Texture identification using image neighborhood hypergraphs

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ABSTRACT

Late developments in image processing have begun to show the promises of discrete formalizations of the digital image and their study from the point of view of combinatorics. In this paper we focus on the modeling of digital image by the means of hypergraph. Such a modeling has already been successfully applied to the problems of segmentation, edge detection, and noise detection. In that work we will focus on the number of possible configurations, one-to-one maps and equivalence classes up to those maps that it is possible to build for a given spatial threshold on the different grids used in image processing. Some results will be presented for texture classification.

Keywords: graph theory-based image modelling, image neighborhood hypergraph, texture classification

1. INTRODUCTION

Graph theory has been applied in many fields of computer vision, especially for segmentation[1] as well as for noise detection[1] or edge detection.[2] The use of techniques arising from hypergraph representation of digital image is also considered promising.[2] In this paper we apply that approach to texture discrimination which has a field intensively studied since many years by numerous researchers.[3] After a presentation of some necessary background concerning hypergraph theory in section 2, we concentrate on the theoretical aspects of texture classification (section 3), and give first experimental results in section 4.

2. THEORETICAL ASPECTS

A *digital image* $\mathcal{I}$ is a map from a subset X (generally finite) from Z^2 in a subset $\mathcal{C}$ of Z^n. Elements of X are called *points*, those of $\mathcal{C}$ *colors*. A couple $(x, I(x))$ where x belongs to X is called a *pixel*. However, the confusion between a point x and $(x, I(x))$ is often made and we will not depart from it since it keeps its meaningfulness. A *tiling* (or a *tessellation*) of R^2 is a partition of R^2. The tilings generally studied are constrained by a limited number of geometric configurations called *tessels*. Given a tiling, the choice of an arbitrary point in the tessels, and the fact to link two points if the tessels share a common side allows to build a *mesh*. For the case of *regular tilings* where the tessels are regular polygons, the center of gravity of the polygon is often chosen and therefore leads to a *regular mesh*. In image processing, three types of meshes are used: *hexagonal*, *triangular* and *square* ones. Because of the current technological devices and the natural data structuring, the last type is the most used and we will therefore restrict our study to this kind of mesh. If a distance on Z^2 defines a undirected, simple, loopless and regular graph on a mesh, that distance will be called a *grid distance*. A *grid* is then a nonempty set of Z^2 with an associated grid distance. On square grids, two distance are mainly used: the *city block* (or square) *distance* for which a given pixel has four neighbors and the *chessboard* (or diamond) *distance* for which a given pixel has eight ones. This paper will only deal with the chessboard distance.

A *hypergraph* H^4 on a set X is an ordered pair (X, E) where E is a set of nonempty subsets of X such that $\bigcup_{e \in E} e = X$. Elements of X are called *vertices*, those of E *hyperedges*. The *size* of a hypergraph is the cardinality of its set of vertices. To any graph may be associated its neighborhood hypergraph defined by: $\big(X, (\{x\} \cup \Gamma(x))_{x \in X}\big)$. Moreover, we will say that the hyperedge $\{x\} \cup \Gamma(x)$ is *generated* by the vertex x and x will be called the *center* of $\{x\} \cup \Gamma(x)$.

Let d be a distance on the set of colors $\mathcal{C}$ and d' be a distance that defines a grid on $X \subset Z^2$. Let α and β be two strict positive reals. A unique neighborhood $\Gamma_{\alpha,\beta}(x)$ for the digital image $\mathcal{I}$ may be associated to any pixel x of X by:

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$$\Gamma_{\alpha,\beta}(x) = \Big\{ y \in X, \quad y \neq x \quad \text{such as} \quad \begin{array}{l} d\left(\mathcal{I}(x), \mathcal{I}(y)\right) < \alpha \\ \text{and} \\ d'\left(x, y\right) < \beta \Big\} \end{array} \tag{1}$$

We will call d' *grid distance* and d *colorimetric distance*. The definition of colorimetric distances highly correlated with the perception of the human brain is always nowadays an extremely delicate domain of study of the neuro-scientific field. However we will assume that such functions exist even if the ones we will use may not appear appropriate with our sensations at the sight of an image. Looking at the previous definition, it appears that the first part of it defines a neighborhood in the space $\mathcal{C}$ whereas the second one only involves the spatial domain. We will therefore respectively call α and β, *colorimetric threshold* and *spatial threshold*. Moreover we will qualify $\Gamma_{\alpha,\beta}(x)$ of *spatiocolorimetric neighborhood*. That notion allows to describe some consistency or homogeneity of a pixel with its environment.

