Self-Assessment for the Master's Program Mathematical Modelling, Simulation, and Optimization (MMSO)

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This self-assessment test is considered as a guideline for applicants to our program to identify whether their mathematical background is rich enough for successfully studying our program.

This document collects some typical questions and exercises from introductory undergraduate mathematical courses. We advise you to carefully work through these exercises to identify any weak point in your mathematical background. We do not presume that any question is obvious or easy for you, but we expect that you understand the questions and that you are able to either correctly answer the exercises or you are able to identify and find solutions for your questions that arise during working on the exercises.

If you feel that you have large gaps in your knowledge, you carefully should think about, whether you will be able to fill those gaps prior to entering our Master's program. If not, we recommend to reconsider your decision of choosing our master program.

1 Analysis

Basic Calculus

1.1 Solve the following (in–)equalities for $x \in \mathbb{R}$

$$\sqrt{x+1} = \sqrt{x-1}, \quad |x+1| < |2-x|\,, \quad \frac{x}{x^2-1} - \frac{1}{x-1} = 1.$$

1.2 Compute the real and imaginary part as well as the absolute value of the complex numbers

$$z_1 := \frac{1}{1-i\sqrt{3}} \quad \text{and} \quad z_2 = (1+i)^3.$$

1.3 Prove by induction:
For all $n \in \mathbb{N}$ it holds that $3^n - 3$ is a multiple of 6.

1.4 Let Ω be an arbitrary set and $A, B \subset \Omega$. Prove or disprove (by providing a counter–example), that

$$\overline{A \cup B} = \bar{A} \cup \bar{B} \,,$$

where $\bar{A} = \Omega \setminus A$ denotes the complement of A.

1.5 Let $(a_n)_{n\in\mathbb{N}}$ be an unbounded sequence of real numbers. Show, that the sequence $(1/a_n)_{n\in\mathbb{N}}$ converges to zero.

1.6 Compute the limit of the series $(a_n)_{n\in\mathbb{N}}$ defined by $a_0 = 1$, $a_{n+1} := \sqrt{1 + a_n}$.

1.7 Check the convergence of the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

You need not to compute the value of the series.

One Dimensional Analysis

1.8 Is the function $f : \mathbb{R} \to \mathbb{R}_+$, $f(x) := |x| - x$ injective? Is f surjective?

1.9 Determine the constants $a, b \in \mathbb{R}$ such that the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) := \begin{cases} \sin(x) + 1 & \text{for } x < 0, \\ ax + b & \text{for } 0 \le x \le 1, \\ (x-1)^2 & \text{for } x > 1 \end{cases}$$

is continuous on entire $\mathbb{R}$. Is this f also continuously differentiable?

1.10 Determine all $x \in \mathbb{R}$ such that the power series

$$\sum_{k=0}^{\infty} (x-2)^k$$

converges.

1.11 Compute the following limits

$$\lim_{x\to\infty} 2^{-x} \sin x, \quad \lim_{x\to 1} \frac{e^x - 1}{x^2 - 1}, \quad \lim_{x\to 0} x \sin(1/x)\,.$$

1.12 Compute the derivatives of

$$f(x) := \frac{\arctan(x)}{x}, \quad g(x) := e^{\sin x}, \quad h(x) := (\sqrt{x})^x.$$

1.13 Compute the roots, extrema, and turning points of the function

$$f(x) := e^{2x} - 5e^x + 6.$$

1.14 Determine the Taylor polynomial of order $k = 3$ for $f(x) = \frac{1}{x+1}$ at $x_0 = 2$.

1.15 Show that for all $n \in \mathbb{N}$, $n \geq 2$ the recursion

$$\int_{-\pi/2}^{\pi/2} (\cos x)^n \, dx = \frac{n-1}{n} \int_{-\pi/2}^{\pi/2} (\cos x)^{n-2} \, dx$$

holds.

