Optimization I

Network Optimization

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Introduction: Definitions and Examples

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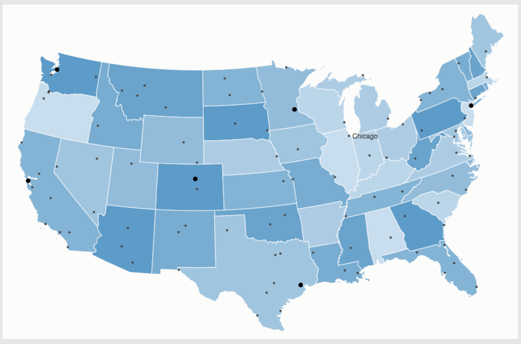
Traveling salesman problem

Example 1.1



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Evacuation

Example 1.2

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Definition 1.3 (Undirected graph)
An undirected graph is a triple $G = (V, E, \gamma)$ with a non-empty set of nodes (or vertices) V, a set of edges (or arcs) E with $V \cap E = \emptyset$ and a map

$$\gamma \colon E \to \{X \colon X \subseteq V \text{ with } 1 \leq |X| \leq 2\},$$

that assigns every edge its corresponding endpoints $\gamma(e) \in V$.

Example 1.4
$\rightarrow$ board

Notes

Definition 1.5 (Directed graph)

A directed graph (or short *digraph*) is a quadrupel $G = (V, R, \alpha, \omega)$ with the following properties:

1. V is a non-empty set of nodes (or vertices).
2. R is the set of arcs.
3. It holds that $V \cap R = \emptyset$.
4. $\alpha\colon R \to V$ and $\omega\colon R \to V$ are maps. $\alpha(e)$ denotes the head of arc e, $\omega(e)$ is the tail of e.

Example 1.6

$\to$ board

Notes

Definition 1.7 (Loops, parallel edges, simple graphs)

Let $G = (V, R, \alpha, \omega)$ be a digraph.

- An arc $r \in R$ is called loop if $\alpha(r) = \omega(r)$.
- A graph G is called loopless if it contains no loops.
- Two arcs $r_1, r_2 \in R$ are called parallel, if $\alpha(r_1) = \alpha(r_2)$ and $\omega(r_1) = \omega(r_2)$.
- A graph G is called simple if it contains neither loops nor parallel arcs.

Example 1.8

$\to$ board

Remark 1.9

In the following, we usually assume that G is simple. In this cases, every arc $r \in R$ is uniquely identified by the pair $(\alpha(r), \omega(r))$.

Notes

Definition 1.10

Let $G = (V, R, \alpha, \omega)$ be a digraph.

- A node $v \in V$ is called *incident* to an arc $r \in R$ if $v \in \{\alpha(r), \omega(r)\}$.
- Two nodes $u, v \in V$ are called *adjacent* iff there exists an arc $r \in R$ that is incident to u and v.
- For a node $v \in V$ we call
  - $\delta_G^+(v) := \{r \in R\colon \alpha(r) = v\}$ the outgoing arcs of v.
  - $\delta_G^-(v) := \{r \in R\colon \omega(r) = v\}$ the ingoing arcs on v.
  - $N_G^+(v) := \{\omega(r)\colon r \in \delta_G^+(v)\}$ the successor set of v.
  - $N_G^-(v) := \{\alpha(r)\colon r \in \delta_G^-(v)\}$ the predecessor set of v.
  - $g_G^+(v) := \left|\delta_G^+(v)\right|$ the out-degree of v.
  - $g_G^-(v) := \left|\delta_G^-(v)\right|$ the in-degree of v.
  - $g_G(v) := g_G^+(v) + g_G^-(v)$ the degree of v.

Remark 1.11

Usually we drop the index G if G is clear from the context.

Example 1.12

$\to$ board

Notes

Complexity of Algorithms

Definition 1.13 (Landau notation)

Let M be the set of all real-valued functions $f : \mathbb{N} \to \mathbb{R}$. Every function $g \in M$ then defines three classes of functions as follows:

$$\mathcal{O}(g) := \{f \in M \colon \exists c \in \mathbb{R}, n_0 \in \mathbb{N} \colon \forall\, n \geq n_0 : f(n) \leq c \cdot g(n)\}$$
$$\Omega(g) := \{f \in M \colon \exists c \in \mathbb{R}, n_0 \in \mathbb{N} \colon \forall\, n \geq n_0 : f(n) \geq c \cdot g(n)\}$$
$$\Theta(g) := \mathcal{O}(g) \cap \Omega(g)$$

We call a function f of *polynomial magnitude* in n, if there exists a polynomial g (e. g. $g(n) = n^3$) such that $f \in \mathcal{O}(g)$.

A function f grows exponentially in n if $f(n)$ cannot be bounded by a polynomial function, e. g. $g(n) = 2^n$.

Notes

Complexity of Algorithms

Model of computation

- For this part of the lecture, we're interested in the running times of the algorithms we're going to develop.
- Our model of computation is the *Unit-Cost RAM* (Random Access Machine); a "machine" that has countable many registers.
- The following operations can be performed in one clock cycle:
  - Reading and writing numbers.
  - Comparison of two registers.
  - Conditional branching.
  - Addition, subtraction, multiplication and division.
- The complexity of an algorithm is a function depending on the input size.

Notes

Complexity of Algorithms

Model of Computation

- Usually, integers are coded as binary strings. $p \in \mathbb{N}$ thus has the coding length of $\lceil \log_2(p+1) \rceil + 1$ bits. We need $\lceil \log_2(p+1) \rceil$ bits to represent $|p|$ and an additional bit for the sign.
- A fractional number p/q with $q \geq 1$ and $p, q \in \mathbb{Z}$ coprime has thus the coding length $\lceil \log_2(p+1) \rceil + \lceil \log_2(q+1) \rceil + 1$.
- We say an algorithm has *worst-case (time) complexity* T, if the running time for all inputs of size ℓ can be bounded from above by $T(\ell)$.

Notes

Definition 1.14 (Adjacency matrix)

Let $G = (V, E)$ be a digraph and $n = |V|$. Then, the *adjacency matrix* $A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$ is a matrix with entries

$$a_{ij} = |\{r \in R \colon \alpha(r) = v_i \text{ and } \omega(r) = v_j\}| .$$

For undirected graphs H we analogously define the adjacency matrix as

$$a_{ij} = |\{e \in E \colon \gamma(e) = \{v_i, v_j\}\}| .$$

Example 1.15

$\rightarrow$ board

Observation 1.16

For undirected graphs the adjacency matrix is symmetric.

Notes

Definition 1.17 (Incidence matrix)

Let $G = (V, R)$ be a loop-less digraph, $n = |V|$ and $R = \{r_1, \dots, r_m\}$. Then, the *incidence matrix* $I \in \mathbb{R}^{n \times m}$ is a matrix with

$$i_{kl} := \begin{cases} 1 & \text{if } \alpha(r_l) = v_k \\ -1 & \text{if } \omega(r_l) = v_k \\ 0 & \text{otherwise.} \end{cases}$$

For undirected graphs we define

$$i_{kl} := \begin{cases} 1 & \text{if } v_k \in \gamma(e_l) \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.18

$\rightarrow$ board

Notes

Remark 1.19

Note that the rows of the incidence matrix I are linearly dependent for every digraph G. If the matrix obtained by removing a row from I has full rank, it is called *full rank incidence matrix*.

Remark 1.20

In the case of an digraph G, the incidence matrix I is *unimodular*, that means every quadratic submatrix has determinant $+1$, -1 or 0.

Observation 1.21

Let $G = (V, E)$ be a (di)graph with $|V| = n$ and $|E| = m$.

- The space needed to save the adjacency matrix is $\Theta(n^2)$ and is independent of the number of edges in G.
- The space needed to save the incidence matrix of G is $\Theta(nm)$.

Notes

Comparison of graph storage techniques

The following table gives the time complexity for three elementary graph operations depending on their storage:

1. Is there an arc from v to w? (short: $(v, w) \in R$?)
2. What is the out-degree of v? (short: $g^+(v) =?$)
3. Exists a node with in-degree $g^-(v) = 0$? (short: $\exists v\colon g^-(v) = 0$?)