It is also interesting to see that such a neighborhood has a useful property: it increases in the sense of inclusion both with α and β. For instance, if we chose two colorimetric thresholds α and α' such as $\alpha < \alpha'$, for a fixed β, $\Gamma_{\alpha,\beta}(x) \subseteq \Gamma_{\alpha',\beta}(x)$. That is due to the fact that if a pixel y is, from the color point of view, close of x at α, it remains close to x at α'. This argument is the same if we chose the spatial thresholds and it is possible to show it also for them. The main interest of that property is that it implies a certain regularity in the modeling that we will now present.

It is now possible to define[1,2] an image neighborhood hypergraph (INH) $\mathcal{H}_{\alpha,\beta}$ on X by:

$$\mathcal{H}_{\alpha,\beta} = \left(X, \left(\{x\} \cup \Gamma_{\alpha,\beta}(x)\right)_{x \in X}\right) \tag{2}$$

It is useful to precise that such a definition is correct since any hyperedge $\Gamma_{\alpha,\beta}(x)$ is nonempty: it contains at least the pixel x. Moreover the union of all the hyperedges is X itself. We will call $\Gamma_{\alpha,\beta}(x)$ hyperedge *centered* in (or *generated* by) the pixel x. As it directly inherits from the spatiocolorimetric neighborhood definition, an INH also increases with α and β in the sense of the hyperedges inclusion. The figure 1 shows the part of an INH for a spatial threshold β of 1 for the chessboard distance. For this figure we did not precise the colorimetric distance and its associated threshold as that is not relevant.

3. DISCRETE TRANSFORMATIONS

3.1. Exact transformations

We will now focus on the transformations of the discrete plane. As we are more particularly interested in the possible transformations of the hyperedges, we will consider those that are bijective and leave the center of a hyperedge invariant. Obviously, for a spatial neighborhood of size β, the number of possible different configurations for the hyperedges is $2^{(2\beta+1)^2-1}$ i.e. nothing but the number of possible choices in a set of $(2\beta+1)^2 - 1$ elements: the center is assumed to be left invariant and is therefore the cause of the -1 term. Moreover we want the transformation affecting a hyperedge to leave the geometric structure of the hyperedge unchanged. More precisely we will therefore only consider isometries i.e. transformations that leave the relative position of one vertex to another unchanged. We should therefore consider two types of mapping: the rotations and the symmetries.

Let us first remark that as the center of a hyperedge is left unchanged, this is also the center of the rotation. As the rotation is an isometry, it is also a bijective map. Checking of a hyperedge is the image of another one by a rotation involves therefore that the number of vertices in each hyperedge is the same. Nevertheless if we consider the possible number of configurations for hyperedges of size k, this obviously leads to a combinatorial explosion as the number of hyperedges of size k in a spatial neighborhood of size β is $\binom{k}{\beta}$. We can however use the fact that as a rotation is an isometry the distance of each vertex to the center should be left unchanged. This

