar{I}|$ from the optimal tableau.

Replace the objective coefficients of the auxiliary objective by the original objective $c^\top x$.

Apply elementary row operations to obtain $t_{0B(i)} = 0$ for all basic columns $B(i)$.

// *Start phase 2*

Notes

Example 6.3

→ board

Notes

Degenerate basic solution/LP

Definition 6.4

- A basic feasible solution x is called *degenerate*, if at least one of the basic variables is equal to 0.
- Since $x_N = 0 \Rightarrow |\{i : x_i = 0\}| > n - m$ for a degenerate BFS x.
- The basis corresponding to a degenerate BFS is also called *degenerate*.
- An LP (in standard form) is called *non-degenerate*, if it is feasible and has no degenerate basic feasible solution.

Notes

Degenerate simplex iteration

Definition 6.5

A simplex iteration is called *degenerate*, if it does not change the basic solution (i. e., the min ratio rule results in 0) and, for this reason, the objective function value does not change, too.

Example 6.6

$\rightarrow$ board

Notes

Theorem 6.7

In a non-degenerate LP, the Simplex Method stops after at most $\binom{n}{m}$ iterations (either it finds an optimal solution or it shows unboundedness).

Proof.

$\rightarrow$ board $\square$

Notes

Degenerate simplex iteration

Remark 6.8

1. Geometric interpretation: Consider a LP with feasible set $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. If a BFS $\bar{x}$ is degenerate, then there exist more than n inequalities , s.t. $A_{i\bullet}\,\bar{x} = b_i$, i. e. the number of constraints of the LP, which are satisfied with equality is greater than n, e. g. 3 lines intersect in one point.
2. Degenerate bases do not necessarily lead to degenerate simplex iterations.
3. Degenerate simplex iterations only occur, if degenerate bases exist.
4. Degenerate simplex iterations only become problematic, if a degenerate basis repeats. $\rightarrow$ infinite loop

Notes

Cycling of the Simplex Algorithm

Definition 6.9

The simplex algorithm is *cycling*, if one simplex tableau $T(B)$ (w. r. t. to some basis) appears in two different iterations.

In this case, the algorithm does not terminate!

Notes

Example 6.10 (Cycling of the Simplex Algorithm, by Beale (1955))

$$
\begin{array}{llcl}
\min & -\tfrac{3}{4}x_1 + 20x_2 - \tfrac{1}{2}x_3 + 6x_4 & & \\
\text{s.t.} & \tfrac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 + x_5 & = & 0 \\
& \tfrac{1}{2}x_1 - 12x_2 - \tfrac{1}{2}x_3 + 3x_4 + x_6 & = & 0 \\
& x_3 + x_7 & = & 1 \\
& x_1, \dots, x_7 & \geq & 0
\end{array}
$$

Choose starting basis $B = \{5, 6, 7\}$ and apply the Simplex-algorithm with the following pivot rule

- Choose the non-basis variable with the smallest reduced costs value $\bar{c}_j < 0$.
- Choose the leaving basis variable according to the min-ratio-rule with the smallest index.

Notes

$$
\begin{array}{c|ccccccc|c}
\hline
 & -3/4 & 20 & -1/2 & 6 & 0 & 0 & 0 & 0 \\
\hline
 & \boxed{1/4} & -8 & -1 & 9 & 1 & 0 & 0 & 0 \\
 & 1/2 & -12 & -1/2 & 3 & 0 & 1 & 0 & 0 \\
 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
\hline
\end{array}
$$

$$
\begin{array}{c|ccccccc|c}
\hline
 & 0 & -4 & -7/2 & 33 & 3 & 0 & 0 & 0 \\
\hline
 & 1 & -32 & -4 & 36 & 4 & 0 & 0 & 0 \\
\longrightarrow & 0 & \boxed{4} & 3/2 & -15 & -2 & 1 & 0 & 0 \\
 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
\hline
\end{array}
$$

$$
\begin{array}{c|ccccccc|c}
\hline
 & 0 & 0 & -2 & 18 & 1 & 1 & 0 & 0 \\
\hline
 & 1 & 0 & \boxed{8} & -84 & -12 & 8 & 0 & 0 \\
\longrightarrow & 0 & 1 & 3/8 & -15/4 & -1/2 & 1/4 & 0 & 0 \\
 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
\hline
\end{array}
$$

$$
\begin{array}{c|ccccccc|c}
\hline
 & 1/4 & 0 & 0 & -3 & -2 & 3 & 0 & 0 \\
\hline
 & 1/8 & 0 & 1 & -21/2 & -3/2 & 1 & 0 & 0 \\
\longrightarrow & -3/64 & 1 & 0 & \boxed{3/16} & 1/16 & -1/8 & 0 & 0 \\
 & -1/8 & 0 & 0 & 21/2 & 3/2 & -1 & 1 & 1 \\
\hline
\end{array}
$$