Storage	Memory	$(v, w) \in R$?	$g^+(v) =?$	$\exists v\colon g^-(v) = 0$?
Adjacency matrix	$\Theta(n^2)$	$\mathcal{O}(1)$	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$
Incidence matrix	$\Theta(nm)$	$\mathcal{O}(m)$	$\mathcal{O}(m)$	$\mathcal{O}(nm)$

Notes

Definition 1.22 (Paths, cycles)

- Let $G = (V, R)$ be a digraph. A *path* in G is a finite sequence

$$P = (v_0, r_1, v_1, \dots, r_k, v_k)$$

  with $k \geq 0$, such that $v_0, v_1, \dots, v_k \in V$ and $r_1, \dots, r_k \in R$ and $\alpha(r_i) = v_{i-1}$ as well as $\omega(r_i) = v_i$ for $i = 1, \dots, k$.
- We define the *starting node* of P as $\alpha(P) = v_0$ and the *end node* as $\omega(P) = v_k$.
- We sometimes call P a v_0-v_k-path.
- The *length* $|P|$ of P is the number of traversed arcs in the path.
- If $\alpha(P) = \omega(P)$ and $k \geq 1$ we call P a *cycle*.
- A path is called *elementary* if it contains—except for the case, that start and end coincide—no node is traversed more than once.

Example 1.23

$\rightarrow$ board

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Definition 1.24 (Connectedness and cuts)

Let $G = (V, E)$ be a graph.

- G is called *connected* if for all $v_1, v_2 \in G$ there exists an v_1-v_2-path in G.
- A *disconnecting set of edges* is a subset $Q \subseteq E$ such that $G \setminus Q := (V, E \setminus Q)$ is unconnected.
- A *cut* Q is minimal disconnecting set of edges, i.e. any subset of Q is not disconnecting.
- A cut can be written in the form $Q = (X, \bar{X})$ as the set of edges with one endpoint in X and one endpoint in $\bar{X}$. ($X, \bar{X} \subseteq V,\ \bar{X} = V \setminus X$)
- Let $s, t \in V$ and $s \neq t$, then a *s-t-cut* is a cut such that $s \in X$ and $t \in \bar{X}$.

Example 1.25

$\rightarrow$ board

Notes

Definition 1.26 (Subgraphs and trees)

Let $G = (V, E)$ be a graph.

- A *subgraph/subdigraph* of G is a graph/digraph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$.
- A *spanning* subgraph/subdigraph of G is a subgraph/subdigraph $G' = (V', E')$ with $V' = V$.
- A graph/digraph is called *acyclic* if it contains no cycle/dicycle.
- A *tree* is a connected, acyclic graph.
- A *spanning tree* is a connected, acyclic, spanning subgraph of G.
- We call vertex $v \in V_T$ *leaf* of a tree $T = (V_T, E_T)$ if there exists exactly one edge in $e \in E_T$, which has v as an endpoint.

Example 1.27

$\rightarrow$ board

Notes

Theorem 1.28

Equivalent characterizations for trees Let $G = (V, E, \gamma$ be an undirected graph. Then, the following statements are equivalent:

1. *G is a tree.*
2. *G contains no elementary cycle, but every supergraph of G with same node-set contains an elementary cycle.*
3. *For every pair $u, v \in V$ exists exactly one elementary path P with $\alpha(P) = u$ and $\omega(P) = v$.*
4. *G is connected and for every edge $e \in E$ the graph $G' = (V, E \setminus \{e\}, \gamma)$ is not connected.*
5. *G is connected and $|E| = |V| - 1$.*
6. *G contains no elementary cycle and $|E| = |V| - 1$.*

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Minimum Spanning Tree Problem

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Minimum Spanning Tree Problem (MST)

Given a graph $G = (V, E)$ and cost coefficients c_{ij} $\forall (i, j) \in E$ on the edges, find a *minimum spanning tree (MST)* $T = (V, E(T))$ of G, i. e.

$$\min_{T \in \mathcal{T}} c(T) = \min_{T \in \mathcal{T}} \sum_{e \in E(T)} c_e$$

where $\mathcal{T}$ denotes the set of all spanning trees of G.

Notes

Minimum Spanning Tree Problem (MST)

Example 2.1

$\longrightarrow$ board

Remark 2.2

The MST problem has many applications in infrastructure design problems or occurs as a subproblem, e.g. design of communication networks, gas networks, highways.

Remark 2.3

In the following, we assume that the graph $G = (V, E)$ is simple. This is no restriction, since a spanning tree never contains loops and in the case of parallel edges only the cheapest edge is included.

Additionally, we assume that G is connected.

Notes

Minimum Spanning Tree Problem (MST)

Definition 2.4

We call a subset $F \subseteq E$ of edges of G *error free*, if there exists a MST F^* with $F \subseteq E(F^*)$. An edge $e \in E$ is called *promising* for F if $F \cup \{e\}$ is error free too.

Theorem 2.5

Let $F \subseteq E$ be error free and (A, B) a cut in G with $\delta(A) \cap F = \emptyset$. If $e \in \delta(A)$ is an edge with minimum weight in $\delta(A)$, then e is promising for F.

Proof.

$\rightarrow$ board $\square$

Corollary 2.6

Let $F \subseteq E$ be error free and U be a connected component of (V, F). If e is an edge with minimum weight in $\delta(U)$, then e is promising for F.

Proof.

Follows directly from Theorem 2.5, since $\delta(U) \cap F = \emptyset$. $\square$

Notes

Kruskal's algorithm

Algorithm 1: Kruskal's Algorithm

Input : An undirected graph $G = (V, E, \gamma)$ with edge weights $c\colon E \to \mathbb{R}$

Output: The set of edges E_F of a MST

```
1 Sort the edges increasingly: $c(e_1) \leq \cdots \leq c(e_m)$
2 $E_F = \emptyset$
3 for $i := 1 \ldots, m$ do
4     if $(V, E_F \cup \{e_i\}, \gamma)$ is acyclic then
5         $E_F := E_F \cup \{e_i\}$
6 return $E_F$
```

Notes

Kruskal's algorithm

Example 2.7

→ board

Theorem 2.8

For a connected graph $G = (V, E, \gamma)$, Kruskal's algorithm computes a minimum spanning tree.

Proof.

→ board □

Notes

Kruskal's algorithm

Complexity of Kruskal's algorithm

- Sorting all m edges takes $\mathcal{O}(m \log m)$ time.
- The algorithm stops after $n - 1$ iterations
- Time to check if acyclic $\mathcal{O}(m)$
- For a naıve implementation we have a worst-case running time of $\mathcal{O}(mn)$

Notes

Kruskal's algorithm

Theorem 2.9

Using the disjoint set data structure, Kruskal's algorithm can be implemented with a worst-case running time of $\mathcal{O}(m \log m)$.

Theorem 2.10

Using even more sophisticated data structures, Kruskal's algorithm can be implemented with a worst-case running time of $\mathcal{O}(m\alpha(n))$ plus the time needed to sort the edges.

Remark 2.11

The function α in Theorem 2.10 is called inverse Ackermann function. α grows extremely slow, it is $\alpha(n) \leq 4$ for $n \leq 10^{684}$.

Notes

The algorithm of Prim

Algorithm 2: The algorithm of Prim

Input : An undirected graph $G = (V, E, \gamma)$ with edge weights $c\colon E \to \mathbb{R}$

Output: The set of edges E_T of a MST

```
1 Choose $s \in V$ arbitrarily, set $E_T = \emptyset$ and $S := \{s\}$
2 while $S \neq V$ do
3     Choose an edge $(u, v) \in \delta(S)$ with minimum cost. Let $u \in S$ and $v \in V \setminus S$
4     Set $E_T := E_T \cup \{(u, v)\}$ and $S := S \cup \{v\}$
5 return $E_T$
```

Notes

The algorithm of Prim

Example 2.12

$\to$ board

Theorem 2.13

For a connected graph $G = (V, E, \gamma)$, the algorithm of Prim computes a MST.

Proof.

Follows from Corollary 2.6 via induction, since S is always a connected component of G. □

Notes

Complexity of Prim's Algorithm

- In every iteration one edge is added to the tree ($\rightsquigarrow n-1$ iterations).
- Every iteration takes $\mathcal{O}(n)$ time to find the minimum cost edge.
- This leads to a worst-case running time of $O(n^2)$ for a naïve implementation.
- more efficient if implemented using a Fibonacci-heap for the not connected vertices sorted wrt. to the cost of adding to the tree: $\mathcal{O}(m + n \log n)$

Notes

Comparison

Algorithm	run time	Properties
Prim	$\mathcal{O}(n^2)$ [$\mathcal{O}(m + n \log n)$ eff. Implem.]	• Iteratively increase the node set S by adding a cost minimal edge • For dense graphs faster than Kruskal. • Efficient implementation: *Fibonacci-heap*
Kruskal	$\mathcal{O}(m\,n)$ [$\mathcal{O}(m\alpha(n))$ eff. Implem.]	• Add iteratively a cost minimal edge which does not form a cycle • For thin graphs faster than Prim. • The sorting algorithm determines the complexity

Notes

IP Formulation

Let $G = (V, E)$ be a graph with $|V| = n$ nodes. We model MST as an integer program:

(Binary) decision variables x_{ij} with

$$x_{ij} := \begin{cases} 1 & \text{if } (i,j) \in E(T) \\ 0 & \text{otherwise} \end{cases}$$

and cost coefficients C_{ij} with

$$C_{ij} := \begin{cases} c_{ij} & \text{if } (i,j) \in E \\ \infty & \text{otherwise} \end{cases}$$

Notes

IP Formulation

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}\, x_{ij} \\
\text{s. t.} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} = n-1 \\
& \sum_{i\in S}\sum_{\substack{j\in S\\ i<j}} x_{ij} \le |S|-1 \quad \forall S \subset V \\
& x_{ij} \in \{0,1\} \\
\text{or} \quad & x_{ij} \in [0,1] \\
\text{or} \quad & x_{ij} \ge 0
\end{aligned}$$

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Notes

IP Formulation

Remark 2.14

- The previous IP formulation can be solved as an LP, since every extreme point of the feasible polyhedron is integral.
- $x \le 1$ follows from second constraint with $|S| = 2$
- Drawback: this formulation has an exponential number of constraints!