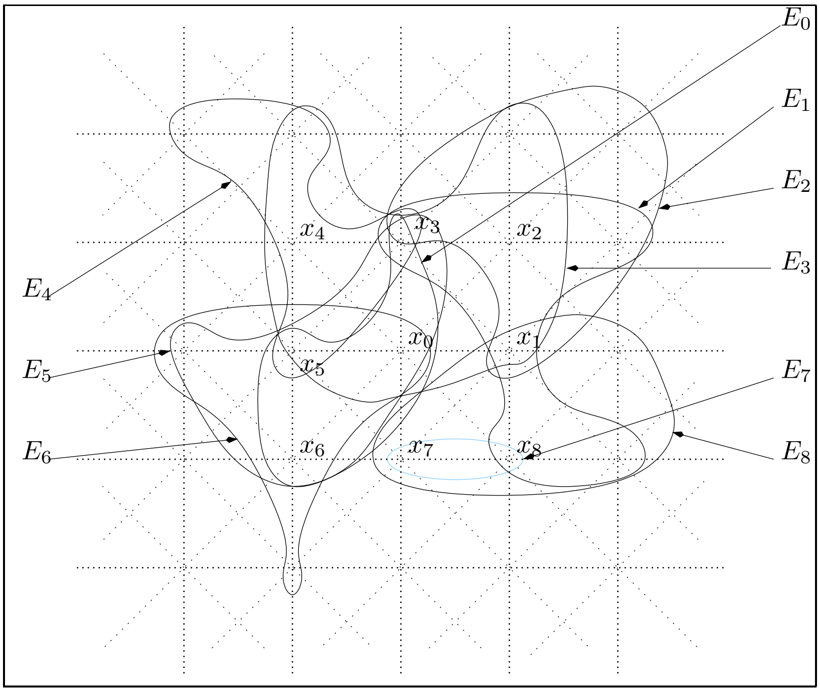

Figure 1. A part of an image neighborhood hypergraph. Each vertex $x_{i i\in\{0..8\}}$ generates the hyperedge E_i for vertices at distance at most 1 from it for the chessboard distance. The colorimetric distance is not precised here.

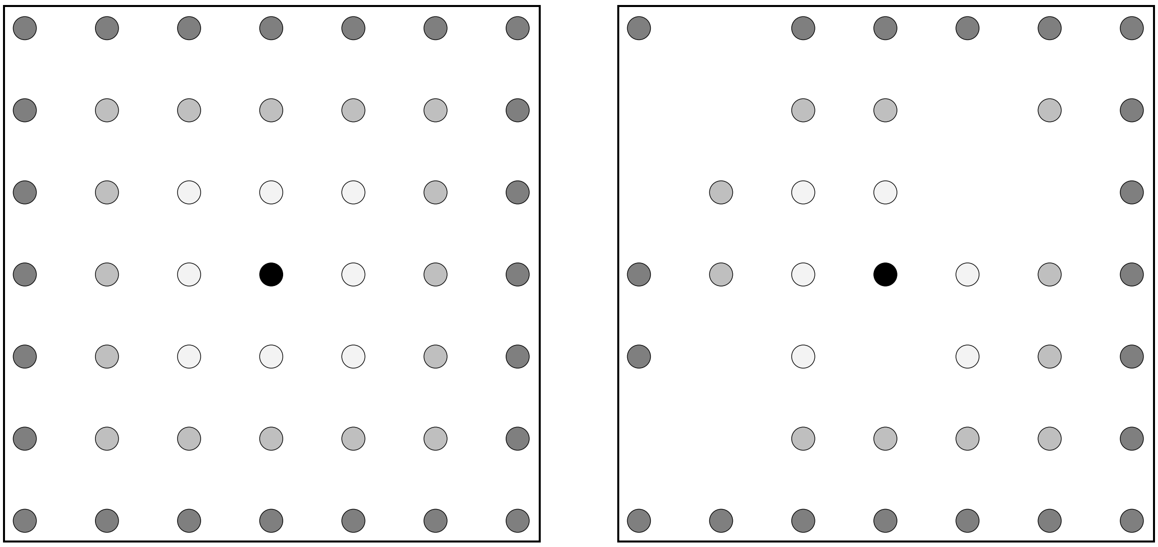

Figure 2. Circles around a hyperedge center. If a hyperedge is the image of an other by a rotation, necessarily their vertices must lie in the same circle around the center of the hyperedge. On the right figure, a hyperedge is shown: it has 20 vertices in its circle of radius 3, 11 in the circle of radius 2, and 6 in radius 1. The circle of radius 0 is not considered as it has always 1 element, the center of the hyperedge and is therefore not useful for our study.

corresponds to check if the discrete circles with increasing radii from the center in each hyperedge are image the one with the other as it can be seen in the figure 2.

To perform that task of finding a possible rotation relationship between two hyperedges efficiently, our idea is the following. We firstly split each hyperedge into its set of circles. We check if the number of vertices in each circle is the same for the two hyperedges. If that first condition holds, then the hyperedges are possibly related by a rotation. We then sort the sets constituted by the circles by increasing number of vertices for a corresponding radii and, in case of equality of the number of that cardinality, we use an increasing radius. We have then to check in each of those sets if the vertices are images by a rotation.