Notes

	$-1/2$	16	0	0	-1	1	0	0
	$-5/2$	56	1	0	$\boxed{2}$	-6	0	0
$\longrightarrow$	$-1/4$	$16/3$	0	1	$1/3$	$-2/3$	0	0
	$5/2$	-56	0	0	-2	6	1	1

	$-7/4$	44	$1/2$	0	0	-2	0	0
	$-5/4$	28	$1/2$	0	1	-3	0	0
$\longrightarrow$	$1/6$	-4	$-1/6$	1	0	$\boxed{1/3}$	0	0
	0	0	1	0	0	0	1	1

	$-3/4$	20	$-1/2$	6	0	0	0	0
	$\boxed{1/4}$	-8	-1	9	1	0	0	0
$\longrightarrow$	$1/2$	-12	$-1/2$	3	0	1	0	0
	0	0	1	0	0	0	1	1

Notes

Cycling

Theorem 6.11 (Bland's pivot rule)

In step (3) of the simplex algorithm, choose

$$j = \min\{j : t_{0j} < 0\}$$

and in step (5), choose r such that

$$B(r) = \min\left\{B(i) : t_{ij} > 0 \text{ and } \frac{t_{in+1}}{t_{ij}} \leq \frac{t_{kn+1}}{t_{kj}} \ \forall k \text{ with } t_{kj} > 0\right\}.$$

Then the simplex algorithm terminates after finitely many pivot operations (i. e., cycling is prevented).

Notes

Example 6.12 (Cycling of the Simplex Algorithm, by Beale (1955))

We apply the Simplex Method with Bland's pivot rule to the problem from Example 6.10.

$$
\begin{array}{|ccccccc|c|}
\hline
-\frac{3}{4} & 20 & -\frac{1}{2} & 6 & 0 & 0 & 0 & 0 \\
\hline
\boxed{\frac{1}{4}} & -8 & -1 & 9 & 1 & 0 & 0 & 0 \\
\frac{1}{2} & -12 & -\frac{1}{2} & 3 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
\hline
\end{array}
$$

$$
\longrightarrow
\begin{array}{|ccccccc|c|}
\hline
0 & -4 & -\frac{7}{2} & 33 & 3 & 0 & 0 & 0 \\
\hline
1 & -32 & -4 & 36 & 4 & 0 & 0 & 0 \\
0 & \boxed{4} & \frac{3}{2} & -15 & -2 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
\hline
\end{array}
$$

Notes

$$\longrightarrow \quad \begin{array}{|ccccccc|c|}\hline 0 & 0 & -2 & 18 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & \boxed{8} & -84 & -12 & 8 & 0 & 0 \\ 0 & 1 & \frac{3}{8} & -\frac{15}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ \hline \end{array}$$

$$\longrightarrow \quad \begin{array}{|ccccccc|c|}\hline \frac{1}{4} & 0 & 0 & -3 & -2 & 3 & 0 & 0 \\ \hline \frac{1}{8} & 0 & 1 & -\frac{21}{2} & -\frac{3}{2} & 1 & 0 & 0 \\ -\frac{3}{64} & 1 & 0 & \boxed{\frac{3}{16}} & \frac{1}{16} & -\frac{1}{8} & 0 & 0 \\ -\frac{1}{8} & 0 & 0 & \frac{21}{2} & \frac{3}{2} & -1 & 1 & 1 \\ \hline \end{array}$$

$$\longrightarrow \quad \begin{array}{|ccccccc|c|}\hline -\frac{1}{2} & 16 & 0 & 0 & -1 & 1 & 0 & 0 \\ \hline -\frac{5}{2} & 56 & 1 & 0 & 2 & -6 & 0 & 0 \\ -\frac{1}{4} & \frac{16}{3} & 0 & 1 & \frac{1}{3} & -\frac{2}{3} & 0 & 0 \\ \boxed{\frac{5}{2}} & -56 & 0 & 0 & -2 & 6 & 1 & 1 \\ \hline \end{array}$$

$$\longrightarrow \quad \begin{array}{|ccccccc|c|}\hline 0 & \frac{24}{5} & 0 & 0 & -\frac{7}{5} & \frac{11}{5} & \frac{1}{5} & \frac{1}{5} \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -\frac{4}{15} & 0 & 1 & \boxed{\frac{2}{15}} & -\frac{1}{15} & \frac{1}{10} & \frac{1}{10} \\ 1 & -\frac{112}{5} & 0 & 0 & -\frac{4}{5} & \frac{12}{5} & \frac{2}{5} & \frac{2}{5} \\ \hline \end{array}$$