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Shortest Path Problems

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Shortest Path Problem

Assumptions for this part:

- A simple digraph $G = (V, R)$. As for spanning trees, this is no loss of generality.
- A function $c\colon R \to \mathbb{R}$ that assigns *weights* or *costs* to every arc in the graph.

Definition 3.1

Let $G = (V, R)$ be a digraph and c be a cost function as above. The *length* (or *weight* or *cost*) $c(P)$ of a path $P = (v_0, r_1, v_1, \dots, r_k, v_k)$ in G is defined by

$$c(P) := \sum_{i=1}^{k} c(r_i).$$

The length of a path $P = (v_0)$ without arcs is thus $c(P) = 0$.

Notes

Shortest (Di-)Path Problem

Shortest (Di-)Path Problem

Let $s \in V$. For all vertices $i \in V \setminus \{s\}$ compute a *shortest path*, i.e. a path P_{si} with *minimum cost/length/weight*

$$c(P_{si}) = \sum_{r \in P_{si}} c(r).$$

Notes

Definition 3.2

The *distance* $\mathrm{dist}_c(u, v, G)$ of two nodes $u, v \in V$ with respect to the weight function c is given by

$$\mathrm{dist}_c(u, v, G) := \inf\{c(P)\colon\ P \text{ is a path from } u \text{ to } v \text{ in } G\}.$$

Remark 3.3

As usual, we set $\inf(\emptyset) = +\infty$.

Remark 3.4

1. If $u \neq v$ and v is not reachable from u, we thus have $\mathrm{dist}_c(u, v) = +\infty$.
2. If u and v coincide there are only two possibilities: Either it holds that $\mathrm{dist}_c(u, u) = 0$ or $\mathrm{dist}_c(u, u) = -\infty$.

Example 3.5

$\to$ board.

Notes

Observation 3.6

Let P be a shortest path from u to v and w is traversed by P. The partial path P_{uw} from u to w is a shortest u-w-path. Analogously for P_{wv}.

Proof.

$\rightarrow$ board $\square$

Lemma 3.7

Let $s \in V$. Then it holds that

$$\mathit{dist}_c(s, v) \leq \mathit{dist}_c(s, u) + c(u, v) \text{ for all } (u, v) \in R.$$

Proof.

$\rightarrow$ board. $\square$

Notes

Definition 3.8 (Reduced costs, potential)

Let $p\colon V \rightarrow \mathbb{R}$ be a node weighting. We define the *reduced costs* $c^p\colon R \rightarrow \mathbb{R}$ by

$$c^p(u, v) := c(u, v) + p(u) - p(v).$$

The node weighting is called *potential* for c, if for all $(u, v) \in R$ it holds that $c^p(u, v) \geq 0$, thus

$$p(v) \leq p(u) + c(u, v) \text{ for all } (u, v) \in R.$$

Observation 3.9

If for $s \in V$ all the values $\operatorname{dist}_c(s, v)$ are finite, then according to Lemma 3.7, $p(v) := \operatorname{dist}_c(s, v)$ is a potential.

Notes

Let $P = (v_0, \dots, v_k)$ be a path from v_0 to v_k. Then it holds that

$$\begin{aligned} c^p(P) &= \sum_{i=0}^{k-1} \Big(c(v_i, v_{i+1}) + p(v_i) - p(v_{i+1}) \Big) \\ &= \sum_{i=0}^{k-1} c(v_i, v_{i+1}) + \sum_{i=0}^{k-1} \Big(p(v_i) - p(v_{i+1}) \Big) \\ &= c(P) + p(v_0) - p(v_k), \end{aligned}$$

since the second sum is a telescoping sum.

Observation 3.10

- In the transition from c to c^p, the length of *every* way from v_0 to v_k differs by the number $p(v_0) - p(v_k)$ that is independent of the way.
- If P is a cycle it holds that $v_0 = v_k$ and thus $c(P) = c^p(P)$.

Notes

Observation 3.11 (Transition to reduced costs)

Let $p\colon V \to \mathbb{R}$ be a node weighting.

1. P is a shortest u-v-path w.r.t. c if and only if P is a shortest u-v-path w.r.t. c^p.
2. For every path P in G it holds that $c(P) = c^p(P) + p(\omega(P)) - p(\alpha(P))$.
3. For every cycle C in G it holds that $c(C) = c^p(C)$.

If we are given a potential p in G, it holds that $c^p(u,v) \geq 0$ for all $(u,v) \in R$.

According to Observation 3.11 we can lead back the problem of finding a shortest path with respect to c to the problem of finding a shortest path with respect to the reduced costs c^p.

Notes

Theorem 3.12

Let $G = (V,R)$ be a simple digraph and $c\colon R \to \mathbb{R}$ be a weight function. A potential p in G exists if and only if there is no cycle with negative length in G. If $c\colon R \to \mathbb{Z}$ is integral, we can choose p to be integral too.

Proof.

$\to$ board. □

Definition 3.13 (Shortest path tree)

Let G be as above. A shortest path tree rooted in s is a tree $T = (V', R')$ with the following properties:

1. V' is the set of nodes that is reachable from s.
2. For all $v \in V' \setminus \{s\}$ the unique path from s to v in T is a shortest path from s to v in G.

Notes

Remark 3.14

A shortest path tree is not necessarily unique.

Example 3.15

$\to$ board.

Theorem 3.16

Let $s \in V$ be in such a way that every reachable cycle in G has non-negative weight. Then a shortest path tree emanating from s exists in G.

Proof.

$\to$ board. □

Notes

The algorithms that we're going to look at maintain a *predecessor graph* G_π that is defined as follows:

Definition 3.17

Let $G_\pi = (V_\pi, R_\pi)$ with

$$V_\pi := \{v \in V : \pi[v] \neq \texttt{nil}\} \cup \{s\}$$
$$R_\pi := \{(\pi[v], v) \in R : v \in V_\pi \wedge v \neq s\}.$$

We will show that G_π is a shortest path tree with respect to s.

Notes

The algorithms that we're going to look at share the same initialization Init from Algorithm 3.

Algorithm 3: Initialization for shortest path algorithms

```
Init($G, s$)
forall $v \in V$ do
    $d[v] := +\infty$
    $\pi[v] := \texttt{nil}$
$d[s] := 0$
```

The values $d[v]$ for $v \in V$ are upper bounds for $\text{dist}_c(s, v)$ that are adjusted during the runtime.

Notes

The algorithms also share the following method Test shown in Algorithm 4.

Algorithm 4: Algorithm that checks for an arc $(u, v) \in R$, whether it can be used to find a shorter path from s to v than the previously known with length $d[v]$

```
Test($u, v$)
if $d[v] > d[u] + c(u, v)$ then
    $d[v] := d[u] + c(u, v)$
    $\pi[v] := u$
```

By calling Test(u, v) for an arc $(u, v) \in R$ it is checked whether the arc (u, v) can be used to construct a shorter way from s to v than the current upper bound $d[v]$.

Notes

Theorem 3.18

An algorithm is initialized with Algorithm 3 and performs an arbitrary number of test steps (Algorithm 4). Then the following holds during the runtime of the algorithm:

1. *For the reduced costs c^d with respect to the node weighting d it holds that $c^d(u, v) \leq 0$ for all $(u, v) \in R_\pi$.*
2. *$d[v] \geq dist_c(s, v)$ for all $v \in V$.*
3. *Every cycle in G_π has negative length with respect to c and c^d.*
4. *If no cycle with negative length in G is reachable from s, the predecessor graph $G_\pi = (V_\pi, R_\pi)$ is a tree with root s and G_π contains for every $v \in V_\pi \setminus \{s\}$ a path from s to v with length at most $d[v]$.*

Proof.

$\rightarrow$ board. □

Notes

Corollary 3.19

If an algorithm after a number of iterations generates $d[v] \leq dist_c(s, v)$ for every $v \in V$, it holds that $d[v] = dist_c(s, v)$ for all $v \in V$ and G_π is a shortest path tree rooted in s.