1.16 Prove or disprove (by a counter–example):
For $a, b \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ continuous, it holds that

$$\int_a^b f(x)g(x) \, dx = \int_a^b f(x) \, dx \int_a^b g(x) \, dx$$

1.17 Compute the (indefinite) integrals

$$\int_0^\infty x \cdot e^{x^2} \, dx, \quad \int (x^2 - x + 1)e^{2x} \, dx, \quad \int x \sin x \, dx.$$

Multi Dimensional Analysis

1.18 Is the function

$$g : \mathbb{R}^2 \to \mathbb{R}, \quad g(x) := \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2} & \text{for } (x_1, x_2) \neq (0,0) \\ 0 & \text{for } (x_1, x_2) = (0,0) \end{cases}$$

continuous? Is it continuously differentiable?

1.19 Compute the extrema of $m_p : \mathbb{R}^2 \to \mathbb{R}$, $m_p(x, y) := x^3 + y^3 - p(xy + 5p)$ depending on the parameter $p \in \mathbb{R}$.

1.20 Consider $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x^2 \sin(y) \log(z^2 + 1)$.
Calculate the gradient ∇ and the Laplacian Δ ($\equiv \nabla^2$) of f.

1.21 Consider $v : \mathbb{R}^3 \to \mathbb{R}^3$. $v(x, y, z) = (x^2, \sin(y), \log(z^2 + 1))^T$.
Calculate the divergence div and rotation rot ($\equiv$ curl) of v.

1.22 Determine the Taylor polynomial of order $k = 2$ for $g(x, y) = e^x \sin y$ at $(x_0, y_0) = (0, 0)$.

1.23 The set of all $x \in \mathbb{R}^2$ with polar coordinates (r, ϕ) given by $r = 1 + \cos \phi$, $0 \leq \phi \leq 2\pi$ defines a planar curve $C \subset \mathbb{R}^2$ called cardioid. Draw a sketch of this curve and determine the curve integral

$$\int_C \sqrt[4]{\|x\|_2} \cdot dx$$

Here $\|\cdot\|_2$ denotes the Euclidean norm of $x \in \mathbb{R}^2$.

1.24 Consider the following pseudocode for bisection to find a root of a given function f
Require: $f(a)f(b) < 0$

```
function BISECTION(f, a, b, ϵ)
    repeat
        c ← ½(a + b)
        if f(a)f(c) < 0 then
            b ← c
        else
            a ← c
        end if
    until |b − a| ≤ ϵ
    return ½(a + b).
end function
```

What does this code do in every step? What is the meaning of the stopping criterion? Why do we state the requirement on the initial data? What is the meaning of the return value?

1.25 Consider the following pseudocode

Require: $f'(x) \neq 0$
Require: $s > 0$

```
function MYFUNCTION(f, x, s, ϵ)
    repeat
        x ← x − s f'(x)^{-1} f(x)
    until |f(x)| ≤ ϵ
    return x
end function
```

Which algorithm is stated? What does this code do in every step? Why do we state the requirement on the initial data x? What is the meaning of the stopping criterion? What is the meaning of the return value?

2 Linear Algebra

Vector spaces

2.1 Let V denote some given vector space over $\mathbb{R}$. What are the defining properties of V, i.e. for any $u, v \in V$ and $\alpha \in \mathbb{R}$, which operations can you perform with these objects?

2.2 Let $\{u, v, w\}$ denote a set of vectors $u, v, w \in V$. What does it mean, that the set $\{u, v, w\}$ is linearly independent?

2.3 Let $\{u, v, w\}$ denote a set of vectors $u, v, w \in V$ and let $x \in V$. What does it mean, that $x \in \operatorname{span}\{u, v, w\}$?

2.4 Given $u = (1, 2, 1, 2)^T$, $v = (1, 0, 1, 0)^T$ and $w = (0, 1, 1, 0)^T$. Show that the set $\{u, v, w\}$ is linearly independent.