Notes

	0	2	0	$\frac{21}{2}$	0	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{5}{4}$
	0	0	1	0	0	0	1	1
$\longrightarrow$	0	-2	0	$\frac{15}{2}$	1	$-\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
	1	-24	0	6	0	2	1	1

An optimal solution of the problem is $x^* = \begin{pmatrix} 1 & 0 & 1 & 0 & \frac{3}{4} & 0 & 0 \end{pmatrix}^\top$ with optimal objective value $z = -\frac{5}{4}$.

Notes

Cycling

Remark 6.13

1. If the simplex algorithm is cycling, then all corresponding pivot operations must be degenerate
2. Degenerate pivot operations do not necessarily have to lead to cycling.
3. Geometric interpretation of cycling: The simplex algorithm gets stuck in one extreme point.

Notes

Contents

Notes

The Dual of a Linear Program

Definition 7.1

Given a linear program in standard form

$$\begin{aligned} \min \ & c^\top x \\ \text{s.t. } & Ax = b \\ & x \geq 0, \end{aligned} \tag{LP}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.

The *dual (linear) program* or *dual problem* (DP) of (LP) is defined as:

$$\begin{aligned} \max \ & b^\top \pi \\ \text{s.t. } & A^\top \pi \leq c \\ & \pi \gtrless 0 \end{aligned} \tag{DP}$$

where $\pi = (\pi_1, \ldots, \pi_m)^\top \in \mathbb{R}^m$ are the *dual variables* and $A^\top \pi \leq c$ are the *dual constraints*.

Notes

Dual Problem

Example 7.2

$\rightarrow$ board

Remark 7.3

In the proof of Theorem 3.9 we have already introduced the dual variables $\pi := c_B^\top A_b^{-1}$.

Observation 7.4

If π is an arbitrary solution of the system

$$A^\top \pi \leq c,$$

then any feasible solution x of (LP) satisfies

$$b^\top \pi = (Ax)^\top \pi = x^\top A^\top \pi \leq x^\top c = c^\top x.$$

Notes

Dual Problem

Theorem 7.5

The dual linear program of (DP) *is* (LP).

Proof.

$\rightarrow$ board $\square$

Notes

Dual Problem

Theorem 7.6

An LP in general form

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & A_{i\bullet}\, x = b_i \quad \forall i \in M \\ & A_{i\bullet}\, x \geq b_i \quad \forall i \in \{1, \dots, m\} \setminus M =: \overline{M} \\ & x_j \geq 0 \quad \forall j \in L \\ & x_j \gtrless 0 \quad \forall j \in \{1, \dots, n\} \setminus L =: \overline{L} \end{aligned}$$

has the dual LP

$$\begin{aligned} \max \quad & b^\top \pi \\ \text{s.t.} \quad & A_{\bullet j}^\top \pi \leq c_j \quad \forall j \in L \\ & A_{\bullet j}^\top \pi = c_j \quad \forall i \in \overline{L} \\ & \pi_i \gtrless 0 \quad \forall i \in M \\ & \pi_i \geq 0 \quad \forall j \in \overline{M} \end{aligned}$$

Notes

Tucker Diagram

$$
\begin{array}{c|c|c|c|c|l}
\multicolumn{1}{c}{} & \multicolumn{2}{c}{\min c^\top x} & \multicolumn{3}{c}{} \\ \cline{2-5}
 & a_{11} \quad \cdots & \cdots \quad a_{1n} & & b_1 & \\
 & \vdots & & = & \vdots & i \in M:\ \pi_i \gtrless 0 \\ \cline{2-5}
\max b^\top \pi & \vdots & & \geq & \vdots & i \in \bar{M}:\ \pi_i \geq 0 \\
 & a_{m1} \quad \cdots & \cdots \quad a_{mn} & & b_m & \\ \cline{2-5}
 & \leq & = & \multicolumn{3}{l}{} \\ \cline{2-3}
 & c_1 \quad \cdots & \cdots \quad c_n & \multicolumn{3}{l}{} \\ \cline{2-3}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{j \in L:\ x_j \geq 0} & \multicolumn{1}{c}{j \in \bar{L}:\ x_j \gtrless 0} & \multicolumn{3}{c}{}
\end{array}
$$

Notes

Dual Problem

Example 7.7

$\rightarrow$ board

Notes

Weak duality theorem

Theorem 7.8 (Weak duality theorem)

Let (LP) *be an LP in general form. If x is feasible for* (LP) *and if π is feasible for* (DP)*, then*

$$b^\top \pi \leq c^\top x$$

Proof.