Proof.

$\rightarrow$ board. □

According to Theorem 3.18 and Corollary 3.19 the “only” thing we have to do is to find a suitable sequence of test-steps, such that $d[v] = \text{dist}_c(s, v)$ holds.

Notes

Algorithm 5: Dijkstra’s algorithm

```
DIJKSTRA($G, c, s$)
Input  : A directed graph $G = (V, R)$ with non-negative arc weight
         function $c\colon R \to \mathbb{R}^+$ and a node $s \in V$
Output: For all $v \in V$ the distance $\text{dist}_c(s, v)$ and a tree of shortest paths
         emanating from $s$
1 INIT($G, s$)
2 PERM := $\emptyset$                      // PERM is the set of permanently marked nodes
3 while $PERM \neq V$ do
4     Choose $u \in Q := V \setminus$ PERM with minimal key $d[u]$.
5     PERM := PERM $\cup \{u\}$
6     forall adjacent nodes $v$ of $u$ with $v \notin PERM$ do
7         TEST($u, v$)
8 return $d[]$ and $G_\pi$                 // $G_\pi$ is spanned by the predecessor pointers $\pi$
```

Notes

Example 3.20 (SPP with nonnegative cost)

$\rightarrow$ board

Theorem 3.21

After termination of Dijkstra's algorithm, it holds that $d[v] = dist_c(s, v)$ for all $v \in V$ and G_π is a shortest path tree rooted in s.

Proof.

$\rightarrow$ board. □

Example 3.22 (SPP with general cost)

$\rightarrow$ board

Notes

Theorem 3.23

A naïve implementation of Dijkstra's algorithm solves the shortest dipath problem with nonnegative costs in $\mathcal{O}(n^2)$ time.

Proof.

$\rightarrow$ board □

Remark 3.24

Using more sophisticated data structures (Fibonacci-Heaps similar as in the algorithm of Prim), Dijkstra's algorithm can be implemented with a worst-case running time of $\mathcal{O}(m + n \log n)$.

Notes

As we have seen in Example 3.22, Dijkstra's algorithm fails if we allow negative weights on the arcs in G.

The following algorithm by Bellman and Ford (Algorithm 6) is capable of dealing with negative weights and is able to detect negative cycles in G.

The algorithm works in n steps. In every step every arc is checked exactly once.

Lemma 3.25

At the end of step $k = 1, 2, \dots, n-1$ in Algorithm 6 it holds for all $v \in V$ that

$$d[v] \leq \min\{c(P)\colon P \text{ is a path from } s \text{ to } v \text{ with at most } k \text{ arcs}\}.$$

Proof.

$\rightarrow$ board. □

Notes

Algorithm 6: Algorithm of Bellman and Ford

```
BELLMAN-FORD($G, c, s$)
Input  : A directed graph $G = (V, R)$ with arc weight function $c\colon R \to \mathbb{R}$
         and a node $s \in V$
Output: For all $v \in V$ the distance $\mathrm{dist}_c(s, v)$ and a tree of shortest paths
         emanating from $s$
INIT($G, s$)
for $k := 1, \ldots, n-1$ do
    forall $(u, v) \in R$ do
        TEST($u, v$)
return $d[]$ and $G_\pi$                    // $G_\pi$ is spanned by the predecessor pointers $\pi$
```

Notes

Example 3.26

$\to$ board.

Theorem 3.27

1. *If G contains no cycle of negative length that is reachable from s, at the termination of Algorithm 6 it holds that $d[v] = dist_c(s, v)$ for all $v \in V$. Moreover, G_π is a shortest path tree rooted in s.*
2. *The algorithm of Bellman and Ford can be implemented with a worst-case running time of $\mathcal{O}(mn)$.*

Proof.

$\to$ board. □

Notes

For the last theorem we assumed that no cycle with negative length is reachable from s. In the following, we show how to modify the algorithm of Bellman and Ford to detect cycles with negative length.

Our modification is that, at the end of the algorithm, we iterate over all arcs once again and check the condition $d[v] \leq d[u] + c(u, v)$. If it holds that $d[v] > d[u] + c(u, v)$ for one arc, we declare this as a certificate for a cycle of negative length.

Example 3.28

$\to$ board.

Notes

Algorithm 7: Algorithm that tests if G contains a cycle with negative length

```
TEST-NEGATIVE-CYCLE($G, c, d$)
Input  : A directed graph $G = (V, R)$ with arc weight function $c\colon R \to \mathbb{R}$
         and the distance values $d$ of the algorithm of Bellman and Ford
Output: The information whether $G$ contains a cycle of negative length or
         not
1 forall $(u, v) \in R$ do
2     if $d[v] > d[u] + c(u, v)$ then
3         return "Yes"
4 return "No"
```

Notes

Theorem 3.29

Algorithm 7 decides in $\mathcal{O}(m + n)$ time after the termination of the algorithm of Bellman and Ford whether G contains a cycle of negative length that is reachable from s or not.

Proof.

$\to$ board. □

Remark 3.30

1. We can decide in $\mathcal{O}(nm + m + n) = \mathcal{O}(nm)$ time whether a graph contains a cycle of negative length or not.
2. In an additional $\mathcal{O}(n)$ time a cycle with negative length can be constructed if we know an arc $(u, v) \in R$ with $d[v] > d[u] + c(u, v)$.

Theorem 3.31

For a directed graph G we can construct a cycle of negative length in $\mathcal{O}(nm)$ time or determine that no such cycle exists.

Notes

In this section we look at the problem to solve the all-pair shortest path problem. In addition to the assumptions at the beginning of this chapter we assume that no cycle with negative length exists in G (this can be checked in $\mathcal{O}(mn)$ time).

In principle, we could determine all distances $\text{dist}_c(u, v)$ by applying n-times the algorithm of Bellman and Ford; once for every node $v \in V$ as start node s. This leads to a worst-case complexity of $\mathcal{O}(n^2 m)$.

In the following, we show how to do better with the concept of *dynamic programming*.

Notes

The idea of the algorithm of Floyd and Warshall is based on a recursion for the distance values.

We number the nodes $V = \{v_1, \dots, v_n\}$ in an arbitrary manner and define for $v_i, v_j \in V$ and $k = 0, 1, \dots, n$ the value $d_k(vi, v_j)$ as the length (w.r.t. c) of a shortest path from v_i to v_j, that traverses only the nodes $\{v_i, v_j, v_1, \dots, v_k\}$.

If no such path exists, we set $d_k(v_i, v_j) := +\infty$.The distance $\text{dist}_c(v_i, v_j)$ then equals $d_n(v_i, v_j)$.

For $k = 0$ the path may only traverse v_i and v_j, it thus holds that

$$d_0(v_i, v_j) := \begin{cases} c(v_i, v_j) & \text{if } (v_i, v_j) \in R \\ \infty & \text{else} \end{cases}.$$

Notes

Let P be a shortest path from v_i to v_j that only traverses $\{v_i, v_j, v1, \dots, v_k\}$. Since, by assumption, G contains no cycle with negative length, we may assume that P is an elementary path.

If v_{k+1} is not traversed by P, it is $c(P) = d_k(v_i, v_j)$. If v_{k+1} is contained in P, we can split up P into two partial paths $P_{v_i,v_{k+1}}$ from v_i to v_{k+1} and P_{v_{k+1},v_j} from v_{k+1} to v_j. Then it holds that $c(P_{v_i,v_{k+1}}) = d_k(v_i, v_{k+1})$ and $c(P_{v_{k+1},v_j}) = d_k(v_{k+1}, v_j)$.

So for $k \geq 1$ we get the following recursion

$$d_k(v_i, v_j) = \min\{d_k(v_i, v_j), \quad d_k(v_i, v_{k+1}) + d_k(v_{k+1}, v_j)\}.$$

This leads to the following algorithm…

Notes

Algorithm 8: Algorithm of Floyd and Warshall

```
FLOYD-WARSHALL($G, c$)
Input  : A directed graph $G = (V, R)$ with arc weight function $c\colon R \to \mathbb{R}$
Output: For all $u, v \in V$ the distance $D_n[u, v] = \text{dist}_c(u, v)$
forall $v_i, v_j \in V$ do
    $D_0[v_i, v_j] := +\infty$
forall $(v_i, v_j) \in R$ do
    $D_0[v_i, v_j] := c(v_i, v_j)$
for $k = 0, \dots, n-1$ do
    for $i = 1, \dots, n$ do
        for $j = 1, \dots, n$ do
            $D_{k+1}[v_i, v_j] := \min\{D_k[v_i, v_j], D_k[v_i, v_{k+1}] + D_k[v_{k+1}, v_j]\}$
return $D_n[]$
```

Notes

Example 3.32

$\rightarrow$ get the example on the lecture's homepage.