2.5 Given $u = (1, 2, 1, 2)^T$, $v = (2, 3, 6, 7)^T$, $w = (0, 2, 0, 2)^T$ $x = (3, 3, 7, 7)^T$. Is the set $\{u, v, w, x\}$ linearly independent? If not, choose a linearly independent subset.

2.6 Let $C([0,1])$ denote the set of all continuous functions $f : [0,1] \to \mathbb{R}$. Show that $C([0,1])$ with the classical summation of functions and the multiplication with real scalars forms a real vector space.

2.7 Given $f_1, f_2, f_3 \in C([0,1])$ with $f_1(x) = x$, $f_2(x) = \sin(\pi x)$, and $f_3(x) = \exp(x)$. Show that the set $\{f_1, f_2, f_3\}$ is linearly independent.

Linear mappings

2.8 What does it mean, that a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ for any $n, m \in \mathbb{N}$ is linear?

2.9 Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x,y) = \begin{pmatrix} x+y \\ xy \\ x-4y \end{pmatrix}$. Is f a linear mapping?

2.10 Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x,y,z) = \begin{pmatrix} 3x - y + z \\ 5x + 3z \end{pmatrix}$.
Give the matrix representation of f.

2.11 Let $A, B \in \mathbb{R}^{n \times n}$ denote two matrices. Does $AB = BA$ hold?

2.12 Given $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Calculate the inverse of A.

2.13 Let $A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ and $b = (1, 0, -2)^T$. Find $x \in \mathbb{R}^3$ such that $Ax = b$.

2.14 Let $A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$. Calculate the determinant $\det(A)$.

Eigenvalues

2.15 Let $A \in \mathbb{R}^{n \times n}$ for arbitrary $n \in \mathbb{N}$. What does it mean that $\lambda \in \mathbb{R}$ is an eigenvalue of A? What does it mean, that $v \in \mathbb{R}^n$ is an eigenvector of A?

2.16 Let $A = \begin{pmatrix} -1 & -3 \\ 2 & 4 \end{pmatrix}$ Calculate all eigenvalues and eigenvectors of A.

2.17 What does it mean, that a bilinear mapping $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric positive definite?

2.18 Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite.
What does this mean for A's eigenvalues?

2.19 Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite and $B \in \mathbb{R}^{n \times n}$ denote a regular matrix. Show that $B^T AB$ is also symmetric positive definite.

Orthogonality

2.20 Let $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denote an inner product. What are its properties?

2.21 Let $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ denote a norm. What are its properties?

2.22 Let $u, v \in \mathbb{R}^n$. What does it mean that they are orthogonal?

2.23 Given $u = (2, 3, 4)^T$ and $v = (1, -2, 1)^T$. Show that u and v are orthogonal.

2.24 How can you use the inner product in $\mathbb{R}^n$ to measure the angle between any two vectors?

2.25 Given $u = (2, 3, 4)^T$ and $v = (1, 1, 1)^T$. Calculate the angle $\measuredangle(u, v)$.

2.26 Let $u = (1, 2, 1, 2)^T$, $v = (1, 0, 1, 0)^T$ and $w = (0, 1, 1, 0)^T$. Consider $V = \text{span}\{u, v, w\}$. Calculate an orthogonal basis for V.

2.27 Let $u = (1, 0, 1)^T$ and $v = (0, 1, 1)^T$ be given. Calculate the cross-product $u \times v$. Give a geometric interpretation of the resulting vector and its length. A sketch might help.

2.28 Consider the following pseudocode for the power iteration

Require: $A \in \mathbb{R}^{n \times n}$ is positive definite

function PowerIteration(A, x, N)
    for $n = 1, ..., N$ **do**
        $x \leftarrow Ax$
        $x \leftarrow \frac{x}{\|x\|}$
    end for
    return x
end function

What does this code do in every step? What is the stopping criterion? Think about other stopping criteria. What does the return value approximate? Let the initial x denote an eigenvector to a real eigenvalue of A. Which sequence (x_n) is generated in this case?