As in each circle there is a limited number of points, it is possible to consider a finite number of rotations. The smallest possible angle θ_r which has a strictly positive measure in the circle of radius r is, as the number of points in such a circle is $(2r+1)^2 - (2(r-1)+1)^2 = 2r$, $\theta_r = \frac{2\pi}{2r}$. Any rotation in that circle is therefore nothing but a rotation whose angle is $k\theta_r$ with k varying between 0 (for the identity) and $2r-1$. Nevertheless, because of the approximation that is made in considering transformation of discrete points by a continuous map, we will not try to estimate during that step the possible rotation angle: a more precise definition of the circles should be given, taking into account the deformation of the grid and the remapping of the data onto it. We will consider that vertices can be seen as successors one from another in the discrete circle as it can be seen in the figure 3. Taking an arbitrary orientation we can define a successor of any vertex in the discrete circle according to the rotation θ_r. Note that composing k times that relationship allow to define the rotation of angle $k\theta_r$. From the computational point of view that becomes interesting since it is possible to build up a table containing all the possible rotations all the circles. That table requires only $2r \times 4r = 8r^2$ to store the circle of size r and therefore $\sum_{r=1}^{\beta} 8r^2 = \frac{4}{3}\beta(\beta-1)(2\beta-1)$ for a hypergraph built with a spatial restriction of β but giving access to possible rotations in a constant time. Moreover inverting the orientation leads to the definition of images of a circle, therefore of hyperedges by a symmetry for only a doubling of the memory cost.

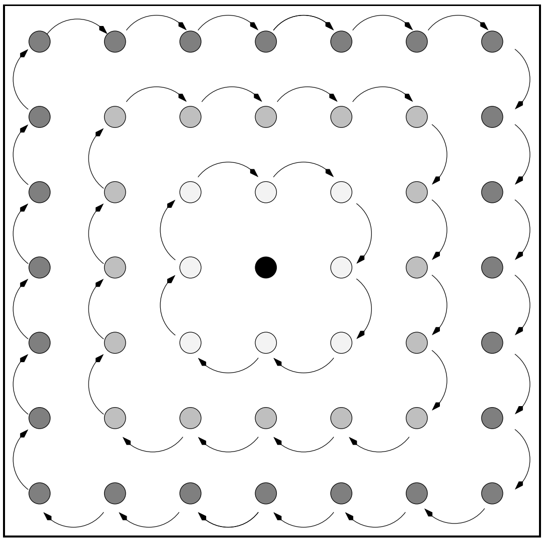

Figure 3. Successors of vertices in each discrete circle around the center of a hyperedge. Composing that relationship allow to define all the possible rotation for a given circle. Inverting that relationship allows to deal with symmetries.

Having defined the rotation R between two hyperedges, we will now consider the following relation $\mathcal{R}$ on the set of hyperedges $\mathcal{E}$:

$$\forall e_1, e_2 \in \mathcal{E} \quad e_1 \mathcal{R} e_2 \Leftrightarrow \exists R \quad e_2 = R(e_1)$$

$\mathcal{R}$ is obviously an equivalence relationship. It is therefore possible to build up the quotient set of $\mathcal{E}$ by $\mathcal{R}$. We will call class representatives of the quotient set *meta-hyperedges*. That set of meta-hyperedges is progressively constructed during a run of the algorithm since it would require $O(2^{((2\beta+1)^2)})$ to build all the possible configurations and to rely them to the appropriate meta-hyperedge.

3.2. Approximating the quotient set

Unfortunately, from the computational point of view the cost of such operations, that is, looking for the image by an exact rotation of one hyperedge into another remains prohibitive. We propose in the following a rough approximation of such a mapping. Taking into account all the possible rotations will, as far as we could see, lead to a combinatorial explosion. The basic idea to solve that problem relies on checking how the hyperedges may check together. Firstly, the vertices constituting a hyperedge may be sorted into circles as we showed it in the previous section. To improve the speed of our method, we propose to rearrange the bits in each circle in such a way that one of the longest chains in the circle comes in front of the sequence defining the circles of the hyperedge. Therefore it becomes easier from the computational point of view to look for sequences of circles in hyperedges that have been already considered as representatives of a class or not.

4. EXPERIMENTS

In that series of experiment we show the result of the classification into approximated meta-hyperedges for different limitation sizes β for the image *Lena* which is often used in image processing (see figure 4). We also show how the classification is influenced by different values of the colorimetric criterion (figure 5): here we choose the L_1 norm in the RGB color space which is in any case definitively not a good choice to give satisfying results from the psycho-visual point of view. In these images, the meta-hyperedges are classified using an arbitrary number and displayed in false colors. Unfortunately the current limitations of displaying facilities forced us to limit the effective number of representatives of classes (only 256 classes can be represented). However the classes index seems to give relevant information concerning the different textured areas in that image.

The table 1 shows the exact number of different classes found for a fixed spatial threshold of 2 using the chessboard distance. The colorimetric distance consists of the absolute difference between the gray level between pixels (i.e. derived from L_1 norm).