Theorem 3.33

If G contains no cycle of negative length, the algorithm of Floyd and Warshall solves the all-pair shortest path problem in $\mathcal{O}(n^3)$ time.

Proof.

Obvious. $\square$

Notes

Network flow problems

Outline

Notes

Flows and cuts

Assumptions for this chapter

Within this chapter, G denotes a finite directed graph $G = (V, R, \alpha, \omega)$ that does not necessarily have to be simple.

Additionally, let $f\colon R \to \mathbb{R}$ be an arbitrary function. We interpret $f(r)$ as *flow value* on the arc r.

Notes

Flows and cuts

Definition 4.1 (Excess of a node *v*)

For some node $v \in V$, we denote by

$$f\left(\delta^+(v)\right) = \sum_{r \in \delta^+(v)} f(r)$$

the amount of flow leaving v and by

$$f\left(\delta^-(v)\right) = \sum_{r \in \delta^-(v)} f(r)$$

the amount of flow entering node v.

We denote by

$$\mathrm{excess}_f(v) := f\left(\delta^-(v)\right) - f\left(\delta^+(v)\right)$$

the *excess* of node $v \in V$.

Notes

Definition 4.2 (Flow, feasible flow, maximum flow)

- Let $s, t \in V$ with $s \neq t$. An (s,t)*-flow* is a function $f\colon R \to \mathbb{R}_+$ with

$$\mathrm{excess}_f(v) = 0 \tag{1}$$

  for all $v \in V \setminus \{s, t\}$.
- The node s is called *source node*, the node t is called *sink node* of flow f. If s and t are clear from the context, we shortly say *flow*.
- We call $\mathrm{val}(f) := \mathrm{excess}_f(t)$ the *flow value* corresponding to f.
- If $u\colon R \to \mathbb{R}$ is a *capacity function* on the arcs of G, we say a flow f is *feasible* if

$$0 \leq f(r) \leq u(r) \tag{2}$$

  for all $r \in R$.
- A feasible flow is called *maximum* (s,t)*-flow* if it has maximum value of all feasible (s,t)-flows.

Notes

Flows and cuts

Example 4.3

$\rightarrow$ board

Remark 4.4

- The conditions $\mathrm{excess}_f(v) = 0$ are called *flow conservation constraints* or *mass balance constraints.* All nodes except s and t are in balance.
- The inequalities $0 \leq f(r) \leq u(r)$ for all $r \in R$ are called *capacity constraints.*

Notes

Flows and cuts

Definition 4.5

Let (S, T) be a cut in G. We define the *forward part* of a cut by

$$\delta^+(S) := \{r \in R \colon \alpha(r) \in S \text{ and } \omega(r) \in T\}$$

and the *backward part* of a cut by

$$\delta^-(S) := \{r \in R \colon \alpha(r) \in T \text{ and } \omega(r) \in S\}.$$

Definition 4.6 (Excess of a cut)

For a set $S \subseteq V$ we define the excess of S by

$$\mathsf{excess}_f(S) := f\left(\delta^-(S)\right) - f\left(\delta^+(S)\right).$$

Notes

Flows and cuts

Lemma 4.7

Let $f \colon R \to \mathbb{R}$ be an arbitrary function and $S \subseteq V$. Then it holds that

$$\mathsf{excess}_f(S) = \sum_{v \in S} \mathsf{excess}_f(v).$$

Proof.

$\to$ board. □

Notes

Flows and cuts

Lemma 4.8

If f is an (s,t)-flow and (S,T) an (s,t)-cut, then it holds that

$$\mathsf{val}(f) = f\left(\delta^+(S)\right) - f\left(\delta^-(S)\right).$$

In particular, it follows that $\mathsf{excess}_f(t) = -\,\mathsf{excess}_f(s)$.

Proof.

$\to$ board. □

Notes

Flows and cuts

Definition 4.9 (Capacity of an (s,t)-cut)

If (S,T) is an (s,t)-cut in G with capacities $u\colon R \to \mathbb{R}_+$ on the arcs, we denote by

$$u\left(\delta^+(S)\right) = \sum_{r\in\delta^+(S)} u(r)$$

the *capacity of the cut* (S,T).

We say (S,T) is a *minimum* (s,t)*-cut*, if it has minimum capacity among all (s,t)-cuts.

Remark 4.10

Intuitively, the capacity of a cut (S,T) is an upper bound for the flow value of a feasible flow f. That this intuition is correct follows from Lemma 4.8.

Notes

Flows and cuts

Lemma 4.11

If f is a feasible (s,t)-flow and (S,T) an (s,t)-cut, then it holds that

$$\operatorname{val}(f) \leq u\left(\delta^+(S)\right).$$

Since f and (S,T) are arbitrary, it follows that

$$\max_{f \text{ is a feasible } (s,t)\text{-flow in } G} \operatorname{val}(f) \leq \min_{(S,T) \text{ is an } (s,t)\text{-cut in } G} u\left(\delta^+(S)\right).$$

Notes

Flows and cuts

Corollary 4.12

If f is a feasible flow and (S,T) a cut in G such that the flow value $\operatorname{val}(f)$ equals the capacity $u\left(\delta^+(S)\right)$ of the cut, then f is a maximum flow and (S,T) is a minimum cut.

Notes

Residual networks and flow augmenting paths

Example 4.13

$\rightarrow$ board.

Definition 4.14 (Residual network)

Let f be a feasible flow in G and l, u be lower and upper capacity bounds with $0 \leq l(r) \leq u(r)$ for all arcs $r \in R$ (we also allow $u(r) = +\infty$).

The *residual network* $G_f = (V, R_f, \alpha', \omega')$ has the same node set as G. The set of arcs is defined as follows:

- If $r \in R$ and $f(r) < u(r)$, G_f contains an arc $+r$ with $\alpha'(+r) = \alpha(r)$ and $\omega'(+r) = \omega(r)$ and *residual capacity* $u_f(+r) := u(r) - f(r)$.
- If $r \in R$ and $f(r) > l(r)$, G_f contains an arc $-r$ with $\alpha'(-r) = \omega(r)$ and $\omega'(-r) = \alpha(r)$ and *residual capacity* $u_f(-r) := f(r) - l(r)$.

Notes

Residual networks and flow augmenting paths

Example 4.15

$\rightarrow$ board.

Remark 4.16

We have introduced the "signs" for the arcs in G_f in order to make clear that these arrows are in the residual network. On the other hand, for example $-r$ emphasizes that this arc is an "flipped" version of r.

In the following we are using σ as a place holder for the sign, thus every arc in G_f can be written as σr for all $r \in R$.

We denote by $-\sigma r$ the inverse arc, thus $-r$ for $+r$ and vice versa.

Notes

Residual networks and flow augmenting paths

Definition 4.17 (Flow augmenting paths)

A path P from s to t in G_f is called *flow augmenting path* for the flow f. The *residual capacity*

$$\Delta(P) := \min_{\sigma r \in P} u_f(\sigma r)$$

of P is the minimum of the residual capacities of its arcs.

Example 4.18

$\rightarrow$ board.

Observation 4.19

If there exists a flow augmenting path for some flow f, f cannot be a maximum flow.

Notes

Residual networks and flow augmenting paths

In Lemma 4.11 we have shown that the capacity of an (s,t)-cut is an upper bound for the flow value of every feasible (s,t)-flow. We re-formulate this result with the help of residual networks.

Lemma 4.20

If f is a feasible (s,t)-flow and f' and maximum (s,t)-flow, it holds that

$$\mathrm{val}(f') \leq \mathrm{val}(f) + u_f(\delta^+_{G_f}(S))$$

for every (s,t)-cut (S,T) in G_f.

Proof.

$\rightarrow$ board. □

Notes

The max-flow-min-cut theorem

Observation 4.19 yields a necessary condition for the maximality of some flow f: there must not exist a flow augmenting path. We're going to show that this condition is also sufficient.

Let f^* be a maximum (s,t)-flow. According to Observation 4.19 there does not exist an flow augmenting path for f^*. This means, that t is not reachable from s in G_f and the two sets

$$S := \left\{v \in V \colon v \text{ is reachable from } s \text{ in } G_f\right\}$$
$$T := \left\{v \in V \colon v \text{ is reachable from } t \text{ in } G_f\right\}$$

are non-empty.

Thus the two sets define a cut (S,T).

Notes

The max-flow-min-cut theorem

Let $r \in \delta^+(S)$ be an arc in the forward part of the cut.

Then it holds that $f^*(r) = u(r)$, because otherwise $+r$ would be an arc in G_{f^*} and $\omega(r)$ would be reachable from s in G_{f^*}, in contradiction to $\omega(r) \in T$ (we have $v \in S$ and thus by definition of S the node v is reachable from s in G_{f^*}).

