Special Topics of Applied Mathematics • SS 2017

Exercise Sheet 5

To be done in the tutorial on 14.06.2017

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Exercise 5.1

Consider finding a compromise solution by maximizing the distance to the nadir point.

a) Let $\|\cdot\|$ be a norm. Show that an optimal solution of the problem

$$\begin{aligned} \max_{x\in X} \quad & \left\| f(x) - y^N \right\| \\ \text{s. t.} \quad & f_k(x) \le y_k^N \text{ for } k = 1, \dots, p \end{aligned} \tag{1}$$

is weakly efficient. Give a condition under which an optimal solution of (1) is efficient.

b) Another probability is to solve

$$\begin{aligned} \max_{x\in X} \quad & \min_{k=1,\dots,p} \left| f_k(x) - y_k^N \right| \\ \text{s. t.} \quad & f_k(x) \le y_k^N \text{ for } k = 1, \dots, p \end{aligned} \tag{2}$$

Prove that an optimal solution of (2) is weakly efficient.

Exercise 5.2

Consider three points in the Euclidean plane, $x^1 = (1,1)^T$, $x^2 = (1,4)^T$ and $x^3 = (4,4)^T$. The l_2^2-location problem is to find a point $x = (x_1, x_2) \in \mathbb{R}^2$ such that the sum of weighted squared distances from x to the three points x^i, $i = 1,2,3$ is minimal. We consider a bicriterion l_2^2-location problem, i.e. two weights for each of the points x^i are given through two weight vectors $w^1 = (1,1,1)$ and $w^2 = (2,1,4)$. The two objectives measuring weighted distances are given by

$$f_k(x) = \sum_{i=1}^{3} w_i^k \left((x_1^i - x_1)^2 + (x_2^i - x_2)^2 \right) \qquad k = 1,2$$

Use Definition 1 and Theorem 2 (you do not need to prove correctness of Theorem 2!) to check the points $x^1 = (2,2)$ and $x^2 = (2,3)$ for Pareto optimality.

Definition 1: Let $X \subseteq \mathbb{R}^n$, $f\colon X \to \mathbb{R}$ and $\hat{x} \in X$.

$$\mathcal{L}_{\le}(f(\hat{x})) := \{x \in X\colon f(x) \le f(\hat{x})\}$$

is called the *level set* of f at $\hat{x}$ and

$$\mathcal{L}_{=}(f(\hat{x})) := \{x \in X\colon f(x) = f(\hat{x})\}$$

is called the *level curve* of f at $\hat{x}$.

Theorem 2: Let $\hat{x} \in X$ be a feasible solution of a multiobjective optimization problem with $p \geq 2$ objective functions and define $\hat{y}_k \coloneqq f_k(\hat{x})$, $k = 1, \ldots, p$. Then $\hat{x}$ is efficient if and only if

$$\bigcap_{k=1}^{p} \mathcal{L}_{\leq}(\hat{y}_k) = \bigcap_{k=1}^{p} \mathcal{L}_{=}(\hat{y}_k).$$

Exercise 5.3

Let

$$\begin{aligned} X &\coloneqq \left\{(x_1, x_2) \in \mathbb{R}\colon -x_1 \leq 0, \quad -x_2 \leq 0, \quad (x_1 - 1)^3 + x_2 \leq 0\right\}, \\ f_1(x) &= -3x_1 - 2x_2 + 3 \text{ and} \\ f_2(x) &= -x_1 - 3x_2 + 1. \end{aligned}$$

Graph X and $Y = f(X)$. Show that $\hat{x} = (1, 0)$ is properly efficient.

Exercise 5.4

In optimization, totally unimodular matrices (see Definition 3 below) are of great interest, since the give a quick way to verify that a linear program is integral (has an integral optimum, when any optimum exists). Specifically, if A is TU and the right hand side b is integral, then linear programs of forms like $\{\min c^T x \mid Ax \leq b, x \geq 0\}$ or $\{\max c^T x \mid Ax \leq b\}$ have integral optima, for any $c \in \mathbb{R}^n$. Hence if A is totally unimodular and b is integral, every extreme point of the feasible region (e.g. $\{x \in \mathbb{R}^n \mid Ax \geq b\}$) is integral and thus the feasible region is an integral polyhedron.

Definition 3 (Total unimodularity): A matrix $A \in \mathbb{R}^{m \times n}$ is *totally unimodular* (TU), if every square submatrix of A has a determinant of $0, +1$ or -1.

Consider the bicriterion linear integer problem

$$\begin{aligned} \min \quad & \begin{matrix} c^T x \\ e^T x \end{matrix} \\ \text{s. t.} \quad & Ax \leq b \end{aligned} \tag{3}$$

with a TU matrix A given in (5) and $e = (1, 1, \ldots, 1) \in \mathbb{R}^n$ the vector of all ones. Show that the constraint matrix $\tilde{A}$ of the scalarized problem (4)

$$\begin{aligned} \min \quad & c^T x \\ \text{s. t.} \quad & ex \leq \varepsilon \\ & Ax \leq b \end{aligned} \tag{4}$$

is not TU.

Use the following totally unimodular matrix for your thoughts

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}. \tag{5}$$

Exercise 5.5 – Presentation Only

Consider the biobjective nonlinear optimization problem

$$\min_{x \in \mathbb{R}^2} \quad \begin{pmatrix} x_1^2 + x_2^3 - 3x_1x_2 + x_1 \\ x_1^3 - 3x_1x_2 + \frac{1}{2}x_2^2 \end{pmatrix}.$$

Determine the ideal and nadir point.

Hint: To determine the (local) minima of a function $f\colon \mathbb{R}^n \to \mathbb{R}$, $\nabla f = 0$ (the gradient of f is zero) is a necessary condition and the positive definiteness of the Hessian matrix H_f in the critical points is a sufficient condition.