$\rightarrow$ board $\square$

Notes

Strong duality theorem

Theorem 7.9 (Strong duality theorem)

Let (LP) *and* (DP) *be a dual pair of linear programs in standard form.*

a) *If one of the two LPs is unbounded, then the corresponding dual program is infeasible.*
b) *If one the two LPs has a finite optimal solution, then so has the other and the optimal objective function values are equal.*
c) (LP) *and* (DP) *may both be infeasible.*

Proof.

$\rightarrow$ board $\square$

Notes

Example 7.10

$\longrightarrow$ board

Notes

Dual pairs

Corollary 7.11 (Possibilities for dual pairs)

primal \ *dual*	*finite optimal solution*	*feasible solution, unbounded objective function value*	*no feasible solution*
finite optimal solution	✓	X	X
feasible solution, unbounded objective function value	X	X	✓
no feasible solution	X	✓	✓

X: cannot happen
✓ : can occur

Notes

Complementary slackness conditions

Theorem 7.12

Let (LP) *be an LP in general form and let* (DP) *be its dual. Furthermore, let* x *and* π *be feasible solutions of* (LP) *and* (DP)*, respectively.*

Then x *and* π *are optimal for* (LP) *and* (DP)*, respectively*

$$\iff \begin{cases} u_i := \pi_i \left(A_{i\bullet}\, x - b_i\right) = 0 & \forall i = 1, \ldots, m \\ v_j := x_j \left(c_j - A_{\bullet j}^\top \pi\right) = 0 & \forall j = 1, \ldots, n \end{cases}$$

Proof.

$\rightarrow$ board $\square$

Notes

Algorithm 7.13 (Dual Simplex Algorithm)

Input: (LP) $\quad \min\{c^\top x : \; A x = b, \; x \geq 0\}$,
basis B, such that $\bar{c}^\top = c^\top - c_B^\top A_B^{-1} A \geq 0$. *// dual feasible*

Determine T_B
while $t_{i,n+1} \ngeq 0, \quad \forall i \in \{1, \dots, m\}$ **do**
  Choose $i \in \{1, \dots, m\}$ with $t_{i,n+1} < 0$
  if $t_{ij} \geq 0, \quad \forall j = \{1, \dots, n\}$ **then**
    return "LP is infeasible."
  else
    Choose $s \in \{1, \dots, n\}$ with

$$\frac{t_{0\,s}}{-t_{i\,s}} = \min\left\{\frac{t_{0\,j}}{-t_{i\,j}} \;:\; t_{i\,j} < 0\right\}$$

    Compute a pivot step with pivot element $-t_{is}$.
return $x = (x_B, x_N)$ with $x_{B(i)} = t_{i,n+1}, \;\; i = 1, \dots, m$ and $x_N = 0$ is a optimal solution.

Notes

Example 7.14

$\longrightarrow$ board.

Notes

Example 7.15

If an optimization problem is solved in practice, it may be found that the modeling was inadequate, i.e. that e.g. a constraint was forgotten. It may also happen that a problem needs to be solved for slightly different input data, e.g. for example, the objective function coefficients c may change due to external conditions.

In this case, based on the already known primal or dual optimal solution, an optimal solution of the modified problem can be determined. The advantage is that usually only a few simplex iterations are necessary.

For a given problem $\min \left\{c^\top x \colon Ax \leq b, x \geq 0\right\}$ with primal optimal solution x^*, dual optimal solution π^* and optimal objective function value $c^\top x^*$, we consider four cases:

1. The objective function coefficients change: $c \rightsquigarrow c'$
2. The right hand side changes: $b \rightsquigarrow b'$
3. A new variable $x_{n+1} \geq 0$ is added.
4. An additional constraint $A_{m+1\,\boldsymbol{\cdot}} \leq b_{m+1}$ with $A_{m+1\,\boldsymbol{\cdot}} \in \mathbb{R}^{1 \times n}$, $b_{m+1} \in \mathbb{R}$ is added.

Notes