This shows that

$$f^*\left(\delta^+(S)\right) = u\left(\delta^+(S)\right). \tag{3}$$

Notes

The max-flow-min-cut theorem

Similarly, for every arc $r \in \delta^-(S)$ it must hold that $f^*(r) = 0$, because otherwise $-r$ would be an arc in G_{f^*} and $\omega(r)$ as well as $\alpha(r)$ would be reachable from s.

Thus it holds that

$$f^*\left(\delta^-(S)\right) = 0. \tag{4}$$

Combining (3) and (4) results in

$$\begin{aligned} u\left(\delta^+(S)\right) &= f^*\left(\delta^+(S)\right) - f^*\left(\delta^-(S)\right) \\ &\overset{\star}{=} \mathrm{val}(f^*), \end{aligned}$$

where we used Lemma 4.8 at $\star$.

Notes

The max-flow-min-cut theorem

Because of Corollary 4.12 f^* is a maximum flow and at the same time (S, T) is a minimum cut, that is a cut with minimum capacity.

This leads to the following, famous theorem by Ford and Fulkerson.

Theorem 4.21 (Max-flow-min-cut theorem)

In a directed graph G with capacities $u\colon R \to \mathbb{R}_+$ the value of a maximum (s,t)-flow is equal to the capacity of a minimum (s,t)-cut:

$$\max_{f \text{ is a feasible } (s,t)\text{-flow in } G} \mathrm{val}(f) = \min_{(S,T) \text{ is an } (s,t)\text{-cut in } G} u\left(\delta^+(S)\right).$$

Notes

The max-flow-min-cut theorem

At the same time, we have proven the following theorem.

Theorem 4.22 (Augmenting path theorem)

A feasible (s,t)-flow f is a maximum flow if and only if there exists no flow augmenting path, in other words, there exists no path from s to t in the residual network G_f.

Notes

The algorithm of Ford and Fulkerson

Assumptions for this part

For the algorithmic part, we assume that G is connected and simple. The assumption that G is simple eases the notation, but again is no restriction: A bunch of parallel arcs $r_1, \dots, r_k$ can be merged to one single arc with capacity $\sum_{i=1}^{k} u(r_k)$.

Also the assumption of connectedness is not a limitation as we otherwise solve the problem in the component that contains s and t (if no such component exists, the maximum flow is obviously 0).

Notes

The algorithm of Ford and Fulkerson

Theorem 4.22 motivates an obvious idea for a simple algorithm to solve the maximum flow problem.

We start with the zero flow $f \equiv 0$ and as long as there are paths from s to t in G_f we augment f along such a path and update the residual network G_f.

If the capacities are integral, this means $u\colon R \to \mathbb{N}$, Algorithm 9 augments the flow in every step by an integer, since if the flow is integral, the residual capacities are integral too.

Moreover, the flow is increased at least by 1 unit. Let $U := \max\{u(r)\colon r \in R\}$. The cut $(s, V \setminus \{s\})$ has at most $n-1$ arcs with capacity at most U. Thus the cut has capacity at most $(n-1)U$.

Notes

The algorithm of Ford and Fulkerson

Algorithm 9: Generic algorithm based on flow augmenting paths

```
Ford-Fulkerson($G, u, s, t$)
Input  : A simple directed graph $G = (V, R)$, a non-negative capacity
         function $u\colon R \to \mathbb{R}$, two nodes $s, t \in V$.
Output: A maximum $(s, t)$-flow $f$ (for a “clever” choice of flow augmenting
         paths)
1 Set $f(r) = 0$ for all $r \in R$
2 while there exists a path from $s$ to $t$ in $G_f$ do
3     Choose such a path $P$
4     Set $\Delta := \min\{u_f(\sigma r)\colon \sigma r \in P\}$        // residual capacity of path $P$
5     Augment $f$ along $P$ by $\Delta$ units
6     Update $G_f$
```

Notes

The algorithm of Ford and Fulkerson

Example 4.23

$\rightarrow$ board.

Theorem 4.24

If all capacities are integral, Algorithm 9 terminates after $\mathcal{O}(nU)$ augmenting steps and $\mathcal{O}\left((m+n)nU\right)$ time with an integral maximum flow. Here, $U := \max\{u(r)\colon r \in R\}$ denotes the maximum capacity in the graph.

Example 4.25

$\rightarrow$ board.

Theorem 4.26 (Integrality theorem)

If all capacities are integral, there exists an integral maximum flow.

Notes

The algorithm of Ford and Fulkerson

Example 4.27

$\rightarrow$ board.

Theorem 4.28

If all capacities are rational numbers, Algorithm 9 terminates after a finite number of steps.

Proof.

$\rightarrow$ Multiply all capacities with the lowest common denominator $K \rightsquigarrow$ solve problem with integral capacities and divide by K. $\square$

Notes

The algorithm of Ford and Fulkerson

Theorem 4.29

For irrational capacities, it may happen that Algorithm 9 does not terminate after a finite number of steps and the "limit flow" (the limit of generated flows) is not even a maximum flow.

Proof.

$\rightarrow$ board. $\square$

Notes

It remains an open question in Algorithm 9 how to choose the flow augmenting path. For the previous theoretical results, it sufficed to use *any* flow augmenting path.

If the flow augmentation occurs along a shortest s-t-path in G_f in Algorithm 9, this leads to the following algorithm by Edmonds and Karp.

Notes

The algorithm of Edmonds and Karp

Algorithm 10: Algorithm of Edmonds and Karp

```
EDMONDS-KARP-MAXFLOW($G, u, s, t$)
Input  : A simple directed graph $G = (V, R)$, a non-negative capacity
         function $u\colon R \to \mathbb{R}$, two nodes $s, t \in V$.
Output: A maximum $(s, t)$-flow $f$.
1 Set $f(r) = 0$ for all $r \in R$
2 while there exists a path from $s$ to $t$ in $G_f$ do
3     Choose such a shortest path $P$
4     Set $\Delta := \min\{u_f(\sigma r)\colon \sigma r \in P\}$     // residual capacity of path $P$
5     Augment $f$ along $P$ by $\Delta$ units
6     Update $G_f$
```

Notes

The algorithm of Edmonds and Karp

Lemma 4.30

Let $G = (V, R)$ be a directed graph. We again denote by $dist(s, t, G)$ the length of a shortest path from s to t in G and by $R_{st}(G)$ the set of all arcs of G, that are part of shortest s-t-paths. Let $R_{st}(G)^{-1} := \{r^{-1}\colon r \in R_{st}(G)\}$, where r^{-1} denotes the inverse arc of r.

Then it holds for the graph G', that emerges from adding all arcs from $R_{st}(G)^{-1}$ to G:

$$dist(s, t, G') = dist(s, t, G)$$
$$R_{st}(G') = R_{st}(G)$$

Proof.

$\to$ board. □

Notes

The algorithm of Edmonds and Karp

Theorem 4.31

Let G be a network with integer, rational or real capacities. The algorithm of Edmonds and Karp terminates after $\mathcal{O}(nm)$ iterations with a maximum flow. The overall complexity of the algorithm is $\mathcal{O}(nm^2)$.

Proof.

$\rightarrow$ board. $\square$

Notes

Lower bounds and b-flows

In some applications it makes sense to introduce additional lower bounds $l\colon R \rightarrow \mathbb{R}_+$ in addition to the upper bounds $u\colon R \rightarrow \mathbb{R}_+$ on every arc $r \in R$.

Finding a maximum flow in this context means to find a maximum flow that also obeys the condition $l(r) \leq f(r) \leq u(r)$ on every arc in G.

Assumption

Let $G = (V, R, \alpha, \omega)$ be a finite directed graph, $l\colon R \rightarrow \mathbb{R}_+$ be a function that assigns a lower bound on the capacities to every arc $r \in R$ and $u\colon R \rightarrow \mathbb{R}_+$ be a function that assigns an upper bound on the capacities to every arc $r \in R$ with $0 \leq l(r) \leq u(r)$ for all $r \in R$.

Does such a flow always exist?

Notes

Lower bounds and b-flows

Example 4.32

$\rightarrow$ board.

We have defined the excess of a node $v \in V$ as

$$\mathsf{excess}_f(v) := f\left(\delta^-(v)\right) - f\left(\delta^+(v)\right)$$

for some flow function f.

For an (s,t)-flow we required the flow to obey the flow conservation constraint $\mathsf{excess}_f(v) = 0$ for all $v \in V \setminus \{s,t\}$. Thus it holds

$$\mathsf{excess}_f(v) = \begin{cases} -\mathsf{val}(f) & \text{if } v = s \\ \mathsf{val}(f) & \text{if } v = t \\ 0 & \text{else} \end{cases}.$$

Notes

Lower bounds and b-flows

Definition 4.33 (b-flow, circulation)

Let $b\colon V \to \mathbb{R}$ be a node labeling. A function $f\colon R \to \mathbb{R}$ is called *b-flow* in G if

$$\text{excess}_f(v) = b(v) \quad \text{for all } v \in V.$$

For the special case $b(v) = 0$ for all nodes $v \in V$, we call this 0-flow a *circulation*.