(T)	Number of different classes according to the number of elements in the meta-hyperedge												
	0	1	2	3	4	5	6	7	8	9	10	11	**(S)**
15	1	2	21	124	419	1141	2215	2807	2719	2281	1822	1467	-
20	1	2	21	106	315	841	1556	2050	2462	2735	2799	2721	-
25	1	2	20	95	244	584	946	1259	1546	1736	1976	2107	-
30	1	2	21	73	174	385	569	752	858	1089	1284	1428	-
35	1	2	19	46	134	279	384	454	580	624	751	877	-
40	1	2	19	41	97	177	202	305	351	416	517	598	-
45	1	2	17	34	70	128	138	182	219	271	338	372	-
50	1	2	14	31	56	66	111	112	133	167	243	255	-
(A)	12	13	14	15	16	17	18	19	20	21	22	23	24
15	1213	922	746	659	553	442	320	263	173	72	19	2	1
20	2512	2212	1853	1571	1144	888	664	435	269	100	21	2	1
25	2416	2428	2489	2341	2108	1756	1305	804	349	104	21	2	1
30	1613	1797	1958	2086	2172	2124	1782	1024	440	122	21	2	1
35	1097	1199	1364	1578	1689	1777	1654	1074	449	124	21	2	1
40	680	791	920	1061	1311	1398	1306	1000	453	126	21	2	1
45	437	473	648	759	834	1034	1027	830	424	125	21	2	1
50	287	348	423	498	570	705	814	636	379	122	21	2	1

Table 1. Number of hyperedges classes in the classification for hyperedges of fixed size 2 using the chessboard distance. The colorimetric threshold **(T)** is varied from 15 to 50 with an increment of 5 using the . **(S)**: Size of the meta-hyperedge

As it is not useful to give in that paper the complete set of tables for the experiments we run (these are available at http://www.uni-koblenz.de/c̃hastel), we choose to display them for that series of experiments, that is, in the case of increasing size of spatial neighborhood from 1 to 6 for the chessboard distance (therefore for size varying from 8 to 180) and for varying colorimetric thresholds from 15 to 45 (see figure 6) using for the

Figure 4. Variation in the classification using different spatial thresholds β varying from 1 to 6 for a constant colorimetric threshold α of 20.

Figure 5. Variation in the classification using a fixed spatial threshold β of 3 for a colorimetric threshold α varying between of 15 and 50.

colorimetric distance the absolute distance between the gray levels between two pixels. In these figures, the classes are numbered increasingly from the smallest meta-hyperedge to the largest one and with an increasing number of vertices in each meta-hyperedge.