If l and u are capacity bounds as in the previous assumption, we say a b-flow f or a circulation β is feasible if $l(r) \leq f(r), \beta(r) \leq u(r)$ for all $r \in R$.

Notes

Lower bounds and b-flows

Remark 4.34

Every (s,t)-flow is a b-flow with $b(s) = -\text{val}(f)$, $b(t) = \text{val}(t)$ and $b(v) = 0$ for all $v \in V \setminus \{s,t\}$.

Furthermore, every function $h\colon R \to \mathbb{R}_+$ is a b-flow for $b(v) := \text{excess}_h(v)$.

A necessary condition for the existence of a b-flow is $\sum_{v \in V} b(v) = 0$, since for every function $f\colon R \to \mathbb{R}_+$ it holds that $\sum_{v \in V} \text{excess}_f(v) = 0$.

Since every circulation β in G is an (s,t)-flow for an arbitrary choice of $s, t \in V$ with flow value $\text{val}(\beta) = 0$, Lemma 4.8 yields the following corollary:

Corollary 4.35

For a circulation β in G and an (S,T) cut it holds that

$$\beta\left(\delta^+(v)\right) = \beta\left(\delta^-(v)\right).$$

Notes

Lower bounds and b-flows

We have seen that a flow with lower and upper bound does not necessarily exist. How can we decide whether such a flow exists, and, if one exists, calculate a feasible b-flow?

Let $G = (V, R, \alpha, \omega)$ be a directed graph with lower and upper bound $l, u\colon R \to \mathbb{R}_+$. We construct a super graph $G' = (V', R', \alpha', \omega')$ in the following way: we add two new nodes s' and t' to this graph.

There exists one arc from s' to every node $v \in V$ and one arc from every $v \in V$ to the new node t'.

$$\begin{aligned} V' &= V \cup \{s', t'\} \\ R' &= R \cup \{(s', v)\colon v \in V\} \cup \{(v, t')\colon v \in V\} \end{aligned}$$

Notes

Lower bounds and b-flows

We set the lower bound $l'(r') := 0$ for all arcs $r' \in R'$ and define the upper bound as follows:

$$\begin{aligned} u'(r) &:= u(r) - l(r) && \text{for all } r \in R \\ u'(s', v) &:= \sum_{r \in \delta^-(v)} l(r) = l\left(\delta^-(v)\right) && \text{for all } v \in V, \text{“minimum inflow”} \\ u'(v, t') &:= \sum_{r \in \delta^+(v)} l(r) = l\left(\delta^+(v)\right) && \text{for all } v \in V, \text{“minimum ouflow”} \end{aligned}$$

For a node $v \in V$ we denote by $\delta^+(v)$ the arcs emanating from v in G and by $\delta^+_{G'}(v) = \delta^+(v) \cup \{(v, t')\}$ the corresponding arcs in G'. We also use the corresponding notion of $\delta^-(v)$ for the incoming arcs.

Notes

Lower bounds and b-flows

Lemma 4.36

Let $f : R \to \mathbb{R}$ be an arbitrary function. Then it holds for every function $f' : R' \to \mathbb{R}$ with the property

$$\begin{aligned} f'(r) &= f(r) - l(r) && \text{for all } r \in R \\ f'(r') &= u'(r') && \text{for all } r' \in R' \setminus R \end{aligned}$$

that $f'(\delta^+_{G'}(v)) = f(\delta^+(v))$ and $f'(\delta^-_{G'}(v)) = f(\delta^-(v))$ for all $v \in V$.

Proof.

$\to$ board. $\square$

Notes

Lower bounds and b-flows

Theorem 4.37

The exists a feasible circulation in G with respect to the lower and upper bounds l and u if and only if the maximum (s', t')-flow in G' has the value $F := \sum_{r \in R} l(r)$.

Proof.

$\to$ board. $\square$

Corollary 4.38

Let G be as in the previous assumption. By computing a maximum flow we either compute a feasible circulation with respect to the lower and upper bounds l and u in G or determine that no such circulation exists.

If such a circulation exists and l and u are integral, the so found circulation is integral too.

Notes

Flow decomposition

Definition 4.39 (Path flow, circulation)

A path flow f along a simple path P in G (with flow value $\delta > 0$) is an $(\alpha(P), \omega(P))$-flow that vanishes on all arcs $r \notin R(P)$, that is so to say

$$f(r) = \begin{cases} \delta & \text{if } r \in R(P) \\ 0 & \text{otherwise.} \end{cases}$$

Analogously we define a circulation β along an elementary cycle C in G as

$$\beta(r) = \begin{cases} \delta & \text{if } r \in R(C) \\ 0 & \text{otherwise.} \end{cases}$$

Notes

Flow decomposition

Let $\mathcal{P}$ be the set of all simple paths in G that are no cycles and $\mathcal{C}$ be the set of all elementary cycles in G.

If f_P for some $P \in \mathcal{P}$ is a path flow and β_C for some $C \in \mathcal{C}$ is a circulation, we can construct an arc label f by

$$f(r) := \sum_{\substack{r \in P \\ P \in \mathcal{P}}} f_P(r) + \sum_{\substack{r \in C \\ C \in \mathcal{C}}} \beta_C(r).$$

It follows, that for all $v \in V$

$$\text{excess}_f(v) = \sum_{P \in \mathcal{P}} \text{excess}_{f_P}(v)$$

holds. This allows us to construct a b-flow from path flows and circulations.

The following theorem shows that the opposite is also true.

Notes

Flow decomposition

Theorem 4.40 (Flow decomposition theorem)

Every b-flow $f : R \to \mathbb{R}_+$ can be expressed as the sum of $m + n$ path flows and circulations with the following properties:

1. *For every path flow f_P P is a path from a node v with $b(v) < 0$ to a node u with $b(u) > 0$.*
2. *In the linear combination there occur at most m circulations.*

If f is integral, all path flows and circulations are also integral.

Proof.

$\rightarrow$ board. $\square$

Example 4.41

$\rightarrow$ board.

Notes

Flow decomposition

Remark 4.42

The method described in the proof of Theorem 4.40 can be implemented with a worst-case time complexity of $\mathcal{O}\left((n+m)^2\right)$. A path or a cycle can be found in $\mathcal{O}(m+n)$ time and there occur at most $n+m$ path flows and cycles.

Remark 4.43

The result further shows that $\mathcal{O}(n+m)$ flow augmenting paths are sufficient to construct a maximum flow. This is substantially less than the bound of $\mathcal{O}(nm)$ iterations in the algorithm of Edmonds and Karp.

However, it is not clear how to turn this into an efficient algorithm, since we already assumed both the knowledge of a maximum flow and that of a flow decomposition.

Notes

Min cost flows

Assumption for this part

In the following let $G = (V, R, \alpha, \omega)$ be a finite graph and l, u be functions that assign a lower and upper capacities to the arcs of G, where it holds that $0 \leq l(r) \leq u(r)$ for all $r \in R$. We also allow the case $u(r) = +\infty$ for $r \in R$. Let $b\colon V \to \mathbb{R}$ be a function that assigns the desired excess to the nodes. Additionally, let $c\colon R \to \mathbb{R}$ be a function that assigns flow costs to the arcs.

Definition 4.44 (Flow cost)

Let $c\colon R \to \mathbb{R}$ be an arc labeling that we call *cost function*. For some b-flow f the *flow costs* are given by

$$c(f) := \sum_{r \in R} c(r) \cdot f(r).$$

We extend the cost function c on the arcs in the residual network by $c(+r) := c(r)$ and $c(-r) := -c(r)$.

Notes

Min cost flows

Min cost flow problem

For a given graph G with capacities $l, u\colon R \to \mathbb{R}_+$, excess values $b\colon V \to \mathbb{R}$ and flow costs $c\colon R \to \mathbb{R}$, the problem of finding a feasible b-flow with minimum flow cost $c(f)$ is called *min cost flow problem*.

Notes

Min cost flows

Remark 4.45

In principle, we also allow negative values for the costs on the arcs $r \in R$.

Additionally, we allow infinite upper flow bounds on the arcs. This allows us to model the maximum flow problem as a special instance of a min cost flow problem.

Let G be a directed graph with lower and upper flow bounds l and u such that $0 \leq l(r) \leq u(r)$ for all $r \in R$ and $s, t \in V$ be the two nodes for those we want to compute a maximum (s, t)-flow. We extend G to G' by adding a new arc r_{ts} from t to s with cost $c(r_{ts}) := -1$ and capacities $l(r_{ts} = 0)$ and $u(r_{ts}) = +\infty$. For all other arcs $r \in R$, we set $c(r) = 0$, for all nodes $v \in V$ we set $b(v) = 0$.