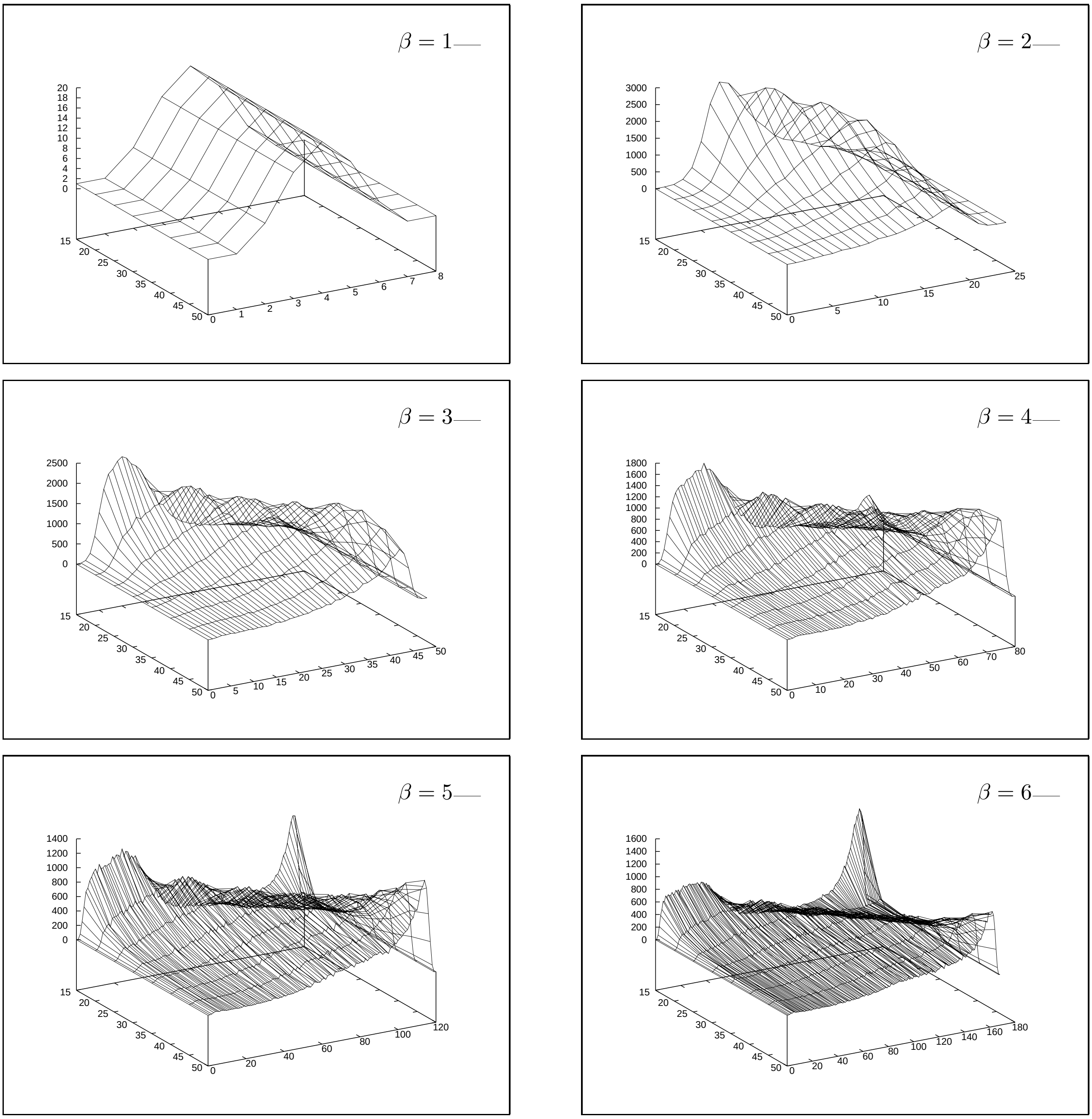

Figure 6. Number of hyperedges classes in the classification using a varying spatial threshold β (from 1 to 6) for the chessboard distance and for a varying colorimetric threshold α (from 15 and to 40). The colorimetric distance consists of the absolute difference between the gray level values between two pixels.

5. CONCLUSION. CURRENT AND PROSPECTIVE WORK

An application of hypergraph theory for texture discrimination has been proposed. It seems that texture identification is made possible by using a classification of pixel neighborhoods based on meta-hyperedges decomposition. The approximation we made in identifying hyperedges to their corresponding meta-hyperedge should soon be solved in using the characterization of mathematical objects called necklaces.[5,6]

Our hope is to study if a given texture has a meaningful number of meta-hyperedge classes is also relevant. The main drawback of that approach is its computational complexity (about forty seconds for a 512×512 image on a 1.8 GHz processor). These combinatorial problems lead unfortunately to the use of small areas surrounding image pixels. We think about possible breakthroughs to increase the current speed and making the image neighborhoods hypergraph model act as a simulator for the human perception.

6. ACKNOWLEDGMENTS

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REFERENCES

1. S. Chastel, *Contribution de la théorie des hypergraphes au traitement des images numériques*. PhD thesis, Université de Saint-Étienne, 2001.
2. A. Bretto and S. Chastel, "A note on valued polyominoes hypergraph image modeling," *International Journal of Computational and Numerical Analysis and Applications* **1**(1), 2002.
3. C. Münzenmeyer, H. Volk, D. Paulus, F. Vogt, and T. Wittenberg, "Multispectral geometrical features for texture analysis and classification," in Franke,[8] pp. 87–94.
4. C. Berge, *Hypergraphs*, North Holland, 1989.
5. F. Ruskey, C. Savage, and T. Min Yih Wang, "Generating necklaces," *J. Algorithms* **13**(3), pp. 414–430, 1992.
6. F. Ruskey and J. Sawada, "An efficient algorithm for generating necklaces with fixed density," *SIAM J. Comput.* **29**(2), pp. 671–684, 1999.
7. R. Diestel, *Graph Theory*, Springer-Verlag, 1997.
8. K.-H. Franke, ed., *8. Workshop Farbbildverarbeitung*, Schriftenreihe des Zentrums für Bild- und Signalverarbeitung e.V. Ilmenau, 1/2002, Ilmenau, 2002.