An (s, t)-flow f in G with $\text{val}(f) = F$ exists if and only if G' contains a cost minimal b-flow with cost $-F$.

Notes

Min cost flows

For the calculation of maximum flows, we augmented an (s, t)-flow along a flow augmenting path P (that is a path in the residual network) by a value of $\delta > 0$.For all arcs $+r \in P$ we augmented the flow value by δ, for all arcs $-r \in P$ we reduced the flow value by δ.

For min cost flows we additionally need the augmentation of a b-flow along a cycle in G_f. Analogously we augment the flow along all arcs $+r \in C$ and reduce the flow along all arcs $-r \in C$. Obviously, the excess remains unchanged in all vertices.

Notes

Min cost flows

Let f be a b-flow and β_C be a circulation in G_f along an elementary cycle $C = (v_0, \sigma_1 r_1, v_1, \ldots, \sigma_k r_k, v_k = v_0)$ in G_f with flow value $\delta > 0$.

Then $f + \beta_C$, defined by

$$(f + \beta_C)(r) := \begin{cases} f(r) + \delta & \text{if } r = r_i \text{ and } \sigma_i = + \\ f(r) - \delta & \text{if } r = r_i \text{ and } \sigma_i = - \\ f(r) & \text{otherwise} \end{cases},$$

is again a b flow in G. The costs $c(f + \beta_C)$ are given by $c(f) + c(\beta_C)$.

With this notion we can formulate and proof the following theorem:

Notes

Min cost flows

Theorem 4.46

Let f and f' be feasible b-flows in G with respect to l and u. f' can be written as the sum of f and at most $2m$ circulations $\beta_{C_1}, \ldots, \beta_{C_p}$ in G_f. It holds that $c(f') = c(f) + \sum_{i=1}^{p} c(\beta_{C_i})$.

Proof.

$\rightarrow$ board. $\square$

Theorem 4.47 (Cycle criterion for cost minimal *b*-flow)

Let G be as in the assumption for this part. f is a cost minimal b-flow if and only if the residual network G_f does not contain a cycle of negative length with respect to the cost function c.

Proof.

$\rightarrow$ board. $\square$

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Notes

Min cost flows

Additional assumptions for the algorithmic part

In addition to the previous assumptions we demand the following three properties for the network flow problem:

1. There exists a feasible b-flow with respect to l and u in G. In particular, this means that $\sum_{v \in V} b(v) = 0$.
2. For all $u, v \in V$ with $u \neq v$ there exists a path from u to v in G that consists only of arcs with infinite capacity (therefore, this path also exist in G_f for an arbitrary $f\colon R \to \mathbb{R}_+$).
3. It is $l(r) = 0$ and $c(r) \geq 0$ for all $r \in R$.

None of these points is a loss of generality. In 1. we ensure that a feasible solution exists. This can be tested by the calculation of a maximum flow. 2. can be enforced by adding new arcs with infinite costs. None of these will be used in an optimal solution.

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Notes

Min cost flows

Theorem 4.47 motivates the following algorithm by Klein to compute min cost flows.

Algorithm 11: Algorithm of Klein to compute min cost flows

```
MinCostFlow-Klein($G, l, u, b, c$)
Input : A simple directed graph $G = (V, R, \alpha, \omega)$ with capacity functions
        $0 \leq l(r) \leq u(r)$ for all $r \in R$, desired excess values $b\colon V \to \mathbb{R}$ and
        flow costs $c\colon R \to \mathbb{R}_+$.
Output: A minimum cost $b$-flow $f$. If $l$, $u$ and $c$ are integral, so is $f$.
1 Compute a feasible $b$-flow $f$          // Can be done by computing a maximum flow
2 while the residual network $G_f$ contains a negative cycle $C$ do
3     Let $\Delta := \min_{\sigma r \in C} c(\sigma r)$ be the minimal residual capacity along $C$
4     Augment $f$ along $C$ by $\Delta$          // Eliminate the negative cycle $C$
5 return $f$
```

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Notes

Min cost flows

Example 4.48

$\to$ board

Theorem 4.49

If all excesses b, capacities u and costs c are integral, the algorithm of Klein terminates after $\mathcal{O}(mUC)$ iterations with an integral min cost b-flow. Here we denote by $U := \max\{u(r)\colon r \in R\}$ the maximum capacity and by $C := \max\{c(r)\colon r \in R\}$ the maximum cost value in G.

Proof.

$\to$ board. □

Corollary 4.50 (Integrality theorem for min cost flows)

If all input data are integral, there exists an integral min cost b-flow.

Notes

Min cost flows

Remark 4.51

Algorithm 11 does not specify how to find an initial start flow and how to find cycles with negative length in the residual network. One can show that cycles C with minimum average weight $c(C)/|C|$ lead to polynomial-time algorithms.

Notes

Min cost flows

Theorem 4.52 (Potential criterion for min cost b-flows)

Let G be as in the assumptions. f is a min cost b-flow if and only if there exists a potential in G_f with respect to the weights c. That means there exits a function $p\colon V \to \mathbb{R}$ with $p(v) \le c(\sigma r) + p(u)$ for all $\sigma r \in G_f$ with $u = \alpha(\sigma r)$ and $v = \omega(\sigma r)$.

Proof.

$\to$ board. □

An equivalent formulation can be stated with reduced costs.

Corollary 4.53 (Reduced cost criterion for min cost b-flows)

Let G be as in the assumptions. f is a min cost b-flow if and only if there exists a node labeling $p\colon V \to \mathbb{R}$ with $c^p(\sigma r) \ge 0$ for all $\sigma r \in G_f$.

Notes

Min cost flows

Definition 4.54 (Pseudo flow)

Let G be as in the assumptions and $l \equiv 0$. A feasible *pseudo flow* in G is a function $f\colon R \to \mathbb{R}$ with $0 \leq f(r) \leq u(r)$ for all $r \in R$.

For a pseudo flow we define the *imbalance* of a node $v \in V$ by

$$\mathsf{imbal}_f(v) := \mathsf{excess}_f(v) - b(v).$$

If $\mathsf{imbal}_f(v) > 0$ we call v a *surplus node.* If $\mathsf{imbal}_f(v) < 0$ we call v a *deficit node.* A node $v \in V$ with $\mathsf{imbal}_f(v) = 0$ is called *satisfied.*

Notes

Min cost flows

For a pseudo flow f we denote by S_f and D_f the set of surplus or deficit nodes. It is

$$\sum_{v \in V} \mathsf{imbal}_f(v) = \underbrace{\sum_{v \in V} \mathsf{excess}_f(v)}_{=0 \text{ (Lemma)}} - \underbrace{\sum_{v \in V} b(v)}_{=0 \text{ by assumption}} = 0,$$

therfore

$$\sum_{v \in S_f} \mathsf{imbal}_f(v) = - \sum_{v \in D_f} \mathsf{imbal}_f(v). \tag{5}$$

(5) yields the following handy observation.

Notes

Min cost flows

Observation 4.55

For a pseudo flow f, the set of surplus nodes S_f is empty if and only if the set of deficit nodes D_f is empty.

Notes

Min cost flows

Algorithm 12: Successive shortest path algorithm

```
SUCCESSIVE-SHORTEST-PATH$(G, u, b, c)$

Input  : A directed graph as in the assumptions.
Output: A minimum cost $b$-flow $f$. If $l$, $u$ and $c$ are integral, so is $f$.

Set $f(r) := 0$ for all $r \in R$ and $p(v) := 0$ for all $v \in V$
Set $\mathrm{imbal}_f(v) := -b(v)$ for all $v \in V$
Compute the set of surplus and deficit nodes
        $S_f = \{v \in V\colon \mathrm{imbal}_f(v) > 0\}$
        $D_f = \{v \in V\colon \mathrm{imbal}_f(v) < 0\}$
while $S_f \neq \emptyset$ do
    Choose a node $s \in S_f$ and a node $t \in D_f$
    Compute the distances $d(v) = \mathrm{dist}_{c^p}(s, f, G_f)$ from $s$ to all other nodes in $G_f$
      with respect to the reduced costs $c^p$
    Let $P$ be a shortest $s$-$t$-path
    Set $\Delta := \min\{u_f(\sigma r)\colon \sigma r \in P\}$
    Update $p := p + d$
    $\varepsilon := \min\{\mathrm{imbal}_f(s), -\mathrm{imbal}_f(t), \Delta\}$
    Augment $f$ along $P$ by $\Delta$
    Update $f$, $G_f$, $S_f$ and $D_f$
return $f$
```

Notes

Min cost flows

Notes

Dynamic network flow problems

Outline

Notes

This section is still under construction.

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